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Quantum Stirling heat engine with squeezed thermal reservoir

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We analyze the performance of a quantum Stirling heat engine (QSHE), using a two-level system and a harmonic oscillator as the working medium, that is in contact with a squeezed thermal reservoir and a cold reservoir. First, we derive closed-form expressions for the produced work and efficiency, which strongly depend on the squeezing parameter $r_{\rm h}$. Then, we prove that the effect of squeezing heats the working medium to a higher effective temperature, which leads to better overall performance. In particular, the efficiency increases with the degree of squeezing, surpassing the standard Carnot limit when the ratio of the temperatures of the hot and cold reservoirs is small. Furthermore, we derive the analytical expressions for the efficiency at maximum work and the maximum produced work in the high and low temperature regimes, and we find that at extreme temperatures the squeezing parameter $r_{\rm h}$ does not affect the performance of the QSHE. Finally, the performance of the QSHE depends on the nature of the working medium.

Keywords: quantum heat engine, open systems, thermodynamics, performance characteristics

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1. Introduction

1.1. Motivation

Over the past few years, the emerging field of quantum thermodynamics has attracted increasing attention in the scientific community, trying to find the quantum origin of the laws of thermodynamics.[1] Fundamental thermodynamic variables such as temperature T, internal energy U, etc., have been investigated extensively in terms of classical, statistical, quantum and relativistic quantum thermodynamics.[2–25] Furthermore, comprehension of the laws of classical thermodynamics is attainable, in the context of quantum heat engines (QHEs), by comparing thermodynamic quantities such as work and efficiency in terms of quantum variables. The QHE was first studied by Scovil and Schulz-DuBois in 1959 using a 3-level maser.[26] Thenceforward, the study of the performance of QHEs[27] has paved the way to new fields of research. In particular, the work and efficiency of QHEs,[28,29] have been analyzed with respect to the role of coherence,[30–36] the impact of entanglement,[37–40] the effect of a many-particle working medium,[41–46] the effect of the degeneracy factor in the energy levels,[47,48] fast Otto-cycles,[49] endoreversible Otto engines with a harmonic oscillator as the working medium,[50] endoreversible engines,[50–52] and an engine based on many body localization as a working fluid.[53]

One of the goals of studying quantum machines is to find ways to improve their overall performance.[54,55] Studies have emphasized that reinforcement-learning[56] and nonequilibrium quantum reservoirs,[57] coherent[58–60] and correlated,[61] can improve the performance of a quantum heat engine. Additionally, the concept of a squeezed thermal reservoir in quantum thermodynamics has started to attract increasing interest. The quantum Otto heat engine whose working medium is in contact with squeezed reservoirs has been studied extensively and presents some very interesting features. In particular, the efficiency at maximum power increases with the degree of squeezing, surpassing the standard Carnot limit.[62–69] In the real world, the efficiency at maximum power is a more important quantity than the maximum efficiency, at which power is zero.[70] The efficiency at maximum power was first studied by Curzon and Ahlborn (CA)[71] and they found that $\eta_{\rm CA} = 1 - \sqrt{T_{\rm c}/T_{\rm h}}$, where $T_{\rm c}$ and $T_{\rm h}$ are the temperatures of the cold and hot reservoirs respectively.

The question arises as to what impact the squeezed thermal bath will have on another thermodynamic cycle, such as the Stirling thermodynamic cycle that has received less attention. The Stirling thermodynamic cycle has two isothermal and two isochoric processes[72] and produces more work than the Otto cycle does under the same parameters or equal efficiency conditions and operates a wider range of coupling strengths.[73]

The aim of this paper is to study the performance of a QHE in which the working medium interacts with a pure cold reservoir and a squeezed thermal reservoir.[74–76] In particular, we analyze the quantum equivalent of the classical Stirling cycle[77–92] in which the hot reservoir is replaced by a thermal squeezed reservoir. We observe that the squeezing parameter $r_{\rm h}$ improves the performance of a QSHE, especially when the ratio of the temperatures of the hot and cold reservoirs is small, which means that the engine has less energetic cost. In order to compare the efficiency of the QSHE with the Carnot limit we introduce an effective temperature so that the steady state of the system under review can be seen as a thermal equi-

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librium state but at temperature $T_{\rm eff}$.[93–96] On the other hand, the generation of squeezing has a thermodynamic cost which depends on the specific configuration employed.[97,98] Generally speaking, when the cost of generating a squeezed state is included in the computation of the efficiency of the engine, the squeezed reservoirs are costly.[99] However, we focus on studying the performance of a QSHE without including the energy losses from the generation of squeezing.

1.2. Analysis and results

Our QHE consists of a working medium that is weakly coupled to a pure cold reservoir and a squeezed thermal reservoir alternately according to the Stirling thermodynamic cycle. We analyze two cases as far as the working medium is concerned, a two-level system and a harmonic oscillator. Our results are the following:

(i) We derive closed-form expressions for the produced work and efficiency which strongly depend on the squeezing parameter $r_{\rm h}$.

(ii) In the case of both working mediums, the produced work and the efficiency of the QSHE increase with the degree of squeezing, see Figs. 1–9.

(iii) The performance of the QSHE, i.e., the efficiency and the total produced work, depends on the nature of the working medium, see Figs. 1–9.

(iv) The effective temperature due to squeezing is always higher than the temperature of the thermal reservoir.

(v) We derived the analytic expressions for the efficiency at maximum work and the maximum produced work in the high and low temperature regimes.

(vi) In the low temperature regime, the efficiency of the QSHE with a two-level system as a working medium coincides with the efficiency of the quantum Otto heat engine.

