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Synchronization of stochastic complex networks with time-delayed coupling

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Noise and time delay are inevitable in real-world networks. In this article, the framework of master stability function is generalized to stochastic complex networks with time-delayed coupling. The focus is on the effects of noise, time delay, and their inner interactions on the network synchronization. It is found that when there exists time-delayed coupling in the network and noise diffuses through all state variables of nodes, appropriately increasing the noise intensity can effectively improve the network synchronizability; otherwise, noise can be either beneficial or harmful. For stochastic networks, large time delays will lead to desynchronization. These findings provide valuable references for designing optimal complex networks in practical applications.

Keywords: stochastic complex network, synchronization, noise, time delay

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1. Introduction

Research on synchronization of complex networks has rapidly developed in recent years and has led to practical applications in many areas such as gene regulatory networks,[1,2] social networks,[3,4] neuronal networks,[5,6] transportation networks,[7–9] and power networks.[10,11] There are many methods to analyze synchronization in complex networks,[12–16] among which, the approach of master stability function (MSF) proposed by Pecora and Carroll in 1998 has attracted much attention.[12] In the MSF analysis, the stability of networks is separated into two independent tasks:[17,18] one is to analyze the synchronized regions determined by the oscillator dynamics and the inner-coupling function; the other is to analyze the eigenvalues of the network Laplacian matrix and the coupling strength. According to the MSF principle, if the product of the coupling strength and the eigenvalues of the network Laplacian matrix fall within the synchronized region, the network is asymptotically synchronized.[12,19] Therefore, existing works on improving the synchronizability of complex networks mainly focus on two aspects. One is to change the network structure, *i.e.*, to change the eigenvalues of the network Laplacian matrix.[20–23] The other is to change the synchronized region, which has been well explored in recent years.[17,24–27] For instance, it was discovered that a network could have different synchronized regions by altering the inner-coupling matrix.[24] It was also found that introducing dual-channel couplings can enlarge the synchronized region.[17] It was moreover reported that synchronized region can be adjusted by introducing noise into the network.[26]

For real networks, due to the physical limitation of transmission speed, time delay is often non-negligible in the process of signal transmission. Applying the method of MSF to analyze the synchronization of networks with time-delayed coupling reveals that synchronizability of networks can be improved by increasing time delays.[28,29] Meanwhile, random factors were taken into account when analyzing network synchronization.[30–37] It was discovered that increasing noise intensity may be beneficial to synchronization when noise passes through all state variables.[26] Nevertheless, there is no systematic study on the influence of both random noise and time delay on network synchronization by using the MSF method.

Motivated by the above discussions, the issue of synchronization for stochastic complex network with time-delayed coupling is investigated in this paper. Based on the MSF framework, an extended master stability equation is derived and consequently a new MSF is formulated. This extended MSF is related to both noise and time delay, different from commonly used network models. Further, the effects of noise and time delay on network synchronization are examined through extensive numerical simulations. The main contributions of this article are summarized as follows.

First, it is discovered that network synchronizability can be enhanced by tuning time delay and noise intensity simultaneously. This is different from existing works which change the network structure or altering the coupling strength. Second, it is found that when there exists time-delayed coupling and noise diffusion through all state variables, network synchronizability can be enhanced by appropriately increasing

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the noise intensity. It is also revealed that when there exists noise in the network, too large time delay might be harmful to network synchronizability. These findings help deepen our understanding of synchronization in stochastic complex networks with time-delayed coupling. Since noise and time delay are common in the real world, the new theoretical results can be useful for practical networks such as smart grids and UAV formations.

The rest of the paper is organized as follows. In Section 2, the research problem is formulated. In Section 3, the effects of noise and time delay on network synchronizability are studied under different inner-coupling functions and diffusion functions. Finally, the paper is concluded in Section 4.

2. Problem formulation

2.1. Model description

Consider the following stochastic oscillator complex network with time-delayed coupling:

$$\begin{aligned} \mathrm{d}x_i(t) = &\left(f(x_i(t)) - c\sum_{j=1}^{N} l_{ij} h(x_j(t-\tau)) \right) \mathrm{d}t \\ &+ \sigma g(x_i(t-\tau))\,\mathrm{d}b(t), \quad i = 1,2,\ldots,N, \end{aligned} \tag{1}$$

where N is the number of nodes, $x_i \in \mathbb{R}^n$ the state of node i, $f(\cdot)\colon \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ a nonlinear and continuously differentiable function representing self-dynamics of each node, $c > 0$ the coupling strength, $h(\cdot)\colon \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ the inner-coupling function, and τ the time delay. Here, we assume that the network is undirected and connected. The Laplacian matrix $L = (l_{ij}) \in \mathbb{R}^{N\times N}$ determines the topology of the network, where l_{ij} is defined as follows: if there is an edge between node i and node j, then $l_{ij} = l_{ji} = -1$; otherwise, $l_{ij} = l_{ji} = 0$ $(j \neq i)$; and the diagonal elements are defined as

$$l_{ii} = -\sum_{j=1, j\neq i}^{N} l_{ij}, \quad i = 1,2,\ldots,N. \tag{2}$$

The stochastic term $\sigma g(x_i(t-\tau))\,\mathrm{d}b(t)$ describes modeling uncertainties for each node, where $b(t)$ is a standard Brownian motion, $g(\cdot)\colon \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is the diffusion function describing how the Brownian motion propagates on each node, and σ is the noise intensity.

