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Citation: Chin. Phys. B, 2023, 32 (**7**): 070203. DOI: 10.1088/1674-1056/acb9f3
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Rapid stabilization of stochastic quantum systems in a unified framework

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(Received 9 November 2022; revised manuscript received 5 February 2023; accepted manuscript online 8 February 2023)

Rapid stabilization of general stochastic quantum systems is investigated based on the rapid stability of stochastic differential equations. We introduce a Lyapunov–LaSalle-like theorem for a class of nonlinear stochastic systems first, based on which a unified framework of rapidly stabilizing stochastic quantum systems is proposed. According to the proposed unified framework, we design the switching state feedback controls to achieve the rapid stabilization of single-qubit systems, two-qubit systems, and N-qubit systems. From the unified framework, the state space is divided into two state subspaces, and the target state is located in one state subspace, while the other system equilibria are located in the other state subspace. Under the designed state feedback controls, the system state can only transit through the boundary between the two state subspaces no more than two times, and the target state is globally asymptotically stable in probability. In particular, the system state can converge exponentially in (all or part of) the state subspace where the target state is located. Moreover, the effectiveness and rapidity of the designed state feedback controls are shown in numerical simulations by stabilizing GHZ states for a three-qubit system.

Keywords: rapid stabilization, state feedback, stochastic quantum systems, switching control

PACS: 02.30.Yy, 42.50.Lc **DOI:** 10.1088/1674-1056/acb9f3

1. Introduction

The Nobel Prize in Physics 2022 was awarded to Alain Aspect, John Clauser, and Anton Zeilinger for their pathfinding work in quantum mechanics and quantum information science. At present, quantum information science has become a hot research field, where the transfer of quantum systems from any initial state to the target state, i.e., the preparation or global stabilization of quantum states, is one of the basic tasks. In order to prepare the target quantum state rapidly and accurately, numerous classical control methods have been introduced into control of quantum systems, e.g., optimal control[1–3] and Lyapunov control.[4–7]

For the better robustness, feedback controls, which need the system information through measurement, are recommended to apply. Although measurement operations can cause collapse of quantum states, quantum filter theory[8,9] helps to overcome this obstacle by using measurement-based feedback control, where quantum filter states described by an Itô stochastic differential equation, instead of the actual quantum states, are used. Based on quantum filter theory, for the stabilization of stochastic quantum systems, switching control strategy, which was proposed in Ref. [10] first, is one of the main control strategies and has been used to achieve global stabilization of eigenstates for N-level quantum systems,[10–12] Bell states for two-qubit systems[13,14] and Greenberger–Horne–Zeilinger (GHZ) state for multi-qubit systems.[15,16] The idea of the switching control strategy is that the state space is divided into two state subspaces, then to design two controls to form the switching control strategy such that one control can make the system state converge to and evolve permanently in the state subspace that contains the target state, and the other control can make the system state converge to the target state. Of course, the state trajectory may travel back and forth between the two subspaces under the switching control. To improve the convergence rate, Vu *et al.* used the coupling Lyapunov functions to design a switching control to reduce the travel number of system trajectory between two subspaces.[13] Recently, Liang *et al.* designed a state feedback to achieve the local exponential stabilization of both eigenstates for N-level quantum angular momentum systems and GHZ states for multi-qubit systems, whose robustness analyses were presented in Ref. [17]. In addition, Cardona *et al.* proposed noise-assisted feedback strategy that can also achieve the exponential stabilization of N-level quantum systems.[18] Considering that state feedback has been applied in the real physical experiment,[19] we mainly focus on the

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state feedback control in this paper.

Due to the rapidity requirement in quantum manipulations, the rapid convergence of quantum states is of great significance. According to the review of related literature, state feedback has been used to achieve the rapid stabilization of stochastic quantum systems, even local exponential stabilization. In order to improve the convergence rate of system states further, we present a unified framework to combine the existing techniques of rapidly stabilizing stochastic quantum systems in this paper, i.e., the reduction of the travel number of system trajectory between state subspaces and the local exponential stabilization of quantum states. For this goal, we study the rapid stabilization of a class of nonlinear stochastic systems and introduce a Lyapunov–LaSalle-like theorem for the general stochastic differential equations by extending the results in Ref. [13]. Based on the proposed Lyapunov–LaSalle-like theorem, a unified framework with five steps for the rapid stabilization of the general stochastic quantum systems is presented and is applied to the rapid stabilization of single-qubit systems, two-qubit systems, and N-qubit systems, which show the effectiveness and universality of the unified framework. According to the proposed unified framework, the state space should be divided into two state subspaces first, and the switching state feedback controls designed based on the constructed coupling Lyapunov functions can make the system trajectory only transit the boundary between the two state subspaces no more than two times, and the target state is globally asymptotically stable in probability. In particular, the system state can converge exponentially in (all or part of) the state subspace where the target state is located. Thus, the main contributions of this paper can be concluded as follows: (i) A Lyapunov–LaSalle-like theorem is introduced for the rapid stabilization of a class of nonlinear stochastic systems. (ii) A unified framework is presented for the rapid stabilization of stochastic quantum systems based on the proposed Lyapunov–LaSalle-like theorem. (iii) The unified framework is applied to the rapid stabilization of single-qubit systems, two-qubit systems, and N-qubit systems.

The rest of this paper is organized as follows. The stochastic differential equation describing the quantum filter state and the problem of rapid stabilization by state feedback are described in Section 2. We introduce a Lyapunov–LaSalle-like theorem for a class of nonlinear stochastic systems in Section 3. In Section 4, the proposed Lyapunov–LaSalle-like theorem is applied to the rapid stabilization of single-qubit systems, two-qubit systems, and N-qubit systems. We take three-qubit systems as an example to verify the effectiveness of the proposed state feedback strategies in numerical simulations in Section 5. Finally, we present conclusions and discuss the possible future directions in Section 6.

Notation $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and complex numbers, respectively; $\mathrm{tr}(\cdot)$ is the trace of matrices; $A^\dagger$ is the conjugate transpose of matrix A; $[A,B]=AB-BA$ is the commutator of matrices A and B; $\mathbb{P}\{\cdot\}$ represents the probability of event. $\mathbb{E}[\cdot]$ denotes the expected value of random variables; $\sigma^x=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $\sigma^y=\begin{bmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{bmatrix}$, $\sigma^z=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ are the Pauli operators along x, y and z axes, respectively, with $\mathrm{i}=\sqrt{-1}$; $\mathrm{diag}[\lambda_1,\ldots,\lambda_n]$ represents the $n\times n$ diagonal matrix with diagonal elements $\lambda_1,\ldots,\lambda_n$; $\otimes$ represents the Kronecker product of matrices. $I_n=\mathrm{diag}[1,\ldots,1]$ is the $n\times n$ identity matrix.