(vii) At extreme temperatures the squeezing parameter $r_{\rm h}$ does not affect the performance of the QSHE.

The structure of this paper is the following. In Section 2 we examine the first case of a QSHE with a two-level system as a working medium. In Section 3, we examine the second case of a QSHE with a harmonic oscillator as the working medium. The conclusions are presented in Section 4.

2. Quantum Stirling heat engine — Two-level atom

In this section, we use a two-level system (TLS) of frequency ω as the working medium, which interacts with a squeezed thermal reservoir in the Born–Markov, rotating-wave approximation. The system Hamiltonian is given by $H = \frac{1}{2}\hbar\omega\sigma^z$. The evolution of the reduced density matrix operator of the system in the interaction picture has the following form:[100]

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial\tau} &= \Gamma[N+1]\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right)\\ &+\Gamma N\left(\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \frac{1}{2}\hat{\sigma}_-\hat{\sigma}_+\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_-\hat{\sigma}_+\right)\\ &-\Gamma M\hat{\sigma}_+\hat{\rho}\hat{\sigma}_+ - \Gamma M^*\hat{\sigma}_-\hat{\rho}\hat{\sigma}_-. \end{aligned} \tag{1}$$

Here Γ is the spontaneous emission rate and σ_+, σ_- are the raising and lowering operators respectively, where

$$N = n\cosh(2r_{\rm h}) + \sinh^2(r_{\rm h}), \tag{2}$$

$$M = -\cosh(r_{\rm h})\sinh(r_{\rm h})\,{\rm e}^{{\rm i}\phi}(2n+1), \tag{3}$$

$$n = \frac{1}{{\rm e}^{\frac{\hbar\omega}{k_{\rm B}T_{\rm h}}} - 1} \tag{4}$$

is the mean number of quanta, $T_{\rm h}$ the temperature, $r_{\rm h}$ the squeeze parameter and ϕ the phase. We notice that the asymptotic state does not depend on the squeezed phase ϕ.

$$\rho = \frac{1}{(2n+1)\cosh(2r)} \times \begin{pmatrix} n\cosh(2r)+\sinh^2(r) & 0 \\ 0 & n\cosh(2r)+\cosh^2(r) \end{pmatrix}, \tag{5}$$

and, equivalently, we have

$$\rho = \frac{1}{(2N+1)}\begin{pmatrix} N & 0 \\ 0 & N+1 \end{pmatrix}. \tag{6}$$

The interaction of a two-level quantum system with a squeezed environment causes a geometric phase (GP) or Berry phase (BP). The dependence of the GP on temperature and squeezing shows complexity.[101] However, equation (1) is valid only when the interaction between the system and the environment is very weak and the environmental correlation time is very small, or, in other words, if the time-scale of the external driving is much larger than the inner relaxation scale of the quantum dynamics of the system.[102–105]

The expectation value of energy is given by

$$U = \langle\hat{E}\rangle = {\rm Tr}(\hat{\rho}\hat{H}), \tag{7}$$

thus,

$$U = -\frac{\hbar\omega}{2[2N+1]}. \tag{8}$$

As far as the interaction of the working medium with the pure cold reservoir is concerned, analogous expressions can be obtained from the above relations by setting these squeezing parameters to zero and the temperature to $T_{\rm c}$.

The quantum Stirling heat engine is the quantum counterpart of the classical Stirling thermodynamic cycle. It consists of two isothermal processes and two isochoric processes. In the first case, as the working medium, we have a two-level system that interacts with a hot squeezed reservoir and a cold

bath alternately so as to perform a quantum Stirling thermodynamic cycle. The general quantum Stirling cycle has the following four steps repeated cyclically:

(i) Isothermal expansion $A(\omega_2,T_\mathrm{h})\to B(\omega_1,T_\mathrm{h})$. Initially, the TLS is in contact with the hot squeezed reservoir at a constant temperature T_h that can be described by a unitary squeezed operator with squeezed parameters r_h and ϕ. During this process the mean number of quanta and the energy gaps must change simultaneously in order for the system to remain in an equilibrium state. This process will occur slowly enough to allow the system to remain in equilibrium with the thermal bath continuously. The frequency reduces from ω_2 to a smaller value ω_1 and the heat transferred is $Q_{AB}=T_\mathrm{h}\Delta S_{AB}$.

(ii) Cold isochoric $B(\omega_1,T_\mathrm{h})\to C(\omega_1,T_\mathrm{c})$. In the second stroke, the TLS is disconnected from the hot squeezed bath and is connected with the cold bath at temperature T_c with fixed frequency.

(iii) Isothermal compression $C(\omega_1,T_\mathrm{c})\to D(\omega_2,T_\mathrm{c})$. In the third stroke, the TLS is still coupled with the cold bath and the frequency increases from ω_1 to a bigger value ω_2. The heat transferred is $Q_{CD}=T_\mathrm{c}\Delta S_{CD}$.

(iv) Hot isochoric $D(\omega_2,T_\mathrm{c})\to A(\omega_2,T_\mathrm{h})$. Finally, at constant frequency the TLS is disconnected from the cold bath and is connected with the hot squeezed bath at temperature T_h with fixed frequency.