2.2. Extended MSF analysis

The objective is to synchronize network (1) onto a desired state $s(t)$, which satisfies $\mathrm{d}s(t) = f(s(t))\mathrm{d}t + \sigma g(s(t-\tau))\mathrm{d}b(t)$. Define $e_i(t) = x_i(t) - s(t)$. The error dynamics is given by

$$\begin{aligned} \mathrm{d}e_i(t) &= \mathrm{d}x_i(t) - \mathrm{d}s(t) \\ &= \left[f(s(t)+e_i(t)) - f(s(t)) \right] \mathrm{d}t \\ &\quad - c\sum_{j=1}^{N} l_{ij} h(s(t-\tau)+e_j(t-\tau))\,\mathrm{d}t \\ &\quad + \sigma\left[g(s(t-\tau)+e_i(t-\tau)) - g(s(t-\tau)) \right] \mathrm{d}b(t), \\ &\quad i = 1,2,\ldots,N. \end{aligned} \tag{3}$$

Let $e(t) = \left[e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{nN}$. The error dynamics (3) can be linearized and rewritten in a compact form as

$$\begin{aligned} \mathrm{d}e(t) = &\left(\left[\mathbb{I}_N \otimes J_f(s(t))\right] e(t) - \left[cL \otimes J_h(s(t-\tau))\right] \cdot e(t-\tau) \right) \mathrm{d}t \\ &+ \sigma\left[\mathbb{I}_N \otimes J_g(s(t-\tau))\right] e(t-\tau)\,\mathrm{d}b(t), \end{aligned} \tag{4}$$

where $\otimes$ is the Kronecker product, $\mathbb{I}_N$ is the $N \times N$ identity martix, and $J_f(s(t))$, $J_h(s(t-\tau))$, and $J_g(s(t-\tau))$ are the Jacobian matrices of f, h, and g evaluated at $s(t)$ and $s(t-\tau)$, respectively.

Because the network is undirected and connected, the Laplacian matrix L is symmetric and diagonalizable, leading to $PLP^{-1} = M$, where M is a diagonal matrix consisting of the eigenvalues of L and the columns of P are the set of the corresponding eigenvectors. Define $\xi(t) := (P \otimes \mathbb{I}_n)\, e(t)$. Then

$$\begin{aligned} \mathrm{d}\xi(t) = &\left(\left[\mathbb{I}_N \otimes J_f(s(t))\right] \xi(t) - \left[cM \otimes J_h(s(t-\tau))\right] \cdot \xi(t-\tau) \right) \mathrm{d}t \\ &+ \sigma\left[\mathbb{I}_N \otimes J_g(s(t-\tau))\right] \xi(t-\tau)\,\mathrm{d}b(t). \end{aligned} \tag{5}$$

The dynamics matrix of system (5) consists of N diagonal blocks of size $n \times n$, where each block is related to one of the eigenvalues of the Laplacian matrix L. Note that the Laplacian matrix L has one zero eigenvalue and $N-1$ positive real eigenvalues satisfying $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$.

Each block of the dynamics matrix of system (5) can be described as

$$\begin{aligned} \mathrm{d}\xi_i(t) = &\left[J_f(s(t))\xi_i(t) - c\lambda_i J_h(s(t-\tau))\xi_i(t-\tau) \right] \mathrm{d}t \\ &+ \sigma J_g(s(t-\tau))\xi_i(t-\tau)\,\mathrm{d}b(t), \quad i = 1,2,\ldots,N, \end{aligned} \tag{6}$$

where $\xi_i(t)$ is the i-th eigenmode related to λ_i.

Let $\eta = c\lambda_i$. An extended master stability equation of the stochastic complex network with time-delayed coupling (1) is given as follows:

$$\begin{aligned} \mathrm{d}y(t) = &\left[J_f(s(t))y(t) - \eta J_h(s(t-\tau))y(t-\tau) \right] \mathrm{d}t \\ &+ \sigma J_g(s(t-\tau))y(t-\tau)\,\mathrm{d}b(t). \end{aligned} \tag{7}$$

According to the MSF method,[12] the criterion that the largest Lyapunov exponent Λ calculated by Eq. (7) is strictly negative guarantees the stability of the synchronous state. The synchronized region $\mathscr{S}$ is thus defined by the range of η for which $\Lambda < 0$. The boundary curve of the synchronized region is determined by the largest Lyapunov exponent. Here, Λ is called the master stability function of Eq. (7).

From the above analyses, the stability of Eq.(1) can be decomposed into two sub-problems. One is to identify the synchronized regions of Eq. (7), which depend on the inherent

dynamics of the isolated node $f(\cdot)$, inner-coupling function $h(\cdot)$, diffusion function $g(\cdot)$, time delay τ, and noise intensity σ. The other one is to analyze the eigenvalue distribution of the Laplacian matrix L.