2. Preliminaries

In this paper, we consider the rapid stabilization of stochastic quantum systems via measurement-based feedback control, where the quantum filter state, instead of the state of the actual quantum system, is used in the feedback control. According to the quantum filter theory,[8,9] the evolution of quantum filter state can be described by the Itô stochastic differential equation, i.e., stochastic master equation (SME), which can be written as[20]

$$\begin{aligned}\mathrm{d}\rho_t=&\left(-\mathrm{i}[H_0,\rho_t]-\sum_{k=1}^{q}\mathrm{i}[H_k,\rho_t]u_{k,t}+\sum_{j=1}^{m}\Gamma_{L_j}\mathcal{D}(L_j,\rho_t)\right)\mathrm{d}t\\&+\sum_{j=1}^{m}\sqrt{\eta_{L_j}\Gamma_{L_j}}\mathcal{H}(L_j,\rho_t)\,\mathrm{d}\omega_{L_j,t},\\\mathrm{d}y_{j,t}=&\sqrt{\eta_{L_j}\Gamma_{L_j}}\,\mathrm{tr}\left(L_j\rho_t+\rho_tL_j^\dagger\right)\mathrm{d}t+\mathrm{d}\omega_{L_j,t},\quad j=1,\ldots,m,\end{aligned}\tag{1}$$

where ρ_t is the density matrix describing quantum state, and $\rho_t\in\mathcal{S}=\left\{\rho_t\in\mathbb{C}^{n\times n}:\rho_t=\rho_t{}^\dagger\geq 0,\mathrm{tr}(\rho_t)=1\right\}$ in which $\mathcal{S}$ represents the state space; H_0 and H_k are free Hamiltonian and control Hamiltonian, respectively; L_j is the measurement operator; Γ_{L_j} and η_{L_j} are the measurement strengths and efficiencies, respectively; $\omega_{L_j,t}$ is the standard Wiener process, while $\mathrm{d}\omega_{L_j,t}$ is the independent Wiener increment that satisfies $\mathrm{d}\omega_{L_i,t}\mathrm{d}\omega_{L_j,t}=\delta_{ij}\mathrm{d}t$, in which δ_{ij} is the Kronecker delta; $u_{k,t}\in\mathbb{R}$ is the control input; $y_{j,t}$ is the measurement record of the output; superoperators $\mathcal{D}(L_j,\rho_t)$ and $\mathcal{H}(L_j,\rho_t)$ are defined as

$$\begin{aligned}\mathcal{D}(L_j,\rho_t)=&\,L_j\rho_tL_j^\dagger-\frac{1}{2}L_j^\dagger L_j\rho_t-\frac{1}{2}\rho_tL_j^\dagger L_j,\\\mathcal{H}(L_j,\rho_t)=&\,L_j\rho_t+\rho_tL_j^\dagger-\mathrm{tr}\left(L_j\rho_t+\rho_tL_j^\dagger\right)\rho_t.\end{aligned}\tag{2}$$

For SME (1), the existence and uniqueness of solutions can be guaranteed by constructing the appropriate control Hamiltonian H_k and the control input $u_{k,t}$.[10] It should be noticed that SME (1) can be used to describe many physical experiment systems, for example, an ensemble of laser-cooled Cesium atoms confined to a cavity mirror,[10,21,22] and a two-qubit system consisting of a couple of two-level atoms placed

in two distant cavities and interacting through a radiation field in a dispersive way.[23] In particular, the theoretical study carried out in Ref. [10] was extended to discrete-time quantum systems,[24,25] and has been applied to the first physical experiment of achieving real-time quantum measurement-based feedback control in the photon box of the Kastler Brossel Laboratory,[19] from which the study of continuous-time quantum systems considered in this paper is the first step towards the practical quantum manipulation and has an important potential practical value.

Measurement-based feedback has been used to achieve the stabilization of stochastic quantum systems, where the feedback architectures of designing the control input $u_{k,t}$ include static output feedback,[26,27] state feedback,[10,12,17,21,28] and noise-assisted feedback.[18,29] Considering the practicability, we focus on the state feedback architecture here. For state feedback, the control input $u_{k,t}$ in Eq. (1) can be written as $u_k(\rho_t)$. Therefore, the goal of this paper is to achieve the rapid stabilization of stochastic quantum systems by using state feedback control $u_k(\rho_t)$, which is described in the following.

Rapid stabilization by state feedback For the quantum filter (1), design the state feedback control $u_k(\rho_t)$ to globally rapidly asymptotically render the system state ρ_t from any initial state ρ_0 to the target state ρ_f almost surely, i.e., $\mathbb{P}\{\lim_{t\to\infty}\rho_t=\rho_f\}=1, \forall\rho_0\in\mathcal{S}$.

In fact, several state feedback strategies have been proposed for the rapid stabilization of stochastic quantum systems, e.g., Refs. [13,16,20,30,31]. However, the state feedback controls designed in Refs. [3,16,20,30,31] are either for a certain class of quantum systems or for a certain class of target quantum states. In this paper, we aim to propose a unified architecture of designing state feedback controls that can be applied to the rapid stabilization of multiple types of stochastic quantum systems and different types of quantum states.

The SME (1) describing quantum filter state is a nonlinear stochastic differential equation. In order to design the state feedback control of rapidly stabilizing stochastic quantum systems, we introduce a Lyapunov–LaSalle-like theorem for the rapid stability in probability of a class of nonlinear stochastic systems in the next section, and then present the unified framework of rapidly stabilizing stochastic quantum systems proposed in this paper and apply the unified framework to Eq. (1) in Section 4, where we take single-qubit systems (eigenstate as the target state), two-qubit systems (Bell state as the target state), and multi-qubit systems (GHZ state as the target state) as examples.

3. Rapid stability of stochastic differential equations

Consider the nonlinear stochastic systems described by the stochastic differential equation

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + g(x_t)\,\mathrm{d}\omega_t, \tag{3}$$

where x_t is the system state; $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times r}$ are Borel-measurable, locally bounded, and locally Lipschitz continuous functions; $f(0,0)=0$, $g(0,0)=0$; ω_t is an r-dimensional standard Wiener process.

Based on the concept of stability in probability as shown in Ref. [32], we have the following theorem, which is an extension of Theorem 1 in Ref. [13].

Theorem 1 For the nonlinear stochastic systems (3), if there exists a partition of $\mathbb{R}^n$, which is denoted as $\{\phi_1,\phi_2\}$, and there exists the coupling Lyapunov functions $V_1(x), V_2(x):\mathbb{R}^n\to\mathbb{R}^+$, which are defined in ϕ_1 and ϕ_2, respectively, and both nonnegative differential and continuously twice differential satisfy the following conditions:

C.I: $V_1(x)$ is constant on the boundary of the sets $\{\phi_1,\phi_2\}$.

C.II: $\alpha_1(|x|)\le V_1(x)\le\alpha_2(|x|), \forall x\in\mathbb{R}^n$.

C.III: $\mathcal{L}V_1(x)\le-\lambda V_1(x), \forall x\in\phi_1$.

C.IV: $\mathcal{L}V_2(x)\le -M, \forall x\in\phi_2$.

Here, $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$-class functions; λ and M are positive constants; $\mathcal{L}$ is the infinitesimal operator associated with Eq. (3) and defined as

$$\mathcal{L}V(x) := \frac{\partial V(x)}{\partial x}f(x) + \frac{1}{2}\mathrm{tr}\left(g^{\mathrm{T}}(x)\frac{\partial^2 V(x)}{\partial x^2}g(x)\right). \tag{4}$$

Then, the equilibrium $x=0$ of the system (3) is globally asymptotically stable in probability, and the system trajectories x_t will only transit through the boundary between ϕ_1 and ϕ_2 no more than two times for any initial state $x_0\in\mathbb{R}^n$. In particular, $\mathbb{E}[V_1(x_t)]\le V_1(x_T)\,\mathrm{e}^{-\lambda(t-T)}$ when $t\ge T$, where T is the time after that x_t evolves in ϕ_1 permanently.

Proof According to Theorem 1 in Ref. [13], if we let

$$W(x) = \lambda V_1(x), \tag{5}$$

it is easy to verify that $W(x)$ is a continuous positive function defined in ϕ_1 due to the fact that $V_1(x)$ is a continuous positive function as well as $\lambda>0$, then the conditions C.1–C.3 of Theorem 1 in Ref. [13] hold true, such that $x=0$ is globally asymptotically stable in probability and x_t transits the boundary between ϕ_1 and ϕ_2 no more than two times.