The entropy for a TLS at thermal equilibrium is

$$S[\hat{\rho}]=-k_\mathrm{B}\,\mathrm{Tr}(\hat{\rho}\log\hat{\rho}), \tag{9}$$

thus

$$\begin{aligned}S=&-\frac{1}{(2n+1)\cosh 2r_\mathrm{h}}k_\mathrm{B}\\&\times\Big((\cosh^2 r_\mathrm{h}+n\cosh 2r_\mathrm{h})\log\frac{2n+1+(\cosh 2r_\mathrm{h})^{-1}}{4n+2}\\&+(n\cosh 2r_\mathrm{h}+\sinh^2 r_\mathrm{h})\log\frac{n+\sinh^2 r_\mathrm{h}\,\mathrm{sech}^2 2r_\mathrm{h}}{2n+1}\Big).\end{aligned} \tag{10}$$

We compute the entropy change in process AB

$$\begin{aligned}S_{AB}=&\,S(B)-S(A)\\=&\,k_\mathrm{B}\log\left[\frac{\cosh\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right)}{\cosh\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)}\right]+F_{r_\mathrm{h}}(\omega_2)\frac{\hbar\omega_2}{2T_\mathrm{h}}\tanh\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)\\&-F_{r_\mathrm{h}}(\omega_1)\frac{\hbar\omega_1}{2T_\mathrm{h}}\tanh\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right)\\&+\frac{1}{2}k_\mathrm{B}\log\left[\frac{1+(1-S_{r_\mathrm{h}}^2)\sinh^2\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)}{1+(1-S_{r_\mathrm{h}}^2)\sinh^2\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right)}\right],\end{aligned} \tag{11}$$

where

$$F_{r_\mathrm{h}}(\omega_i)=S_{r_\mathrm{h}}\left\{1+\frac{k_\mathrm{B}T_\mathrm{h}}{\hbar\omega_i}\log\left[\frac{1+S_{r_\mathrm{h}}+\mathrm{e}^{-\frac{\hbar\omega_i}{k_\mathrm{B}T_\mathrm{h}}}(1-S_{r_\mathrm{h}})}{1+S_{r_\mathrm{h}}+\mathrm{e}^{\frac{\hbar\omega_i}{k_\mathrm{B}T_\mathrm{h}}}(1-S_{r_\mathrm{h}})}\right]\right\}, \tag{12}$$

$$S_{r_\mathrm{h}}=\mathrm{sech}(2r_\mathrm{h}). \tag{13}$$

Consequently, the heat from this process is

$$Q_{AB}=T_\mathrm{h}\Big[S(B)-S(A)\Big], \tag{14}$$

and from Eq. (11) we obtain

$$\begin{aligned}Q_{AB}=&\,k_\mathrm{B}T_\mathrm{h}\log\left[\frac{\cosh\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right)}{\cosh\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)}\right]+F_{r_\mathrm{h}}(\omega_2)\frac{\hbar\omega_2}{2}\tanh\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)\\&-F_{r_\mathrm{h}}(\omega_1)\frac{\hbar\omega_1}{2}\tanh\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right)\\&+\frac{1}{2}k_\mathrm{B}T_\mathrm{h}\log\left[\frac{1+(1-S_{r_\mathrm{h}}^2)\sinh^2\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)}{1+(1-S_{r_\mathrm{h}}^2)\sinh^2\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right)}\right].\end{aligned} \tag{15}$$

The heat absorbed in the isothermal process AB using a thermal squeezed reservoir (15) has the usual form but with coefficients that contain the squeezing parameter r_h. In the limit $r_\mathrm{h}\to 0$, Eq. (15) takes the form

$$\begin{aligned}Q_{AB}=&\,k_\mathrm{B}T_\mathrm{h}\log\left[\frac{\cosh\frac{\hbar\omega_1}{2T_\mathrm{h}}}{\cosh\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}}\right]+\frac{\hbar\omega_2}{2}\tanh\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)\\&-\frac{\hbar\omega_1}{2}\tanh\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right).\end{aligned} \tag{16}$$

The work output per cycle according to the first law of thermodynamics is Refs. [3,106]

$$Q_{AB}=\Delta U_{AB}+W_{AB}. \tag{17}$$

The expectation value of energy from Eq. (8) is

$$\begin{aligned}\langle\hat{H}\rangle_A&=-\frac{\hbar\omega_2}{2}S(r_\mathrm{h})\tanh\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right),\\\langle\hat{H}\rangle_B&=-\frac{\hbar\omega_1}{2}S(r_\mathrm{h})\tanh\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right),\\\langle\hat{H}\rangle_C&=-\frac{\hbar\omega_1}{2}\tanh\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{c}}\right),\\\langle\hat{H}\rangle_D&=-\frac{\hbar\omega_2}{2}\tanh\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{c}}\right).\end{aligned} \tag{18}$$

The expectation value of work is

$$\begin{aligned}W_{AB}=&\,k_\mathrm{B}T_\mathrm{h}\log\left[\frac{\cosh\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right)}{\cosh\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)}\right]\\&+\frac{1}{2}k_\mathrm{B}T_\mathrm{h}\log\left[\frac{1+(1-S_{r_\mathrm{h}}^2)\sinh^2\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)}{1+(1-S_{r_\mathrm{h}}^2)\sinh^2\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right)}\right]\\&+G(\omega_2)\tanh\left(\frac{\hbar\omega_2}{2k_\mathrm{B}T_\mathrm{h}}\right)-G(\omega_1)\tanh\left(\frac{\hbar\omega_1}{2k_\mathrm{B}T_\mathrm{h}}\right),\end{aligned} \tag{19}$$

where

$$G(\omega_i)=S_{r_\mathrm{h}}\frac{k_\mathrm{B}T_\mathrm{h}}{\hbar\omega_i}\log\left[\frac{1+S_{r_\mathrm{h}}+\mathrm{e}^{-\frac{\hbar\omega_i}{k_\mathrm{B}T_\mathrm{h}}}(1-S_{r_\mathrm{h}})}{1+S_{r_\mathrm{h}}+\mathrm{e}^{\frac{\hbar\omega_i}{k_\mathrm{B}T_\mathrm{h}}}(1-S_{r_\mathrm{h}})}\right]. \tag{20}$$