3. Main results

Now, the objective is to analyze the effects of noise and time delay on the synchronization of stochastic complex networks with time-delayed coupling under different inner-coupling functions and diffusion functions. Without loss of generality, consider the Rössler system and Lorenz system as the self-dynamics of the nodes, respectively. The Rössler system is given as follows:[38]

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 + px_2, \\ \dot{x}_3 = q + (x_1 - r)x_3, \end{cases} \tag{8}$$

which has a chaotic attractor when $p = q = 0.2$ and $r = 9$. The Lorenz system is given as follows:[39]

$$\begin{cases} \dot{x}_1 = \rho\,(x_2 - x_1)\,, \\ \dot{x}_2 = \gamma x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = x_1 x_2 - \beta x_3, \end{cases} \tag{9}$$

which has a chaotic attractor when $\rho = 10$, $\beta = 1$, and $\gamma = 28$. In the simulation, the master stability equation (7) is discretized with time step 0.01. The noise is described by a Brownian motion with correspondingly-defined mean 0 and variance 0.01. The system averages the largest Lyapunov exponent in the exponential separation caused by each iteration in multiple iterations, denoted as Λ.

Next, the synchronized regions under different noises and time delays are simulated for inner-coupling functions and diffusion functions under different conditions. From the extended master stability equation (7), when the time delay τ is fixed, the largest Lyapunov exponent Λ can be obtained from (σ, η). Similarly, when the noise intensity σ is fixed, Λ can also be obtained from (τ, η). According to the MSF principle, when the largest Lyapunov exponent is negative, the synchronous state of the network is stable. Therefore, the task is to calculate Λ under different conditions and the region with $\Lambda < 0$ is called as the synchronized region.

3.1. Effect of the noise

First, the impact of noise on the synchronization of the stochastic oscillator complex network of Rössler systems is discussed. In this network, the time delay τ is set as 0.1. Then, the synchronized region of the network with noise is analyzed under different inner-coupling functions and diffusion functions.

In the first scenario, figure 1 shows the largest Lyapunov exponents $\Lambda(\sigma, \eta)$ of the network with different η and noise intensity σ. Figure 1(a) is drawn when the inner-coupling function $h(x_i(t-\tau)) = x_i(t-\tau)$ and the diffusion function $g(x_i(t-\tau)) = x_i(t-\tau)$. This means that the connection between nodes passes through all state variables and noise diffuses to all state variables of the nodes. In Figs. 1, 3, 5, and 7 below, the region of $\Lambda < 0$ is denoted in green color and that of $\Lambda > 0$ is denoted in red color. It can be observed that with the increase of the noise intensity, the synchronized region first emerges and then disappears. In particular, when $\sigma = 0.6$, the Lyapunov exponents corresponding to η are all less than zero, but when $\sigma = 2.7$, the largest Lyapunov exponent starts to become greater than 0, but there is no synchronized region when $\sigma = 3.2$. It can be seen that when noise diffuses and nodes are linked through all the state variables, appropriately increasing noise is beneficial to the synchronization of the network.

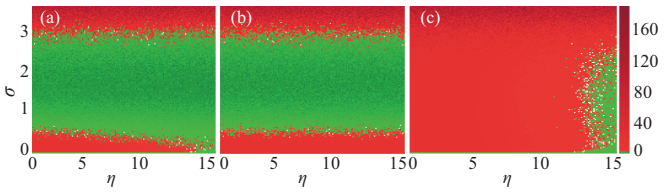


Fig. 1. The largest Lyapunov exponents $\Lambda(\sigma, \eta)$ for the stochastic complex network of Rössler systems with time delay $\tau = 0.1$: (a) $h(x_i(t-\tau)) = g(x_i(t-\tau)) = x_i(t-\tau)$; (b) $h(x_i(t-\tau)) = [x_{i1}(t-\tau), 0, 0]^{\mathrm{T}}$, $g(x_i(t-\tau)) = x_i(t-\tau)$; (c) $h(x_i(t-\tau)) = x_i(t-\tau)$, $g(x_i(t-\tau)) = [x_{i1}(t-\tau), 0, 0]^{\mathrm{T}}$.

To illustrate this observation in more details, figure 2 demonstrates several groups of MSF curves under different noise intensities. In the left panel of the figure, it can be observed that the small noise intensity shifts down the MSF curve at the beginning. According to the common classification,[18] the synchronized region switches from type Ψ_0 to type Ψ_1. Here, type Ψ_0 means that the synchronized region is an empty set. Type Ψ_1 means that the synchronized region is a bounded region. For the curve corresponding to $\sigma = 1$, $\Lambda < 0$ for all η, which means that all nodes in the network reach synchronization. However, as the noise intensity continuously increases, the MSF curve for $\sigma = 3.2$ is changed to be above all the other curves and the network fails to synchronize.

Next, the situation of $h(x_i(t-\tau)) = [x_{i1}(t-\tau), 0, 0]^{\mathrm{T}}$ and $g(x_i(t-\tau)) = x_i(t-\tau)$ is simulated, with results shown in Fig. 1(b). It can be observed that the synchronizability of the network is enhanced with properly increasing the noise intensity. The same can also be observed from Fig. 2(b). With the increase of the noise intensity, the MSF curves are moving downward at first but when increased to $\sigma = 3.2$, the curve moves to above all the others.