On the other hand, according to Theorem 3 in Ref. [18], if $V_1(x)$, T, and ϕ_1 are regarded as $V(x)$, t_0, and S, respectively, we can directly conclude that $\mathbb{E}[V_1(x_t)]\le V_1(x_T)\,\mathrm{e}^{-\lambda(t-T)}$. The proof is completed.

Remark 1 In Theorem 1, it is necessary that $V_1(x)$ is constant on the boundary between ϕ_1 and ϕ_2, regarded as an independent condition, which is denoted as condition C.I. From the proof of Theorem 1 in Ref. [13], the condition C.I is used to guarantee that there is no sliding motion of the system trajectory on the boundary and help to prove the conclusion that x_t will evolve in ϕ_1 permanently after a finite time for any initial state x_0. Therefore, when applying Theorem 1, we need to deliberately check the condition C.I to avoid misusing Theorem 1 to obtain wrong conclusions.

4. Rapid stabilization of stochastic quantum systems

In this section, we present the unified framework of rapidly stabilizing stochastic quantum systems based on Theorem 1 and Ref. [13] first, and then apply the unified framework to the rapid stabilization of single-qubit systems, two-qubit systems, and N-qubit systems, to demonstrate the effectiveness and universality of the proposed unified framework.

4.1. The unified framework of rapidly stabilizing stochastic quantum systems

Comparing SME (1) and the stochastic differential equation (3), SME (1) is a special class of stochastic differential equation with $f(\rho_t) = -\mathrm{i}[H_0,\rho_t] - \sum_k^q \mathrm{i}[H_k,\rho_t]u_{k,t} + \sum_j^m \Gamma_{L_j}\mathcal{D}(L_j,\rho_t)$ and $g(\rho_t) = \sum_j^m \sqrt{\eta_{L_j}\Gamma_{L_j}}\mathcal{H}(L_j,\rho_t)$, thus Theorem 1 is suitable for SME (1). According to Theorem 1 in this paper and Theorem 1 in Ref. [13], we proposed a unified framework for the rapid stabilization of stochastic quantum systems, which is shown in Fig. 1.

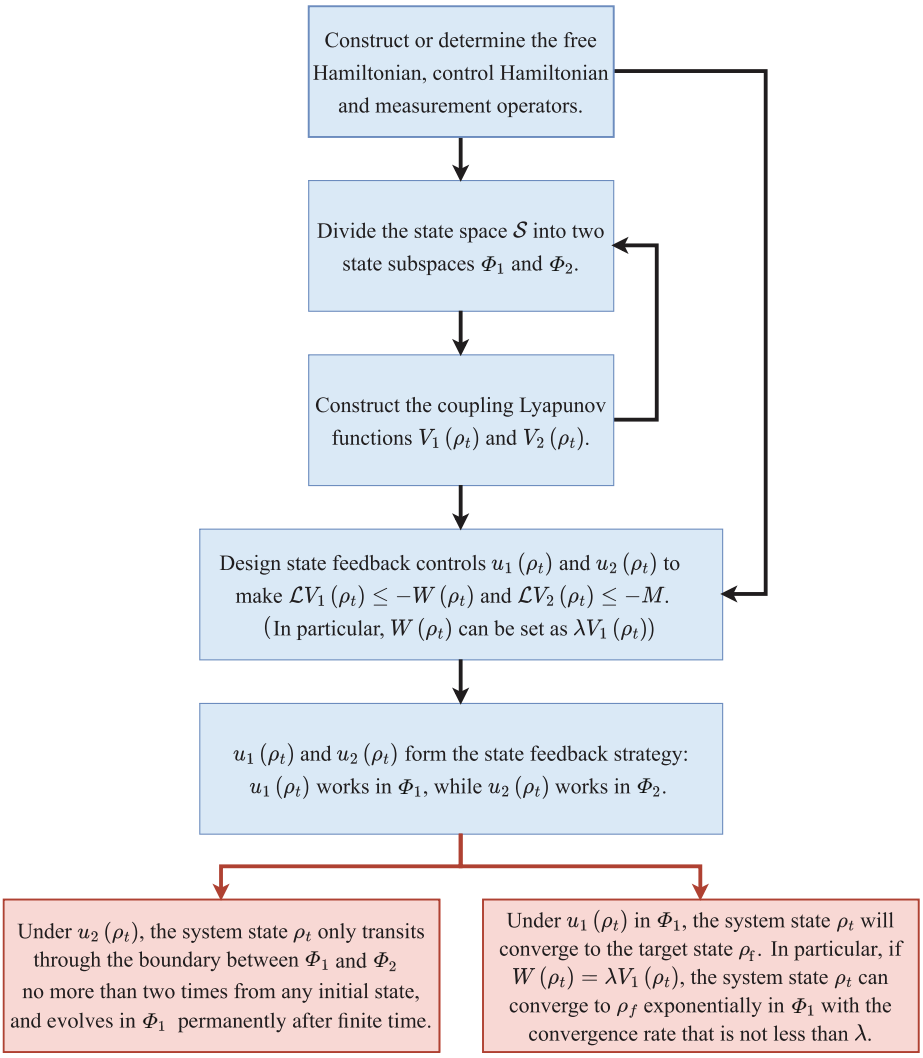


Fig. 1. The diagram of the unified framework proposed in this paper for the rapid stabilization of stochastic quantum systems, where the blue blocks represent the operations, the pink blocks represent the results of stabilization control, while the lines with arrow describe the relationships between different blocks.

From Fig. 1, it is necessary to perform five steps for the design of state feedback controls to rapidly stabilize stochastic quantum systems. The specific details of each step are described as follows.

Step 1 Construct or determine Hamiltonians and measurement operators. For the stabilization of stochastic quantum systems, the solvability is the first problem to be solved. The system Hamiltonians (including free Hamiltonian H_0 and

control Hamiltonians H_k) and measurement operators L_j need to be constructed reasonably to guarantee the solvability of the stabilization problem of quantum systems. For example, in Ref. [11], under the setting of Hamiltonians and measurement operator presented in Assumptions 2–3 of Ref. [11] the target state belongs to system equilibria, and the controlled quantum system has a finite number of equilibrium states due to the fact that the measurement operator in Ref. [11] is non-degenerate. However, if the measurement operator is degenerate, the number of the system equilibria may be infinite, as shown in Refs. [16,30,31]. On the other hand, if the controlled quantum system is the actual physical system, the first step is also to determine Hamiltonians and measurement operators. No matter what is the case, Hamiltonians and measurement operators should be constructed or determined to ensure that the target state belongs to the system equilibria, such that the solvability of the stabilization problem of quantum systems is guaranteed.

Step 2 Divide the state space $\mathcal{S}$ into two state subspaces, which are denoted as Φ_1 and Φ_2. The state space $\mathcal{S}$ can be divided artificially and directly, or can also be divided based on the Lyapunov function $V_1(\rho_t)$ constructed in Step 3. No matter what is the division method, the partition of $\mathcal{S}$ should satisfy the following conditions:

If the number of system equilibria is finite, the target state is located in Φ_1 while the other equilibria are located in Φ_2.

If the number of system equilibria is infinite, the target state is only the state that simultaneously located in the LaSalle invariant set and Φ_1.

Step 3 Construct the coupling Lyapunov functions $V_1(\rho_t)$ and $V_2(\rho_t)$. $V_1(\rho_t)$ is defined in Φ_1, while $V_2(\rho_t)$ is defined in Φ_2. Both $V_1(\rho_t)$ and $V_2(\rho_t)$ should be nonnegative differentiable and twice continuous differentiable. In particular, $V_1(\rho_t)$ must be constant on the boundary between the state subspaces Φ_1 and Φ_2. Moreover, $V_1(\rho_t)$ can be used to divide the state space $\mathcal{S}$ in Step 2.