Following the same procedure as before we compute the heat and work in the third stroke

$$\begin{aligned} Q_{CD} &= T_{\rm c}\Big[S(D)-S(C)\Big] \\ &= k_{\rm B}T_{\rm c}\log\left[\frac{\cosh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right)}{\cosh\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm c}}\right)}\right]+\frac{\hbar\omega_1}{2}\tanh\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm c}}\right) \\ &\quad -\frac{\hbar\omega_2}{2}\tanh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right), \end{aligned} \tag{21}$$

$$W_{CD} = k_{\rm B}T_{\rm c}\log\left[\frac{\cosh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right)}{\cosh\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm c}}\right)}\right]. \tag{22}$$

In the quantum isochoric process, no work is done, and thus the absorbed or released heat is

$$\begin{aligned} Q_{BC} &= U(C)-U(B) \\ &= \frac{\hbar\omega_1}{2}\left[\operatorname{sech}(2r_{\rm h})\tanh\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm h}}\right)-\tanh\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm c}}\right)\right], \end{aligned} \tag{23}$$

$$\begin{aligned} Q_{DA} &= U(A)-U(D) \\ &= \frac{\hbar\omega_2}{2}\left[\tanh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right)-\operatorname{sech}(2r_{\rm h})\tanh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm h}}\right)\right]. \end{aligned} \tag{24}$$

The total work is computed from Eqs. (19) and (22), $W = W_{AB}+W_{CD}$, thus

$$\begin{aligned} W &= k_{\rm B}T_{\rm h}\log\left[\frac{\cosh\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm h}}\right)}{\cosh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm h}}\right)}\right]+k_{\rm B}T_{\rm c}\log\left[\frac{\cosh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right)}{\cosh\left(\frac{\omega_1}{2k_{\rm B}T_{\rm c}}\right)}\right] \\ &\quad +\frac{k_{\rm B}T_{\rm h}}{2}\log\left[\frac{1+(1-S_{r_{\rm h}}^2)\sinh^2\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm h}}\right)}{1+(1-S_{r_{\rm h}}^2)\sinh^2\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm h}}\right)}\right] \\ &\quad +G(\omega_2)\tanh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm h}}\right)-G(\omega_1)\tanh\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm h}}\right). \end{aligned} \tag{25}$$

In Figs. 1 and 2 we plot the produced work $W/(k_{\rm B}T_{\rm c})$ of the QSHE as a function of the ratio of frequencies ω_2/ω_1 and temperatures $T_{\rm h}/T_{\rm C}$ respectively, with different values of the squeezing parameter $r_{\rm h}$. We notice that the squeezing parameter $r_{\rm h}$ plays a pivotal role in the work production due to the fact that the work is maximum even if the ratio of hot and cold temperatures is small, meaning less energetic cost.

The system absorbs heat only during the first isothermal stroke and the final isochoric stroke, thus, the total heat absorbed during a complete cycle

$$Q_H = Q_{AB}+Q_{DA}. \tag{26}$$

When the average heat $\langle\hat{Q}\rangle_H \geq 0$ it is absorbed from the hot squeezed reservoir, and when the average heat $\langle\hat{Q}\rangle_C \leq 0$ it is absorbed from the cold thermal reservoir. The mean work $\langle\hat{W}\rangle_{ab} \geq 0$ expresses the consumed work by the system and $\langle\hat{W}\rangle_{cd} \leq 0$ means that the system produces work.

The efficiency of the QHE is the proportion of the total work per cycle with respect to the heat absorbed from the hot thermal reservoir:

$$\eta_{\rm S} = \frac{\langle\hat{W}\rangle_{\rm total}}{\langle\hat{Q}\rangle_H} = 1+\frac{Q_{BC}+Q_{CD}}{Q_{AB}+Q_{DA}}. \tag{27}$$

In Fig. 3, we plot the efficiency $\eta_{\rm S}$ of our cycle as a function of the squeezing parameter $r_{\rm h}$ and compare it with the Carnot and Curzon–Ahlborn efficiencies.[71] We notice that the squeezing parameter $r_{\rm h}$ increases the efficiency, especially when the ratio of temperatures is small, and the efficiency tends to the Curzon–Ahlborn efficiency $\eta_{\rm CA}$, which is the efficiency at maximum work.[107] On the other hand, when the ratio of temperatures $T_{\rm h}/T_{\rm C}$ increases the squeezing parameter cannot improve the performance of the QSHE so as to surpass the Carnot efficiency $\eta_{\rm C}$.

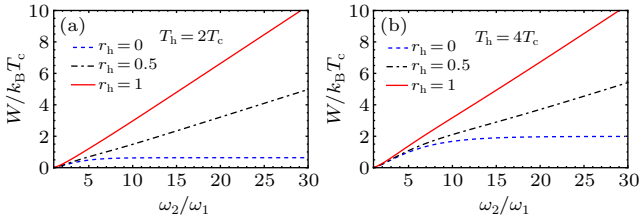


Fig. 1. Total work output $W/(k_{\rm B}T_{\rm c})$ as a function of the frequency modulation ω_2/ω_1, for different values of the squeezed parameter $r_{\rm h}$.