Through the above analysis, it can be concluded that as long as the noise diffuses through all the variables of the nodes, the synchronizability of the network can be improved by properly increasing the noise intensity.

An interesting phenomenon is observed from Figs. 1(a) and 1(b): when the noise intensity σ increases to a certain

level, the network is synchronized even if $\eta = 0$, implying that the network is disconnected, which is called noise-induced synchronization.[41–46] This phenomenon has been confirmed in the biological research literature.[46]

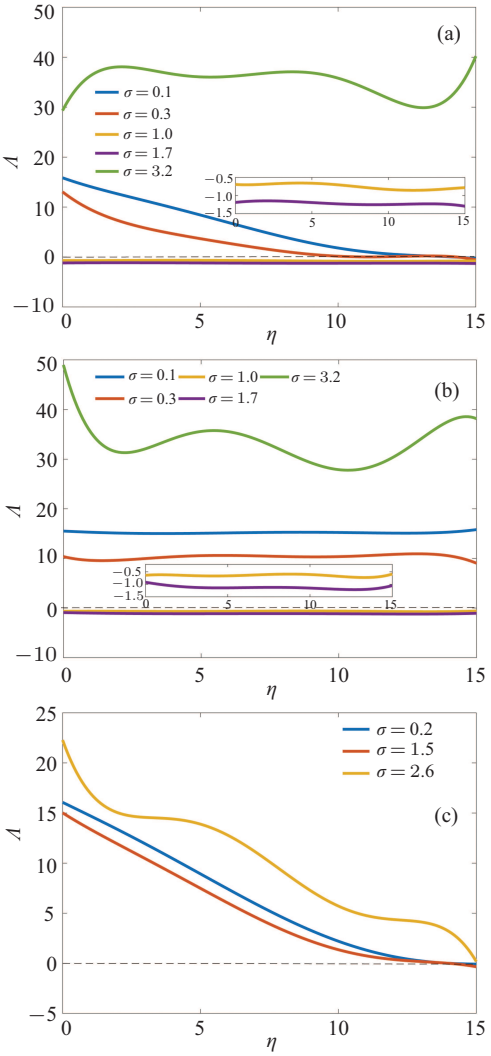


Fig. 2. The MSF curves for the stochastic complex network of Rössler systems with time delay $\tau = 0.1$ and different noise intensities σ. (a) $h(x_i(t-\tau)) = g(x_i(t-\tau)) = x_i(t-\tau)$; (b) $h(x_i(t-\tau)) = [x_{i1}(t-\tau), 0, 0]^{\mathrm{T}}$, $g(x_i(t-\tau)) = x_i(t-\tau)$; (c) $h(x_i(t-\tau)) = x_i(t-\tau)$, $g(x_i(t-\tau)) = [x_{i1}(t-\tau), 0, 0]^{\mathrm{T}}$.

Lastly, consider the case that the inner-coupling function $h(x_i(t-\tau))$ is $x_i(t-\tau)$ and the diffusion function $g(x_i(t-\tau))$ is $[x_{i1}(t-\tau), 0, 0]^{\mathrm{T}}$. Diffusion function in this form means that the noise only affects the first state variable of the node. Different from Figs. 1(a) and 1(b), it can be seen from Fig. 1(c) that only when η is large enough, the synchronized region emerges, but when the noise intensity is too large, the synchronized region disappears. Figure 2(c) also confirms this result. As illustrated in the figure, the MSF curve of $\sigma = 0.2$ is above the curve with $\sigma = 1.5$, but from the curve corresponding to $\sigma = 2.6$, it can be seen that it is above all curves and there is no synchronized region. Therefore, it can be concluded that when noise affects a subset of the state variables, noise can be beneficial or harmful to the synchronizability of stochastic oscillator complex networks with time-delayed coupling.

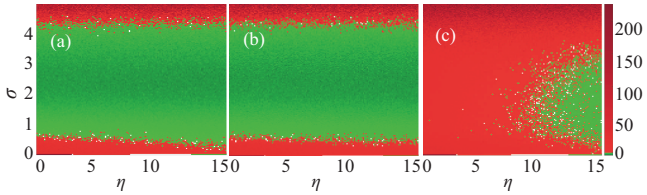


Fig. 3. The largest Lyapunov exponents $\Lambda(\sigma, \eta)$ for the stochastic complex network of Lorenz systems with time delay $\tau = 0.05$: (a) $h(x_i(t-\tau)) = g(x_i(t-\tau)) = x_i(t-\tau)$; (b) $h(x_i(t-\tau)) = [x_{i1}(t-\tau), x_{i2}(t-\tau), 0]^{\mathrm{T}}$, $g(x_i(t-\tau)) = x_i(t-\tau)$; (c) $h(x_i(t-\tau)) = x_i(t-\tau)$, $g(x_i(t-\tau)) = [x_{i1}(t-\tau), x_{i2}(t-\tau), 0]^{\mathrm{T}}$.