Step 4 Design state feedback controls $u_1(\rho_t)$ and $u_2(\rho_t)$ to make $\mathcal{L}V_1(\rho_t) \le -W(\rho_t)$ and $\mathcal{L}V_2(\rho_t) \le -M$. In particular, $W(\rho_t)$ can be set as $\lambda V_1(\rho_t)$. If the designed state feedback control $u_1(\rho_t)$ cannot make $\mathcal{L}V_1(\rho_t) \le -\lambda V_1(\rho_t)$ hold in all of Φ_1, we can design another state feedback control denoted as $\hat{u}_1(\rho_t)$ to make $\mathcal{L}V_1(\rho_t) \le -W(\rho_t)$ hold in the state subspace denoted as $\hat{\Phi}_1$, where $u_1(\rho_t)$ cannot make $\mathcal{L}V_1(\rho_t) \le -\lambda V_1(\rho_t)$ hold, and $u_1(\rho_t)$ still works in $\Phi_1 \backslash \hat{\Phi}_1$.

Step 5 $u_1(\rho_t)$ and $u_2(\rho_t)$ form the state feedback strategy: $u_1(\rho_t)$ works in Φ_1, while $u_2(\rho_t)$ works in Φ_2. If $u_1(\rho_t)$ cannot make $\mathcal{L}V_1(\rho_t) \le -\lambda V_1(\rho_t)$ hold in all of Φ_1, $u_1(\rho_t)$, $\hat{u}_1(\rho_t)$ and $u_2(\rho_t)$ form the state feedback strategy: $u_1(\rho_t)$ works in $\Phi_1 \backslash \hat{\Phi}_1$, $u_2(\rho_t)$ works in Φ_2, while $\hat{u}_1(\rho_t)$ works in $\hat{\Phi}_1$.

Following the above design process, the state feedback strategy can be applied to achieve:

Under $u_2(\rho_t)$, the system state ρ_t only transits through the boundary between Φ_1 and Φ_2 no more than two times from any initial state, and evolves in Φ_1 permanently after finite time.

Under $u_1(\rho_t)$ in Φ_1, the system state ρ_t will converge to the target state $\rho_{\rm f}$. In particular, if $W(\rho_t) = \lambda V_1(\rho_t)$, the system state ρ_t can converge to $\rho_{\rm f}$ exponentially in Φ_1 with the convergence rate that is not less than λ, which means that the target state is stabilized globally and rapidly. In Subsections 4.2, 4.3, and 4.4, we apply the unified framework to achieve the rapid stabilization of single-qubit systems, two-qubit systems, and N-qubit systems, respectively.

4.2. Rapid stabilization of single-qubit systems

For single-qubit systems, the time evolution of n-dimensional quantum filter state is described by the following SME:[11,33]

$$\begin{aligned} {\rm d}\rho_t &= (-{\rm i}[H_0,\rho_t] - {\rm i}[H_1,\rho_t]u_t + \mathcal{D}(L_{\rm e},\rho_t))\,{\rm d}t \\ &\quad + \sqrt{\eta_{L_{\rm e}}}\mathcal{H}(L_{\rm e},\rho_t)\,{\rm d}\omega_{L_{\rm e},t}, \\ {\rm d}y_t &= \sqrt{\eta_{L_{\rm e}}}\,{\rm tr}\left(L_{\rm e}\rho_t + \rho_t L_{\rm e}^{\dagger}\right){\rm d}t + {\rm d}\omega_{L,t}, \end{aligned} \tag{6}$$

which corresponds to the case of $q=1$, $m=1$, $L_1 = L_{\rm e}$, and $\Gamma_{L_{\rm e}} = 1$ in SME (1).

Similar to Ref. [11], the target state $\rho_{\rm f}$ is set as an eigenstate of the measurement operator $L_{\rm e}$. For the rapid stabilization of single-qubit systems (6), we follow the five steps in the unified framework as shown in Subsection 4.1 to design the state feedback control u_t.

To guarantee the solvability, the measurement operator $L_{\rm e}$ is self-adjoint, non-degenerate, and commutative with H_0, while the control Hamiltonian H_1 is connected, under which the system equilibria are all the eigenstates of the measurement operator $L_{\rm e}$ and the target state belongs to the system equilibria, such that the number of the system equilibria is finite, i.e., n.

Now, we should divide the state space $\mathcal{S}$ into two subspaces. Consider the following coupling Lyapunov functions

$$\begin{aligned} V_{\rm e}(\rho_t) &= 1 - {\rm tr}(\rho_t\rho_{\rm f}) + c_{\rm e}U(\rho_t), \\ V_{\rm c}(\rho_t) &= a + {\rm tr}(X\rho_t), \end{aligned}$$

where $c_{\rm e}$ is nonnegative constant; $U(\rho_t) = {\rm tr}\left(L_{\rm e}^2\rho_t\right) - {\rm tr}^2(L_{\rm e}\rho_t)$; $a \ge \max\limits_{\rho\in\mathcal{S}} |{\rm tr}(X\rho)|$. Then, $V_{\rm e}(\rho_t)$ and $V_{\rm c}(\rho_t)$ are regarded as $V_1(\rho_t)$ and $V_2(\rho_t)$, respectively, and the state space $\mathcal{S}$ can be divided into $\{\Phi_{{\rm e},1}, \Phi_{{\rm e},2}\}$ based on $V_{\rm e}(\rho_t)$, i.e.,

$$\begin{aligned} \Phi_{{\rm e},1} &= \{\rho\in\mathcal{S} : V_{\rm e}(\rho_t) \le k_{\rm e}\}, \\ \Phi_{{\rm e},2} &= \mathcal{S}\backslash\Phi_{{\rm e},1}, \end{aligned} \tag{7}$$

with $0 < k_{\rm e} \leq 1$, from which $\rho_{\rm f} \in \Phi_{\rm e,1}$ and $\rho_i \in \Phi_{\rm e,2}$, where ρ_i is the eigenstates except the target state $\rho_{\rm f}$. It is obvious that $V_{\rm e}(\rho_t)$ is constant on the boundary between $\Phi_{\rm e,1}$ and $\Phi_{\rm e,2}$.

Based on $V_{\rm e}(\rho_t)$ and $V_{\rm c}(\rho_t)$, the feedback controls $u_{\rm e,1}(\rho_t)$ and $u_{\rm e,2}(\rho_t)$, which are regarded as $u_1(\rho_t)$ and $u_2(\rho_t)$, can be designed. According to Itô's product rule and Eq. (4), the infinitesimal generator associated with Eq. (6) acting on $V_{\rm e}(\rho_t)$ and $V_{\rm c}(\rho_t)$ are

$$\mathcal{L}V_{\rm e}(\rho_t) = g_{\rm e}(\rho_t)u_{\rm e,1}(\rho_t) - 4c_{\rm e}\eta_{L_{\rm e}}U^2(\rho_t), \tag{8}$$

$$\mathcal{L}V_{\rm c}(\rho_t) = {\rm tr}(-{\rm i}[X,H_0]\rho_t) + {\rm tr}(-{\rm i}[X,H_1]\rho_t)u_{\rm e,2}(\rho_t) + {\rm tr}(X\mathcal{D}(L_{\rm e},\rho_t)), \tag{9}$$

respectively, with

$$g_{\rm e}(\rho_t) = {\rm tr}\left(-{\rm i}[H_1,\rho]\left(c_{\rm e}\left(L_{\rm e}^2 - 2L_{\rm e}\,{\rm tr}(L_{\rm e}\rho)\right) - \rho_{\rm f}\right)\right).$$