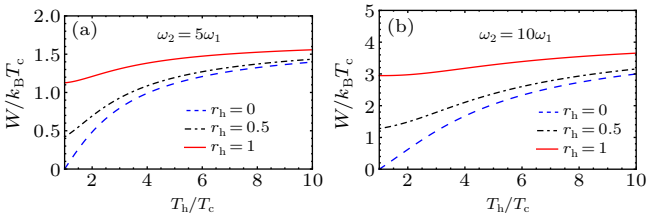


Fig. 2. Total work output $W/(k_{\rm B}T_{\rm c})$ as a function of the temperature $T_{\rm h}/T_{\rm c}$, for different values of the squeezed parameter $r_{\rm h}$.

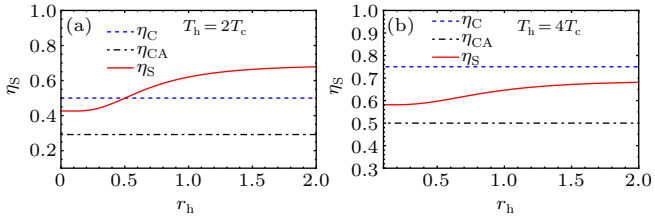


Fig. 3. Efficiency of the heat engine as a function of the squeezing parameter $r_{\rm h}$ in comparison to the Carnot $\eta_{\rm C}$ and Curzon–Ahlborn $\eta_{\rm CA}$ efficiencies. Others parameters are $\omega_2 = 5\omega_1$.

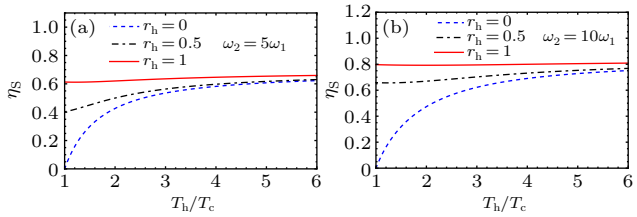


Fig. 4. Efficiency of the heat engine as a function of the temperatures $T_{\rm c}/T_{\rm h}$ for different ratios of frequencies.

In Fig. 5 we plot the efficiency of the QSHE as a function of the squeezing parameter $r_{\rm h}$ and ratio of frequencies ω_2/ω_1.

Moreover, we compare it with the Carnot efficiency and show that the efficiency increases with the increase of $\omega_1 = k_B T_c/\hbar$ as well as with r_h, and for some values of these parameters the QSHE surpasses the Carnot limit.

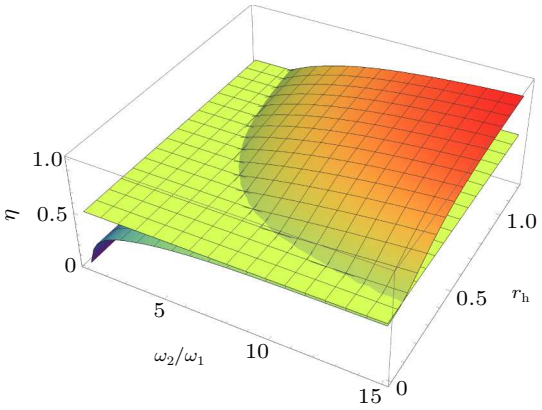


Fig. 5. Efficiency of the heat engine as a function of ω_2/ω_1 and the squeezing parameter r_h for $T_h = 2T_C$ in comparison with the Carnot efficiency.

We highlight that the efficiency of the QSHE with squeezed thermal reservoir surpasses the standard Carnot limit when the ratio of the hot and cold baths' temperatures is small. This interesting result is caused when the squeezing parameter $r_h \to 1$, and its mechanisms can be explained as follows. At the end of the quantum isochoric process, the asymptotic state of the two-level system is given by Eq. (6). When $r_h \to 0$ the asymptotic state of the two-level system is given by

$$\rho = \frac{1}{(2n+1)} \begin{pmatrix} n & 0 \\ 0 & n+1 \end{pmatrix}, \tag{28}$$

where n is given by Eq. (4). In contrast, when $r_h > 0$, the asymptotic state of the two-level system is given again by Eq. (28) but with an effective temperature that contains the squeezing parameter r_h and the temperature T,

$$T^{\text{eff}} = \frac{\hbar\omega}{2k_B \operatorname{arctanh}\left[\frac{\tanh\left(\frac{\hbar\omega}{2k_B T_h}\right)}{\cosh(2r_h)}\right]}, \tag{29}$$

and T^{eff} is always higher than T due to the squeezed parameters r_h.[93–96]

Let us now proceed to examine the high temperature limit, $\omega/T_h \to 0$. The total produced work in the first order approximation is

$$W_{\text{ht}} = \frac{\hbar^2(\omega_2^2 - \omega_1^2)}{8k_B}\left[S_{r_h}(S_{r_h} - 2)\frac{1}{T_h} + \frac{1}{T_c}\right]. \tag{30}$$

We use the second derivative test with respect to the ratio of frequencies ω_2/ω_1 [108] in order to find the maximum value of the produced work. In this case the produced work does not have a local maximum value.