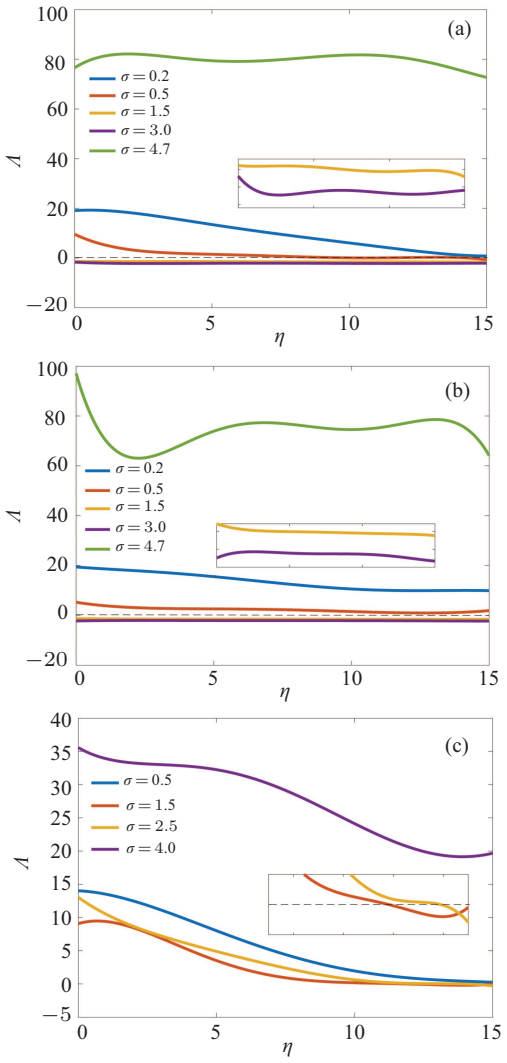


Fig. 4. The master stability function curves for the stochastic complex network of Lorenz systems with time delay $\tau = 0.05$ and different noise intensities σ. (a) $h(x_i(t-\tau)) = g(x_i(t-\tau)) = x_i(t-\tau)$; (b) $h(x_i(t-\tau)) = [x_{i1}(t-\tau), x_{i2}(t-\tau), 0]^{\mathrm{T}}$, $g(x_i(t-\tau)) = x_i(t-\tau)$; (c) $h(x_i(t-\tau)) = x_i(t-\tau)$, $g(x_i(t-\tau)) = [x_{i1}(t-\tau), x_{i2}(t-\tau), 0]^{\mathrm{T}}$.

To verify and confirm the above observation of the relationship between noise and network synchronization, the Lorenz system is used for simulation with the time delay τ set as 0.05. From Figs. 3(a) and 4(a), the same conclusion can be drawn: when noise diffuses and nodes are linked through all the state variables, the network synchronizability firstly increases with the increase of the noise intensity, but when it increases to $\sigma = 4.3$, the synchronizability begins to reduce.

To proceed, consider the case that the network nodes are connected by the first two state variables (see Figs. 3(b) and 4(b)), which leads to the same conclusion as before: increasing the noise intensity properly can improve the synchronizability of the network. Finally, consider the case when the inner-coupling function $h(x_i(t-\tau))$ is $x_i(t-\tau)$ and the diffusion function $g(x_i(t-\tau))$ is $[x_{i1}(t-\tau),x_{i2}(t-\tau),0]^{\mathrm{T}}$. It is found that the synchronized region first expands and then narrows down with the increase of the noise intensity. Therefore, this simulation further proves that the effect of noise on synchronization can be beneficial or harmful when noise propagates through a subset of the state variables.

3.2. Effect of the time delay

This section analyzes the influence of time delay on the synchronization of the network with different inner-coupling functions and diffusion functions. First, consider the situation that the nonlinear function of the node is Rössler system and the noise intensity σ is set as 1. As in the above analyses, we first consider the case when $h(x_i(t-\tau)) = g(x_i(t-\tau)) = x_i(t-\tau)$. As illustrated in Fig. 5(a), the synchronized region disappears with the increase of the time delay. From the MSF curves, one can see that the time delay causes the curves to move up. From Figs. 5(a) and 6(a), one can observe that when the noise diffuses are across all state variables of every node and the nodes are linked through all variables, larger time delay may reduce the synchronizability of the network.

The same conclusion can be drawn when $h(x_i(t-\tau)) = [x_{i1}(t-\tau),0,0]^{\mathrm{T}}$ and $g(x_i(t-\tau)) = x_i(t-\tau)$. From Fig. 5(b), as the time delay increases, the synchronized region disappears. From Fig. 6(b), the MSF curves move upward as the time delay increases. This implies that large time delay may be harmful to network synchronization when connections between nodes pass through a subset of state variables.