From Eqs. (8) and (9), the feedback controls $u_{\rm e,1}(\rho_t)$ and $u_{\rm e,2}(\rho_t)$ can be designed as

$$u_{\rm e,1}(\rho_t) = \frac{-\bar{\lambda}_{\rm e}V_{\rm e}(\rho_t) + 4c_{\rm e}\eta_{L_{\rm e}}U^2(\rho_t)}{g_{\rm e}(\rho_t)}, \quad \bar{\lambda}_{\rm e} > 0, \tag{10}$$

$$u_{\rm e,2}(\rho_t) = \frac{-\bar{M}_{\rm e} + {\rm tr}({\rm i}[X,H_0]\rho_t) - {\rm tr}(X\mathcal{D}(L_{\rm e},\rho_t))}{{\rm tr}(-{\rm i}[X,H_1]\rho_t)}, \quad \bar{M}_{\rm e} > 0, \tag{11}$$

respectively, which make $\mathcal{L}V_{\rm e}(\rho_t) \leq -\lambda_{\rm e}V_{\rm e}(\rho_t)$ and $\mathcal{L}V_{\rm c}(\rho_t) \leq -M_{\rm e}$ hold, where $0 < \lambda_{\rm e} \leq \bar{\lambda}_{\rm e}$ and $0 < M_{\rm e} \leq \bar{M}_{\rm e}$. In order to ensure that the feedback controls are implementable, the denominators of $u_{\rm e,1}(\rho_t)$ and $u_{\rm e,2}(\rho_t)$ cannot be zero, i.e.,

$$g_{\rm e}(\rho_t) \neq 0 \tag{12}$$

and

$${\rm tr}(-{\rm i}[X,H_1]\rho_t) \neq 0. \tag{13}$$

Fortunately, Eq. (13) can hold by constructing X based on the algorithm proposed in Ref. [16], and Eq. (12) holds in the state subspace $\Phi_{\rm e,1}\setminus\Phi_{\rm e,3}$, where $\Phi_{\rm e,3} = \{\rho \in \Phi_{\rm e,1} : g_{\rm e}(\rho) \neq 0\} \subset \hat{\Phi}_{\rm e,1} = \{\rho \in \Phi_{\rm e,1} : |g_{\rm e}(\rho)| \leq \varepsilon_{\rm e}\}$ with $\varepsilon_{\rm e} > 0$. For the state subspace $\hat{\Phi}_{\rm e,1}$, we design another state feedback control as

$$\hat{u}_{\rm e,1}(\rho_t) = -\frac{s_{\rm e}g_{\rm e}(\rho_t)}{|g_{\rm e}(\rho_t)| + \eta_{\rm e}}, \tag{14}$$

where $s_{\rm e} > 0$ is the strength of the feedback control, and $\eta_{\rm e} > 0$ is a parameter used to adjust the hardness of the control. The state control (14) is called the approximate bang-bang control, which was proposed in Ref. [5] first for closed quantum systems. Here, $\hat{u}_{\rm e,1}(\rho_t)$ can make $\mathcal{L}V_{\rm e}(\rho_t) \leq -W_{\rm e}(\rho_t)$ hold, with $W_{\rm e}(\rho_t) = \frac{s_{\rm e}g_{\rm e}^2(\rho_t)}{|g_{\rm e}(\rho_t)|+\eta_{\rm e}} + 4c_{\rm e}\eta_{L_{\rm e}}U^2(\rho_t)$ being a continuous positive definite function defined on $\mathcal{S}$.

Finally, $u_{\rm e,1}(\rho_t)$, $u_{\rm e,2}(\rho_t)$, and $\hat{u}_{\rm e,1}(\rho_t)$ form the switching state feedback strategy, which can be expressed as

$$u_{\rm e}(\rho_t) = \begin{cases} u_{\rm e,1}(\rho_t), & \text{if } \rho_t \in \Phi_{\rm e,1}\setminus\hat{\Phi}_{\rm e,1}, \\ u_{\rm e,2}(\rho_t), & \text{if } \rho_t \in \Phi_{\rm e,2}, \\ \hat{u}_{\rm e,1}(\rho_t), & \text{if } \rho_t \in \hat{\Phi}_{\rm e,1}. \end{cases} \tag{15}$$

From Theorem 1, under $u_{\rm e,2}(\rho_t)$, the system state ρ_t only transits through the boundary between $\Phi_{\rm e,1}$ and $\Phi_{\rm e,2}$ no more than two times from any initial state, and evolves in $\Phi_{\rm e,1}$ permanently after finite time. In $\Phi_{\rm e,1}$, $u_{\rm e,1}(\rho_t)$ and $\hat{u}_{\rm e,1}(\rho_t)$ make $\mathcal{L}V_{\rm e}(\rho_t) \leq -\lambda_{\rm e}V_{\rm e}(\rho_t)$ and $\mathcal{L}V_{\rm e}(\rho_t) \leq -W_{\rm e}(\rho_t)$ hold, which means that the system trajectory will converge to the invariant set $\mathcal{I}_{\rm e} = \{\rho \in \Phi_1 : \mathcal{L}V_{\rm e}(\rho) = 0\}$ from LaSalle's invariance principle. Due to the fact that the target state $\rho_{\rm f}$ is only the equilibrium in $\Phi_{\rm e,1}$, thus $\mathcal{I}_{\rm e} = \{\rho_{\rm f}\}$ and the system state ρ_t will converge to the target state $\rho_{\rm f}$ with probability 1, thus the target state $\rho_{\rm f}$ is globally asymptotically stable in probability under the switching control $u_{\rm e}(\rho_t)$. In addition, the system state ρ_t can converge exponentially in $\Phi_{\rm e,1}\setminus\hat{\Phi}_{\rm e,1}$ with the convergence rate that is not less than $\lambda_{\rm e}$ due to $\mathcal{L}V_{\rm e}(\rho_t) \leq -\lambda_{\rm e}V_{\rm e}(\rho_t)$ and $\lambda_{\rm e} \leq \bar{\lambda}_{\rm e}$ according to Theorem 1 in this paper and Theorem 3 in Ref. [18].

Remark 2 As shown in Eq. (15), we use different feedback controls in the corresponding different state subspaces. However, under switching state feedback $u_{\rm e}(\rho_t)$, there is no sliding motion of the system trajectory on the boundary $\Lambda_{\rm e}$ between $\Phi_{\rm e,1}$ and $\Phi_{\rm e,2}$ as well as the boundary $\hat{\Lambda}_{\rm e}$ of $\hat{\Phi}_{\rm e,1}$, which are defined as $\Lambda_{\rm e} = \{\rho \in \mathcal{S} : V_{\rm e}(\rho) = k_{\rm e}\}$ and $\hat{\Lambda}_{\rm e} = \{\rho \in \Phi_{\rm e,1} : |g_{\rm e}(\rho)| = \varepsilon_{\rm e}\}$, respectively. The conclusions can be proved by contradiction as follows.

If there exists sliding motion of ρ_t on $\Lambda_{\rm e}$, then $\mathcal{L}V_{\rm e}(\rho_t) = 0$ for some $\rho_t \in \Lambda_{\rm e}$. On the other hand, $\mathcal{L}V_{\rm e}(\rho_t) \leq -\lambda_{\rm e}V_{\rm e}(\rho_t)$ or $\mathcal{L}V_{\rm e}(\rho_t) \leq -W_{\rm e}(\rho_t)$ holds in $\Phi_{\rm e,1}$. It is obvious that both $V_{\rm e}(\rho_t)$ and $W_{\rm e}(\rho_t)$ are positive definite on $\Phi_{\rm e,1}$ and $\rho_{\rm f}$ is not on $\Lambda_{\rm e}$, such that $V_{\rm e}(\rho_t) > 0$ and $W_{\rm e}(\rho_t) > 0$ for $\forall \rho_t \in \Lambda_{\rm e}$, therefore $\mathcal{L}V_{\rm e}(\rho_t) < 0$, which contradicts with $\mathcal{L}V_{\rm e}(\rho_t) = 0$. Thus, there is no sliding motion of the system trajectory on the boundary $\Lambda_{\rm e}$.