Next, we compute the efficiency in the high temperature regime in the second order approximation

$$\eta_{\text{mw}}^{\text{ht}} = 1 + \frac{2\frac{\omega_1^2}{\omega_2^2}S_{r_h}(1-\eta_C) - 1) - (1 - \frac{\omega_1^2}{\omega_2^2})}{2 - (1-\eta_C)\hbar S_{r_h}\left[2 - \hbar S_{r_h}(1-\eta_C)(1-\frac{\omega_1^2}{\omega_2^2})\right]}. \tag{31}$$

Let us now proceed to examine the low temperature limit $\omega/T_C \to \infty$. The total produced work is

$$W_{\text{lt}} = \frac{1}{2}\hbar(\omega_2 - \omega_1) + \frac{\hbar^2(\omega_2^2 - \omega_1^2)}{8k_B T_h}\left[S_{r_h}(S_{r_h} - 2)\right], \tag{32}$$

where the first and second terms are the first and second order approximations respectively. The maximum work is when

$$\omega_2 = \frac{2k_B T_h}{\hbar}\frac{1}{S_{r_h}(2 - S_{r_h})}. \tag{33}$$

The efficiency in the low temperature regime is

$$\eta_{\text{mw}}^{\text{lt}} = 1 - \frac{\omega_1}{\omega_2}, \tag{34}$$

which coincides with the efficiency of a quantum Otto heat engine,[109,110] and is independent of the squeezing parameter r_h as it depends on the ratio of the frequencies only. The efficiency at maximum work is taken from Eqs. (33) and (34).

The second order approximation of the efficiency, which depends on squeezing parameter r_h, is

$$\eta_{\text{mw}}^{\text{lt}} = 1 - \frac{\omega_1^2}{\omega_2^2}\frac{\frac{4k_B T_h}{\omega_1} - 2\hbar S_{r_h}}{\frac{4k_B T_h}{\omega_2} - \hbar S_{r_h}\left[2 - S_{r_h}(1 - \frac{\omega_1^2}{\omega_2^2})\right]}. \tag{35}$$

3. Quantum Stirling heat engine — Harmonic oscillator

In this section we examine the harmonic oscillator as the working medium of the QSHE. First, we discuss the thermal squeezed state of the harmonic oscillator. For a harmonic oscillator of mass m and frequency ω, the squeezed thermal state results from the application of the squeezing operator to the thermal equilibrium state. The density matrix is as follows:

$$\rho_{\text{sq}} = \hat{S}(\xi)\rho_G\hat{S}^\dagger(\xi) = \hat{S}(\xi)\frac{e^{-\beta\hat{H}}}{Z(\beta)}\hat{S}^\dagger(\xi), \tag{36}$$

where $\hat{\rho}_G$ is the Gibbs state of the harmonic oscillator, and $\hat{S}(\xi)$ is the squeezed operator,

$$\hat{S}(\xi) = e^{\frac{1}{2}(\xi^*\alpha^2 - \xi\alpha^{\dagger 2})}, \tag{37}$$

and $\xi = re^{i\theta}$ is a complex number with $r \geq 0$ called the squeezing parameter with $\theta \in [0, 2\pi]$. The internal energy of the harmonic oscillator in a squeezed thermal state is[111]

$$\langle\hat{H}\rangle = \hbar\omega\left(n + \tfrac{1}{2}\right)\cosh(2r_h), \tag{38}$$

where

$$n = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}. \tag{39}$$

In an analogous way, as in the first part of this work, we construct the four strokes of the QSHE with the harmonic oscillator as the working medium,[112] and then we compute the expectation value of energy of each process, thus

$$\begin{aligned}\langle\hat{H}\rangle_A &= \frac{\hbar\omega_2}{2S(r_{\rm h})}\coth\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm h}}\right),\\ \langle\hat{H}\rangle_B &= \frac{\hbar\omega_1}{2S(r_{\rm h})}\coth\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm h}}\right),\\ \langle\hat{H}\rangle_C &= \frac{\hbar\omega_1}{2}\coth\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm c}}\right),\\ \langle\hat{H}\rangle_D &= \frac{\omega_2}{2}\coth\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right).\end{aligned} \tag{40}$$

Consequently, the heat in the isothermal process AB is

$$\begin{aligned}Q_{AB} &= k_{\rm B}T_{\rm h}\sum_{i=1}^{2}(-1)^{i+1}\Big\{F_{r_{\rm h}}(\omega_i)\log[F_{r_{\rm h}}(\omega_i)]\\ &\quad-(F_{r_{\rm h}}(\omega_i)-1)\log[F_{r_{\rm h}}(\omega_i)-1]\Big\},\end{aligned} \tag{41}$$

where

$$F_{r_{\rm h}}(\omega) = \frac{1}{2}\left[1+S_{r_{\rm h}}^{-1}\coth\left(\frac{\hbar\omega}{2k_{\rm B}T_{\rm h}}\right)\right]. \tag{42}$$

The expectation value of work in this process is obtained from the first law of thermodynamics

$$\begin{aligned}W_{AB} &= k_{\rm B}T_{\rm h}\sum_{i=1}^{2}(-1)^{i+1}\Big\{F_{r_{\rm h}}(\omega_i)\log[F_{r_{\rm h}}(\omega_i)]\\ &\quad-(F_{r_{\rm h}}(\omega_i)-1)\log[F_{r_{\rm h}}(\omega_i)-1]\Big\}\\ &\quad+\frac{\hbar\omega_2}{2S(r_{\rm h})}\coth\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm h}}\right)-\frac{\hbar\omega_1}{2S(r_{\rm h})}\coth\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm h}}\right).\end{aligned} \tag{43}$$