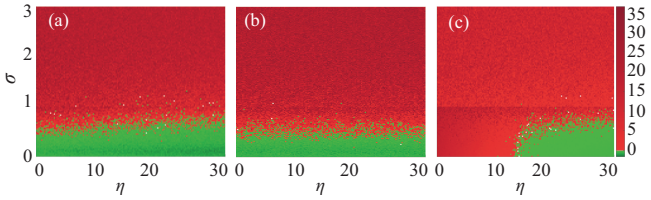


Fig. 5. The largest Lyapunov exponents $\Lambda(\tau,\eta)$ for the stochastic complex network of Rössler systems with noise intensity $\sigma = 1$. (a) $h(x_i(t-\tau)) = g(x_i(t-\tau)) = x_i(t-\tau)$; (b) $h(x_i(t-\tau)) = [x_{i1}(t-\tau),0,0]^{\mathrm{T}}$, $g(x_i(t-\tau)) = x_i(t-\tau)$; (c) $h(x_i(t-\tau)) = x_i(t-\tau)$, $g(x_i(t-\tau)) = [x_{i1}(t-\tau),0,0]^{\mathrm{T}}$.

Finally, we consider the situation that the diffusion function is $[x_{i1}(t-\tau),0,0]^{\mathrm{T}}$. From the color block diagram (Fig. 5(c)), one can also see that the synchronizability decreases with the increase of time delay, and when $\tau = 0.6$, the synchronized region disappears. In Fig. 6(c), the MSF curves change from type Ψ_1 to type Ψ_0 and keep rising with the increase of the time delay.

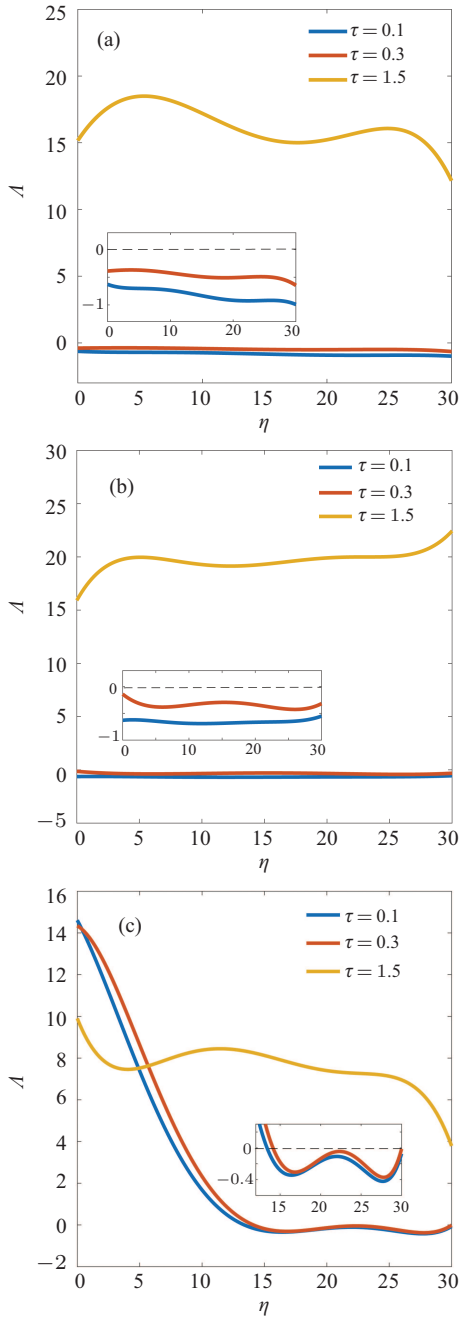


Fig. 6. The MSF curves for the stochastic complex network of Rössler systems with noise intensity $\sigma = 1$ and different time delay τ. (a) $h(x_i(t-\tau)) = g(x_i(t-\tau)) = x_i(t-\tau)$; (b) $h(x_i(t-\tau)) = [x_{i1}(t-\tau),0,0]^{\mathrm{T}}$, $g(x_i(t-\tau)) = x_i(t-\tau)$; (c) $h(x_i(t-\tau)) = x_i(t-\tau)$, $g(x_i(t-\tau)) = [x_{i1}(t-\tau),0,0]^{\mathrm{T}}$.

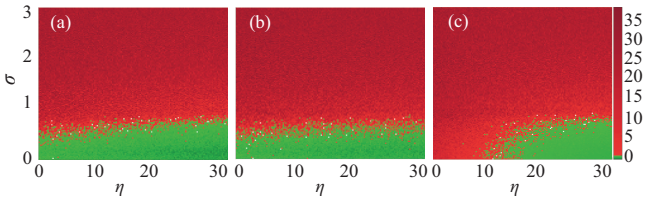


Fig. 7. The largest Lyapunov exponents $\Lambda(\tau,\eta)$ for the stochastic complex network of Lorenz systems with noise intensity $\sigma = 0.8$. (a) $h(x_i(t-\tau)) = g(x_i(t-\tau)) = x_i(t-\tau)$; (b) $h(x_i(t-\tau)) = [x_{i1}(t-\tau),x_{i2}(t-\tau),0]^{\mathrm{T}}$, $g(x_i(t-\tau)) = x_i(t-\tau)$; (c) $h(x_i(t-\tau)) = x_i(t-\tau)$, $g(x_i(t-\tau)) = [x_{i1}(t-\tau),x_{i2}(t-\tau),0]^{\mathrm{T}}$.

In summary, no matter what forms the linear inner-coupling function and diffusion function adopt, too large time

delay is harmful to the synchronization of stochastic oscillator complex networks with time-delayed coupling.