Due to $\hat{\Lambda}_{\rm e} \in \hat{\Phi}_{\rm e,1}$, $\hat{u}_{\rm e,1}(\rho_t)$ works in $\hat{\Lambda}_{\rm e}$. If there is no sliding motion of the system trajectory on the boundary $\hat{\Lambda}_{\rm e}$, then $\mathcal{L}V_{\rm e}(\rho_t) = 0$ for some $\rho_t \in \hat{\Lambda}_{\rm e}$, which means that some $\rho_t \in \hat{\Lambda}_{\rm e}$ can make $U(\rho_t) = 0$ to ensure $W_{\rm e}(\rho_t) = 0$. However, the states that make $U(\rho_t) = 0$ hold are the all eigenstates of the measurement operator $L_{\rm e}$. On the other hand, for the all eigenstates of $L_{\rm e}$, only $\rho_{\rm f}$ is located in $\Phi_{\rm e,1}$ and $\rho_{\rm f} \notin \hat{\Lambda}_{\rm e} \subset \Phi_{\rm e,1}$, thus there does not exist any state $\rho_t \in \hat{\Lambda}_{\rm e}$ to make $\mathcal{L}V_{\rm e}(\rho_t) = 0$ hold, which means that there is no sliding motion of the system trajectory on the boundary $\hat{\Lambda}_{\rm e}$.

4.3. Rapid stabilization of two-qubit systems

For two-qubit systems, the time evolution of quantum filter state is described by the following SME:[13]

$${\rm d}\rho_t = \left(-{\rm i}[H_1,\rho_t]u_t + \sum_{j=1}^{2}\Gamma_{L_j}\mathcal{D}(L_j,\rho_t)\right){\rm d}t$$

$$+\sum_{j=1}^{2}\sqrt{\eta_{L_j}\Gamma_{L_j}}\mathcal{H}(L_j,\rho_t)\,\mathrm{d}\omega_{L_j,t},$$

$$\mathrm{d}y_{j,t} = \sqrt{\eta_{L_j}\Gamma_{L_j}}\,\mathrm{tr}\left(L_j\rho_t+\rho_t L_j^{\dagger}\right)\mathrm{d}t+\mathrm{d}\omega_{L_j,t},\ j=1,2, \tag{16}$$

which corresponds to the case of $H_0=0$, $q=1$, $m=2$ in SME (1). Bell states are the maximally entangled states of two-qubit quantum systems, which are written as

$$\phi^{\pm}=\frac{1}{2}\begin{bmatrix}1&0&0&\pm1\\0&0&0&0\\0&0&0&0\\\pm1&0&0&1\end{bmatrix},\quad \psi^{\pm}=\frac{1}{2}\begin{bmatrix}0&0&0&0\\0&1&\pm1&0\\0&\pm1&1&0\\0&0&0&0\end{bmatrix},$$

and the target state ρ_{f} is set as ϕ^+ in this subsection.

To guarantee the solvability, the measurement operators are set as $L_1=\sigma_1^z\otimes I_2-I_2\otimes\sigma_2^z$ and $L_2=\sigma_1^x\otimes\sigma_2^x$, under which the system state ρ_t will converge to one of the two Bell states $\{\phi^+,\phi^-\}$ almost surely, i.e., the system equilibria are ϕ^+ and ϕ^-, and the target state ρ_{f} is one of the system equilibria.

In order to divide the state space $\mathcal{S}$ into two subspaces, we use the following Lyapunov function:

$$V_{\mathrm{B}}(\rho_t)=1-\mathrm{tr}(\rho_t\rho_{\mathrm{f}})+c_{\mathrm{B}}(U_1(\rho_t)+U_2(\rho_t)),$$

where c_{B} is nonnegative constant and $U_j(\rho_t)=\mathrm{tr}\left(L_j^2\rho_t\right)-\mathrm{tr}^2(L_j\rho_t)$, $j=1,2$. Based on $V_{\mathrm{B}}(\rho_t)$, the state space $\mathcal{S}$ is divided into $\{\Phi_{\mathrm{B},1},\Phi_{\mathrm{B},2}\}$, i.e.,

$$\Phi_{\mathrm{B},1} = \{\rho\in\mathcal{S}: V_{\mathrm{B}}(\rho_t)\le k_{\mathrm{B}}\},$$

$$\Phi_{\mathrm{B},2} = \mathcal{S}\backslash\Phi_{\mathrm{B},1}, \tag{17}$$

with $0<k_{\mathrm{B}}<1$ and $1-k_{\mathrm{B}}+c_{\mathrm{B}}\max_{\rho\in\mathcal{S}}(U_1(\rho)+U_2(\rho))<\frac{1}{4}$. Therefrom, $\rho_{\mathrm{f}}\in\Phi_{\mathrm{B},1}$ and all other Bell states are in $\Phi_{\mathrm{B},2}$.

Then, $V_{\mathrm{B}}(\rho_t)$ and $V_{\mathrm{c}}(\rho_t)$ form the coupling Lyapunov functions, based on which we can design the feedback controls $u_{\mathrm{B},1}(\rho_t)$ and $u_{\mathrm{B},2}(\rho_t)$ for the rapid stabilization of Bell states. The infinitesimal generator associated with Eq. (16) acting on $V_{\mathrm{B}}(\rho_t)$ and $V_{\mathrm{c}}(\rho_t)$ are

$$\mathcal{L}V_{\mathrm{B}}(\rho_t) = g_{\mathrm{B}}(\rho_t)u_{\mathrm{B},1}(\rho_t)-4c_{\mathrm{B}}\Bigg(\sum_{j=1}^{2}\eta_{L_j}\Gamma_{L_j}U_j^2(\rho_t) +\sum_{j=1}^{2}\eta_{L_j}\Gamma_{L_j}U_{12}^2(\rho_t)\Bigg), \tag{18}$$

$$\mathcal{L}V_{\mathrm{c}}(\rho_t) = \mathrm{tr}(-\mathrm{i}[X,H_1]\rho_t)u_{\mathrm{B},2}(\rho_t) +\sum_{j=1}^{2}\mathrm{tr}\left(X\Gamma_{L_j}\mathcal{D}(L_j,\rho_t)\right), \tag{19}$$

where

$$g_{\mathrm{B}}(\rho_t)=\mathrm{tr}(\mathrm{i}\rho_{\mathrm{f}}[H_1,\rho_t])+2c\sum_{j=1}^{2}\mathrm{tr}(L_j\rho_t)\,\mathrm{tr}(-\mathrm{i}L_j[H_j,\rho_t])$$

and $U_{12}(\rho_t)=\prod_{j=1}^{2}\mathrm{tr}(L_j\rho_t)$. From Eqs. (18) and (19), the control laws $u_{\mathrm{B},1}(\rho_t)$ and $u_{\mathrm{B},2}(\rho_t)$ can be designed as

$$u_{\mathrm{B},1}(\rho_t)=\frac{-\bar{\lambda}_{\mathrm{B}}V_{\mathrm{B}}(\rho_t)+4c_{\mathrm{B}}\left(\sum_{j=1}^{2}\eta_{L_j}\Gamma_{L_j}U_j^2(\rho_t)+\sum_{j=1}^{2}\eta_{L_j}\Gamma_{L_j}U_{12}^2(\rho_t)\right)}{g_{\mathrm{B}}(\rho_t)},\quad \bar{\lambda}_{\mathrm{B}}>0, \tag{20}$$