In the quantum isochoric processes, no work is done, and thus the absorbed or released heat is

$$\begin{aligned}Q_{BC} &= U(C)-U(B)\\ &= \frac{\hbar\omega_1}{2}\left[\coth\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm c}}\right)-\frac{1}{S(r_{\rm h})}\coth\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm h}}\right)\right],\end{aligned} \tag{44}$$

$$\begin{aligned}Q_{DA} &= U(A)-U(D)\\ &= \frac{\hbar\omega_2}{2}\left[\frac{1}{S(r_{\rm h})}\coth\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm h}}\right)-\coth\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right)\right].\end{aligned} \tag{45}$$

Following the same procedure as before we compute the heat and work in the third stroke

$$\begin{aligned}Q_{CD} &= T_{\rm c}\Big[S(D)-S(C)\Big]\\ &= k_{\rm B}T_{\rm c}\log\left[\frac{\sinh\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm c}}\right)}{\sinh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right)}\right]+\frac{\hbar\omega_2}{2}\coth\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right)\\ &\quad-\frac{\hbar\omega_1}{2}\coth\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm c}}\right).\end{aligned} \tag{46}$$

Consequently, the total produced work is

$$\begin{aligned}W_{\rm total} &= k_{\rm B}T_{\rm h}\sum_{i=1}^{2}(-1)^{i+1}\Big\{F_{r_{\rm h}}(\omega_i)\log[F_{r_{\rm h}}(\omega_i)]\\ &\quad-(F_{r_{\rm h}}(\omega_i)-1)\log[F_{r_{\rm h}}(\omega_i)-1]\Big\}\\ &\quad+k_{\rm B}T_{\rm c}\log\left[\frac{\sinh\left(\frac{\hbar\omega_1}{2k_{\rm B}T_{\rm c}}\right)}{\sinh\left(\frac{\hbar\omega_2}{2k_{\rm B}T_{\rm c}}\right)}\right].\end{aligned} \tag{47}$$

In Figs. 6 and 7, we show that the total produced work increases monotonically from zero as a function of the frequency ratio with respect to the increase of the squeezing parameter $r_{\rm h}$.

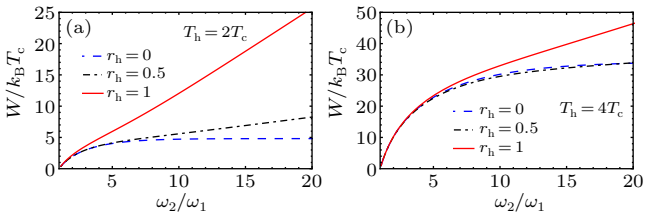


Fig. 6. Total work output $W/(k_{\rm B}T_{\rm c})$ as a function of the frequency modulation, ω_2/ω_1, for different values of the squeezed parameter $r_{\rm h}$ and temperature ratio.

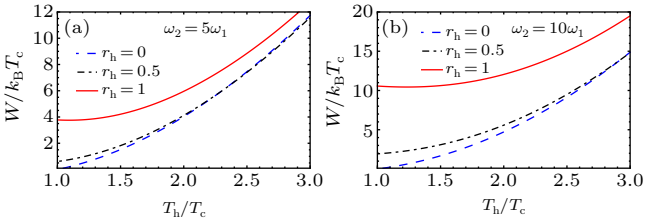


Fig. 7. Total work output, $W/(k_{\rm B}T_{\rm c})$ as a function of temperatures ratio, $T_{\rm h}/T_{\rm C}$, for different values of the squeezed parameter $r_{\rm h}$ and frequency ratio.

Furthermore, we compute the efficiency of the QSHE

$$\eta = \frac{\langle\hat{W}\rangle_{\rm total}}{\langle\hat{Q}\rangle_H} = 1+\frac{Q_{BC}+Q_{CD}}{Q_{AB}+Q_{DA}}. \tag{48}$$

Due to the fact that the analytical expression of the efficiency of the QSHE is complicated, we plot the numerical results in Figs. 8 and 9. In Fig. 8 we plot the efficiency as a function of squeezing parameter $r_{\rm h}$ and compare it with the Curzon–Ahlborn $\eta_{\rm CA}$ efficiency and the efficiency of the classical Stirling heat engine, which is the same as the Carnot efficiency $\eta_{\rm C}$. We show that increasing $r_{\rm h}$ can improve the efficiency of the QSHE so as to approach and surpass the Carnot limit.

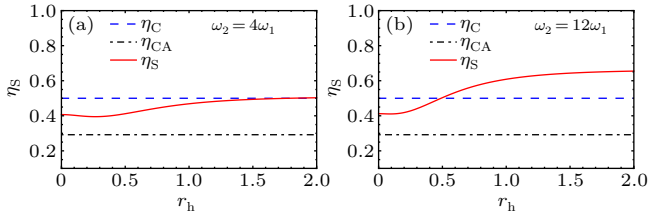


Fig. 8. Efficiency η as a function of the squeezed parameter $r_{\rm h}$ for different frequencies of the oscillator $\omega_1=k_{\rm B}T_{\rm c}/\hbar$ in comparison to the Carnot $\eta_{\rm C}$ and Curzon-Ahlborn $\eta_{\rm CA}$ efficiencies. Other parameters are $T_{\rm h}=2T_{\rm c}$.

In Fig. 9 we plot the efficiency of the QSHE as a function of the squeezing parameter $r_{\rm h}$ and ratio of frequencies ω_2/ω_1. Moreover, we compare it with the Carnot efficiency and show that the efficiency increases with the increase of $\omega_1 = k_{\rm B}T_{\rm c}/\hbar$ as well as with $r_{\rm h}$ and that for some values of these parameters the QSHE surpasses the Carnot limit.