Similarly, simulations are carried out on the network of Lorenz systems and also the special cases that nodes are connected by and noise propagates through the first two state variables of every node. It can be seen from Fig. 7 that regardless of the form of the inner-coupling function and the diffusion function, the synchronized region decreases with the increase of the time delay, and the MSF curves (Fig. 8) also show an upward trend with the increase of the time delay. These observations lead to the same conclusion as in the Rössler system setting: if there is noise in a network, the synchronizability of the network will be reduced with the increase of the time delay.

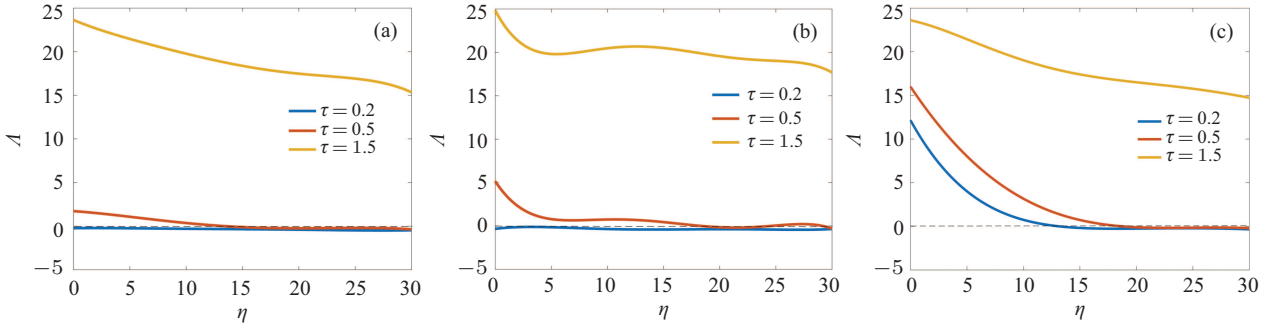


Fig. 8. The MSF curves for the stochastic complex network of Lorenz systems with noise intensity $\sigma = 0.8$ and different time delay τ. (a) $h(x_i(t-\tau)) = g(x_i(t-\tau)) = x_i(t-\tau)$; (b) $h(x_i(t-\tau)) = [x_{i1}(t-\tau), x_{i2}(t-\tau), 0]^{\mathrm{T}}$, $g(x_i(t-\tau)) = x_i(t-\tau)$; (c) $h(x_i(t-\tau)) = x_i(t-\tau)$, $g(x_i(t-\tau)) = [x_{i1}(t-\tau), x_{i2}(t-\tau), 0]^{\mathrm{T}}$.

4. Conclusion

In this article, the MSF framework has been extended to stochastic oscillator complex networks with time-delayed coupling. The effects of noise and time delay on network synchronization have been analyzed under different inner-coupling functions and different diffusion functions. In simulations, the chaotic Rössler system and Lorenz system have been adopted as the self-dynamics of the nodes. It is found that the influence of noise on network synchronization is related to the diffusion function when there exists time-delayed coupling. Only when noise diffuses through all state variables of the node, may appropriately increasing noise be beneficial to synchronization. In other cases, noise can be beneficial or harmful depending on the network parameter settings. When there is a certain noise in the network, no matter how the connection between nodes and the form of noise diffuses, large time delay is harmful to network synchronization.

It is worthwhile to mention that, although numerical simulations are performed with two chaotic systems, the extension to other general systems, such as Chen system[47] and Lü system,[48] is straightforward. In the future, it will be interesting to consider introducing pinning control[49–52] to stochastic complex networks with time-delayed coupling. Moreover, other types of noise [34,35,53] can be considered to investigate their effects on the network synchronizability. The synchronization of stochastic complex networks with parameter mismatches[54,55] or heterogeneous node dynamics[56] are challenging but also worthy of future investigations.

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References

[1] Qiao Y, Yan H, Duan L and Miao J 2020 *Neural Networks* **126** 1
[2] Anbalagan P, Hincal E, Ramachandran R, Baleanu D, Cao J and Niezabitowski M 2021 *AIMS Mathematics* **6** 4526
[3] Shahal S, Wurzberg A, Sibony I, Duadi H, Shniderman E, Weymouth D, Davidson N and Fridman M 2020 *Nat. Commun.* **11** 3854
[4] Wang L, Yan B, Li G, Ma Y and Yang L 2021 *Expert Systems with Applications* **170** 114550
[5] Han F, Lu Q, Wiercigroch M, Fang J and Wang Z 2012 *Int. J. Nonlinear Mech.* **47** 1161
[6] Han F, Wang Z, Fan H and Gong T 2015 *Chin. Phys. Lett.* **32** 040502
[7] Wu J, Liu M, Sun H, Li T, Gao Z and Wang D Z W 2015 *Transportation Research Part C-Emerging Technologies* **51** 1
[8] Liu T and Ceder A 2016 *Transportation Research Record* **2539** 84
[9] Feng S, Hu B, Nie C, Shen X and Ci Y 2016 *Chin. Phys. B* **25** 030504
[10] Wu J, Yu X and Li X 2019 *IEEE Trans. Circ. Syst. I: Regular Papers* **66** 3123
[11] Mingotti A, Peretto L and Tinarelli R 2018 *IEEE Trans. Instrum. Measur.* **67** 600
[12] Pecora L M and Carroll T L 1998 *Phys. Rev. Lett.* **80** 2109
[13] Wang X F and Chen G 2002 *IEEE Transactions on Circuits and Systems-I* **49** 54
[14] Wu X, Mao B, Wu X and Lü J 2022 *SIAM Journal on Control and Optimization* **60** 189
[15] Gu H, Wang X, Liu K and Lü J 2021 *IEEE Transactions on Systems, Man, and Cybernetics: Systems* **51** 26766
[16] Yang C, Cao J, Huang T, Zhang J and Qiu J 2018 *Sci. China-Inform. Sci.* **61** 052203
[17] Yang W, Huang Z G, Wang X, Huang L, Yang L and Lai Y C 2015 *New J. Phys.* **17** 023055