$$u_{\mathrm{B},2}(\rho_t)=\frac{-\bar{M}_{\mathrm{B}}-\sum_{j=1}^{2}\mathrm{tr}\left(X\Gamma_{L_j}\mathcal{D}(L_j,\rho_t)\right)}{\mathrm{tr}(-\mathrm{i}[X,H_1]\rho_t)},\quad \bar{M}_{\mathrm{B}}>0, \tag{21}$$

which make $\mathcal{L}V_{\mathrm{B}}(\rho_t)\le-\lambda_{\mathrm{B}}V_{\mathrm{B}}(\rho_t)$ and $\mathcal{L}V_{\mathrm{c}}(\rho_t)\le-M_{\mathrm{B}}$ hold, where $0<\lambda_{\mathrm{B}}\le\bar{\lambda}_{\mathrm{B}}$ and $0<M_{\mathrm{B}}\le\bar{M}_{\mathrm{B}}$. In order to ensure that the denominator of $u_{\mathrm{B},2}(\rho_t)$ is not zero, i.e., $\mathrm{tr}(-\mathrm{i}[X,H_1]\rho_t)\neq0$, we can construct X and H_1 by making $-\mathrm{i}[X,H_1]=I_4-4\rho_{\mathrm{f}}$ such that $\mathrm{tr}(-\mathrm{i}[X,H_1]\rho)>0$ for $\forall\rho\in\Phi_{\mathrm{B},2}$. We can use the similar skill in Subsection 4.2 to make the denominator $g_{\mathrm{B}}(\rho_t)$ of $u_{\mathrm{B},1}(\rho_t)$ notzero, i.e., design another feedback control as $\hat{u}_{\mathrm{B},1}(\rho_t)=-\frac{s_{\mathrm{B}}g_{\mathrm{B}}(\rho_t)}{|g_{\mathrm{B}}(\rho_t)|+\eta_{\mathrm{B}}}$ with $s_{\mathrm{B}}>0$ and $g_{\mathrm{B}}>0$, which works in the state subspace $\hat{\Phi}_{\mathrm{B},1}=\{\rho\in\Phi_{\mathrm{B},1}:|g_{\mathrm{B}}(\rho)|\le\varepsilon_{\mathrm{B}}\}$ with $\varepsilon_{\mathrm{B}}>0$.

Therefore, $u_{\mathrm{B},1}(\rho_t)$, $u_{\mathrm{B},2}(\rho_t)$, and $\hat{u}_{\mathrm{B},1}(\rho_t)$ form the state feedback strategy of rapidly stabilizing Bell states, which can be expressed as

$$u_{\mathrm{B}}(\rho_t)=\begin{cases}u_{\mathrm{B},1}(\rho_t), & \text{if } \rho_t\in\Phi_{\mathrm{B},1}\backslash\hat{\Phi}_{\mathrm{B},1},\\ u_{\mathrm{B},2}(\rho_t), & \text{if } \rho_t\in\Phi_{\mathrm{B},2},\\ \hat{u}_{\mathrm{B},1}(\rho_t), & \text{if } \rho_t\in\hat{\Phi}_{\mathrm{B},1}.\end{cases} \tag{22}$$

Through a discussion similar to that in Subsection 4.2, we can conclude that the target state ρ_{f} is globally asymptotically stable in probability under the switching state feedback control $u_{\mathrm{B}}(\rho_t)$ and the system state ρ_t can converge exponentially in $\Phi_{\mathrm{B},1}\backslash\hat{\Phi}_{\mathrm{B},1}$ with the convergence rate that is no less than λ_{B}.

4.4. Rapid stabilization of N-qubit systems

In this subsection, we consider the rapid stabilization of N-qubit systems, where $N\ge3$. Without loss of generality, the target state is set as the GHZ state, i.e.,

$$\rho_{\mathrm{f}}=|\phi_{\mathrm{f}}\rangle\langle\phi_{\mathrm{f}}|=\frac{1}{2}\begin{bmatrix}1&0&\dots&0&1\\0&0&\dots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\dots&0&0\\1&0&\dots&0&1\end{bmatrix}_{n\times n}, \tag{23}$$

where $|\phi_{\mathrm{f}}\rangle=\frac{1}{\sqrt{2}}(|00\cdots0\rangle+|11\cdots1\rangle)$ and $n=2^N$. Similar to Ref. [31], a diagonal measurement operator is chosen as $L_{\mathrm{G}}=\mathrm{diag}\left[l_1,l_2,\dots,l_{n/2},l_{n/2},\dots,l_2,l_1\right]$, where $l_i\neq l_j$ for

$i,j\in\{1,2,\ldots,n/2\}$ and $i\neq j$, which makes the target state $\rho_{\rm f}$ be one of the eigenstates of $L_{\rm G}$. Due to the fact that $L_{\rm G}$ is degenerate, two control Hamiltonians should be used, such that the corresponding SME describing N-qubit systems is described as

$$\begin{aligned}
{\rm d}\rho_t &= \big(-{\rm i}\left[H_0,\rho_t\right]-{\rm i}\left[H_1,\rho_t\right]-{\rm i}\left[H_2,\rho_t\right]u_{2,t}\\
&\quad+\Gamma_{L_{\rm G}}\mathcal{D}\left(L_{\rm G},\rho_t\right)\big){\rm d}t\\
&\quad+\sqrt{\eta_{L_{\rm G}}\Gamma_{L_{\rm G}}}\mathcal{H}\left(L_{\rm G},\rho_t\right){\rm d}\omega_{L_{\rm G},t}\\
&=\left(-{\rm i}\left[\hat{H}_0,\rho_t\right]-{\rm i}\left[H_2,\rho_t\right]u_{2,t}+\Gamma_{L_{\rm G}}\mathcal{D}\left(L_{\rm G},\rho_t\right)\right){\rm d}t\\
&\quad+\sqrt{\eta_{L_{\rm G}}\Gamma_{L_{\rm G}}}\mathcal{H}\left(L_{\rm G},\rho_t\right){\rm d}\omega_{L_{\rm G},t},\\
{\rm d}y_t &= \sqrt{\eta_{L_{\rm G}}\Gamma_{L_{\rm G}}}{\rm tr}\left(L_{\rm G}\rho_t+\rho_t L_{\rm G}^{\dagger}\right){\rm d}t+{\rm d}\omega_{L_{\rm G},t},
\end{aligned}\tag{24}$$

where $\hat{H}_0=H_0+H_1$ and H_2 can be regarded as the virtual free Hamiltonian and control Hamiltonian, respectively, and $u_{2,t}$ represents the state feedback control that needs to be designed. SME (24) corresponds to the case of $q=2$, $m=1$, $L_1=L_{\rm G}$, and $u_{1,t}=1$ in SME (1).