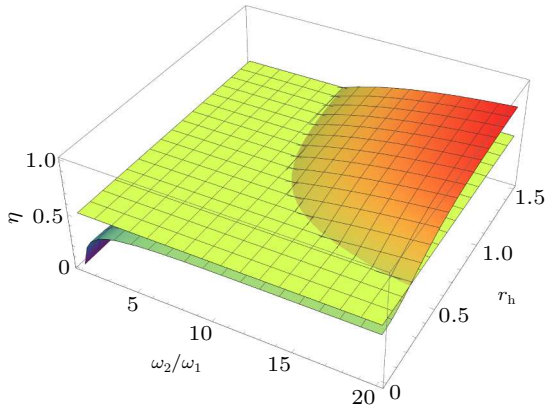


Fig. 9. Efficiency of the QSHE with respect to ω_2/ω_1 and the squeezing parameter $r_{\rm h}$ in comparison with the Carnot efficiency. The oscillator's frequency is $\omega_1 = k_{\rm B}T_{\rm c}/\hbar$.

Let us now proceed to examine the high temperature limit $\omega/T_{\rm h} \to 0$. The total produced work is

$$W_{\rm ht} = k_{\rm B}(T_{\rm h}-T_{\rm c})\log\left(\frac{\omega_2}{\omega_1}\right)+\frac{\hbar}{12k_{\rm B}T_{\rm h}S_{r_{\rm h}}}(\omega_2^2-\omega_1^2) +\frac{\hbar^2}{24}(\omega_1^2-\omega_2^2)\left(\frac{S_{r_{\rm h}}}{k_{\rm B}T_{\rm h}S_{2r_{\rm h}}}+\frac{1}{k_{\rm B}T_{\rm c}}\right), \tag{49}$$

where the first term is the first order approximation and the rest of the terms are derived from the second-order approximation. Following the same procedure as in the first part of this work, we find that the maximum work is when

$$\omega_2 = \frac{2\sqrt{3k_{\rm B}T_{\rm c}\eta_{\rm C}}}{\hbar\sqrt{1+S_{2r_{\rm h}}^{-1}S_{r_{\rm h}}^2(1-\eta_{\rm C})}}. \tag{50}$$

Consequently, the efficiency at maximum work in the high temperature regime is

$$\eta_{\rm mw}^{\rm ht} = \frac{\eta_{\rm C}}{\eta_{\rm C}-1+S_{r_{\rm h}}^{-1}+\log\left(\frac{\omega_2}{\omega_1}\right)}\log\left(\frac{\omega_2}{\omega_1}\right). \tag{51}$$

Let us now proceed to examine the low temperature limit $\omega/T_{\rm C} \to \infty$. The total produced work is

$$W_{\rm lt} = k_{\rm B}T_{\rm h}\log\left(\frac{\omega_2}{\omega_1}\right)+\frac{1}{2}\hbar(\omega_1-\omega_2) +\frac{\hbar}{12}(\omega_1^2-\omega_2^2)\left(\frac{1}{S_{r_{\rm h}}}-\frac{S_{r_{\rm h}}}{2S_{2r_{\rm h}}}\frac{\hbar}{k_{\rm B}T_{\rm h}}\right), \tag{52}$$

where the first two terms are the first order approximation. The maximum work is when

$$\omega_2 = \frac{1+S_{2r_{\rm h}}}{2}\frac{k_{\rm B}T_{\rm h}}{\hbar}\left[-3+\sqrt{3(11-4S_{r_{\rm h}}^2)}\right]. \tag{53}$$

Consequently, the efficiency at maximum work in the low temperature regime is

$$\eta_{\rm mw}^{\rm lt} = 1+\frac{\hbar\omega_1-2k_{\rm B}T_{\rm h}S_{r_{\rm h}}^{-1}}{-\hbar\omega_2+2k_{\rm B}T_{\rm h}[S_{r_{\rm h}}^{-1}+\log\left(\frac{\omega_2}{\omega_1}\right)]}. \tag{54}$$

4. Conclusion

In this paper, we have investigated a quantum Stirling heat engine (QSHE) with a two-level system and a harmonic oscillator as the working medium that is in contact with a squeezed thermal reservoir and a cold reservoir. We obtained closed-form expressions for the produced work and efficiency which depend strongly on the squeezing parameter $r_{\rm h}$. Then we proved that the effect of squeezing heats the working medium to a higher effective temperature leading to a better overall performance. In particular, the efficiency increases with the degree of squeezing surpassing the standard Carnot limit when the ratio of the temperatures of the hot and cold reservoirs is small, which means that the engine has less energetic cost. Additionally, we derived the analytic expressions for the efficiency at maximum work and the maximum produced work in the high and low temperature regimes. We found that the efficiency in the low temperature regime coincides with the efficiency of a quantum Otto heat engine and is independent of the squeezing parameter $r_{\rm h}$, depending only on the ratio of frequencies. We noticed that in the first order approximation the efficiency at maximum work and the maximum produced work are independent of the squeezing parameter $r_{\rm h}$. In contrast, the aforementioned result in the second order approximation depends on the squeezing parameter $r_{\rm h}$. This means that at extreme temperatures the squeezing parameter $r_{\rm h}$ does not affect the performance of the QSHE. This phenomenon is expected because the system loses its quantum features and the state becomes classical, i.e., a mixture of coherent states, losing its entanglement potential and other quantum features.[113,114] Finally, the performance of the QSHE depends on the nature of the working medium. Thus, in future work, we will extend the corresponding analysis for different working mediums, such as spin-1/2 systems[29,115,116] and coupled oscillators.[117]

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