[18] Xiang L, Chen F and Chen G 2014 *Nonlinear Dynamics* **78** 1609
[19] Wang X, Lai Y C and Lai C H 2007 *Phys. Rev. E* **75** 056205
[20] Jalili M and Rabiee H R 2009 *International Conference on Information Management and Engineering* 22–26
[21] Jalili M 2013 *IEEE Transactions on Neural Networks and Learning Systems* **24** 1009
[22] Jalili M and Yu X 2016 *IEEE Transactions on Network Science and Engineering* **3** 106
[23] Gadpale P, U B, Malviya Y, Mate L and Bhil S 2018 2018 *International Conference on Computational and Characterization Techniques in Engineering & Sciences (CCTES)* 137–142
[24] Checco P D, Kocarev L, Maggio G M and Biey M 2004 *IEEE International Symposium on Circuits and Systems* **4** 800
[25] Bogojeska A, Mirchev M, Mishkovski I and Kocarev L 2014 *IEEE Transactions on Circuits and Systems I: Regular Papers* **61** 522
[26] Rossa F D and DeLellis P 2020 *Phys. Rev. E* **101** 052211
[27] Fan H, Wang Y, Chen M and Wang X 2016 *Sci. China-Technological Sciences* **59** 428
[28] Shrii M M, Senthilkumar D V and Kurths J 2012 *Europhys. Lett.* **98** 10003
[29] Jalili M 2011 *Neurocomputing* **74** 1551
[30] Wang G, Cao J and Lu J 2010 *Physica A* **389** 1480
[31] Meng J H and Riecke H 2018 *Sci. Rep.* **8** 6949
[32] Maisel B and Lindenberg K 2019 *Physica A* **552** 123186
[33] Zhang C and Yang Y 2020 *Physica A* **545** 123496
[34] Rossa F D and DeLellis P 2020 *IEEE Transactions on Control of Network Systems* **7** 1678
[35] Shi X, Wang Q and Lu Q 2008 *Cognitive Neurodynamics* **2** 195
[36] Sun J, Xiang L and Chen G 2023 *Frontiers of Physics* **11** 1129844
[37] Han F, Lu Q, Wiercigroch M and Ji Q 2009 *Int. J. Nonlinear Mech.* **44** 298
[38] Rössler O E 1976 *Phys. Lett. A* **57** 397
[39] Lorenz E N 1963 *Journal of the Atmospheric Sciences* **20** 130
[40] Maritan and Banavar 1994 *Phys. Rev. Lett.* **72** 1451
[41] Nakao H, Arai K and Kawamura Y 2007 *Phys. Rev. Lett.* **98** 184101
[42] Nakada K, Yakata S and Kimura T 2012 *IEEE Transactions on Magnetics* **48** 4558
[43] Wu Y, Gong Y and Wang Q 2014 *Eur. Phys. J. B* **87** 198
[44] Kawamura Y and Nakao H 2016 *Phys. Rev. E* **94** 032201
[45] Sarkar M 2020 arXiv: 2004.00294v1, Corpus ID: 214743456
[46] Zhou C and Kurths J 2002 *Phys. Rev. Let.* **88** 230602
[47] Chen G and Ueta T 1999 *Int. J. Bifur. Chaos* **9** 1465
[48] Lü J and Chen G 2002 *Int. J. Bifur. Chaos* **12** 659
[49] Chen G 2014 *International Journal of Control, Automation, and Systems* **12** 221
[50] Zhao X, Zhou J and Lu J A 2019 *IEEE Transactions on Cybernetics* **49** 4262
[51] Ruan X, Feng J, Xu C, Wang J and Zhao Y 2022 *IEEE Transactions on Circuits and Systems II: Express Briefs* **69** 2211
[52] Liu X, Ho D W C, Song Q and Xu W 2019 *IEEE Transactions on Cybernetics* **49** 2398
[53] Ray A, Chowdhury A R and Ghosh D 2013 *Physica A* **392** 4837
[54] Sorrentino F and Porfiri M 2011 *Eurphys. Lett.* **93** 50002
[55] Sun J, Bollt E M and Nishikawa T 2009 *Eurphys. Lett.* **85** 60011
[56] Chen F, Feng G, Liu L and Ren W 2015 *IEEE Transactions on Automatic Control* **60** 547