Here, we choose the distance function as $V_1\left(\rho_t\right)$, i.e.,

$$V_{\rm G}\left(\rho_t\right)=1-{\rm tr}\left(\rho_t\rho_{\rm f}\right),\tag{25}$$

which is used to form the coupling Lyapunov functions with $V_{\rm c}\left(\rho_t\right)$ for design of the state feedback controls $u_{{\rm G},1}\left(\rho_t\right)$ and $u_{{\rm G},2}\left(\rho_t\right)$ of rapidly stabilizing GHZ states and to divide the state space $\mathcal{S}$ into two state subspaces as

$$\begin{aligned}
&\Phi_{{\rm G},1}=\left\{\rho\in\mathcal{S}:V_{\rm G}\left(\rho_t\right)\leq k_{\rm G}\right\},\\
&\Phi_{{\rm G},2}=\mathcal{S}\backslash\Phi_{{\rm G},1},
\end{aligned}\tag{26}$$

where $0<k_{\rm G}<1$. The infinitesimal generators associated with Eq. (24) acting on $V_{\rm G}\left(\rho_t\right)$ and $V_{\rm c}\left(\rho_t\right)$ are

$$\mathcal{L}V_{\rm G}\left(\rho_t\right)={\rm tr}\left({\rm i}\left[\hat{H}_0,\rho_t\right]\rho_{\rm f}\right)+{\rm tr}\left({\rm i}\left[H_2,\rho_t\right]\rho_{\rm f}\right)u_{{\rm G},1}\left(\rho_t\right),\tag{27}$$

$$\begin{aligned}
\mathcal{L}V_{\rm c}\left(\rho_t\right) &= -{\rm tr}\left({\rm i}X\left[\hat{H}_0,\rho_t\right]\right)-{\rm tr}\left({\rm i}\left[X,H_2\right]\rho_t\right)u_{{\rm G},2}\left(\rho_t\right)\\
&\quad+\Gamma_{L_{\rm G}}{\rm tr}\left(X\mathcal{D}\left(L_{\rm G},\rho_t\right)\right).
\end{aligned}\tag{28}$$

Because $L_{\rm G}$ is degenerate, the number of system equilibria is infinite, so we construct the Hamiltonian H_1 such that $\left[\hat{H}_0,\rho_{\rm f}\right]=0$ holds to simplify the control design. Similar to Subsections 4.2 and 4.3, the switching state feedback control can be designed as

$$u_{\rm G}\left(\rho_t\right)=\begin{cases}
u_{{\rm G},1}\left(\rho_t\right)=-\frac{\lambda_{\rm G}V_{\rm G}\left(\rho_t\right)}{{\rm tr}\left({\rm i}\left[H_2,\rho_t\right]\rho_{\rm f}\right)}, & \text{if } \rho_t\in\Phi_{{\rm G},1}\backslash\hat{\Phi}_{{\rm G},1},\\
u_{{\rm G},2}\left(\rho_t\right)=\frac{M_{\rm G}+\Gamma_{L_{\rm G}}{\rm tr}\left(X\mathcal{D}\left(L_{\rm G},\rho_t\right)\right)}{{\rm tr}\left({\rm i}\left[X,H_2\right]\rho_t\right)}, & \text{if } \rho_t\in\Phi_{{\rm G},2},\\
\hat{u}_{{\rm G},1}\left(\rho_t\right)=-\frac{s_{\rm G}{\rm tr}\left({\rm i}\left[H_2,\rho_t\right]\rho_{\rm f}\right)}{\left|{\rm tr}\left({\rm i}\left[H_2,\rho_t\right]\rho_{\rm f}\right)\right|+\eta_{\rm G}}, & \text{if } \rho_t\in\hat{\Phi}_{{\rm G},1},
\end{cases}\tag{29}$$

where $s_{\rm G}>0$, $\eta_{\rm G}>0$, and

$$\hat{\Phi}_{{\rm G},1}=\left\{\rho\in\Phi_{{\rm G},1}:\left|{\rm tr}\left({\rm i}\left[H_2,\rho\right]\rho_{\rm f}\right)\right|\leq\varepsilon_{\rm G}\right\}$$

with $\varepsilon_{\rm G}>0$, which makes $\mathcal{L}V_{\rm G}\left(\rho_t\right)\leq-\lambda_{\rm G}V_{\rm G}\left(\rho_t\right)$ or $\mathcal{L}V_{\rm G}\left(\rho_t\right)\leq-W_{\rm G}\left(\rho_t\right)$ and $\mathcal{L}V_{\rm c}\left(\rho_t\right)\leq-M_{\rm G}$ hold with $0<\lambda_{\rm G}\leq\bar{\lambda}_{\rm G}$, $0<M_{\rm G}\leq\bar{M}_{\rm G}$ and $W_{\rm G}\left(\rho_t\right)=\frac{s_{\rm G}{\rm tr}^2\left({\rm i}\left[H_2,\rho_t\right]\rho_{\rm f}\right)}{\left|{\rm tr}\left({\rm i}\left[H_2,\rho_t\right]\rho_{\rm f}\right)\right|+\eta_{\rm G}}$. For the feedback control $u_{\rm G}\left(\rho_t\right)$ in Eq. (29), $\hat{u}_{{\rm G},1}\left(\rho_t\right)$ is used to avoid the denominators of $u_{{\rm G},1}\left(\rho_t\right)$ too small, while the denominators of $u_{{\rm G},1}\left(\rho_t\right)$ cannot be zero by constructing X and H_2 based on the algorithm in Ref. [16].

From Theorem 1, under $u_{{\rm G},2}\left(\rho_t\right)$, the system state ρ_t only transits through the boundary between $\Phi_{{\rm G},1}$ and $\Phi_{{\rm G},2}$ no more than two times from any initial state, and evolves in $\Phi_{{\rm G},1}$ permanently after finite time. In the state subspace $\Phi_{{\rm G},1}$, according to LaSalle's invariance principle, the system state ρ_t will converge to the invariant sets $\mathcal{I}_{\rm G}=\{\rho\in\Phi_{{\rm G},1}:V_{\rm G}\left(\rho\right)=0\}$ and $\hat{\mathcal{I}}_{\rm G}=\{\rho\in\Phi_{{\rm G},1}:{\rm tr}\left({\rm i}\left[H_2,\rho\right]\rho_{\rm f}\right)=0\}$ under $u_{{\rm G},1}\left(\rho_t\right)$ and $\hat{u}_{{\rm G},1}\left(\rho_t\right)$, respectively. It is obvious that $\mathcal{I}_{\rm G}=\{\rho_{\rm f}\}$. On the other hand, $\hat{\mathcal{I}}_{\rm G}=\{\rho_{\rm f},\rho_i\}$, where ρ_i represent the eigenstates of $L_{\rm G}$ except $\rho_{\rm f}$ from the analysis in Ref. [31]. The states in $\hat{\mathcal{I}}_{\rm G}$, which are denoted as $\rho_{\hat{\mathcal{I}}_{\rm G}}$, satisfy $\mathcal{D}\left(L_{\rm G},\rho_{\hat{\mathcal{I}}_{\rm G}}\right)=0$ and $\mathcal{H}\left(L_{\rm G},\rho_{\hat{\mathcal{I}}_{\rm G}}\right)=0$, such that SME (24) becomes ${\rm d}\rho_t=-{\rm i}\left[\hat{H}_0,\rho_t\right]{\rm d}t$, from which one can construct H_1 to make $\left[\hat{H}_0,\rho_i\right]\neq0$, thereby only $\rho_{\rm f}$ is the equilibrium state that located in the invariant set $\hat{\mathcal{I}}_{\rm G}$ under the condition $\left[\hat{H}_0,\rho_{\rm f}\right]=0$. Thus, the target state $\rho_{\rm f}$ is globally asymptotically stable in probability under the feedback control $u_{\rm G}\left(\rho_t\right)$ and the system state ρ_t can converge exponentially in $\Phi_{{\rm G},1}\backslash\hat{\Phi}_{{\rm G},1}$ with the convergence rate that is not less than $\lambda_{\rm G}$ from Theorem 1 in this paper and Theorem 3 in Ref. [18].

Remark 3 The stabilization of multi-qubit systems was also studied in Ref. [31], where the proposed feedback control is a special case of the feedback control $u_{\rm G}\left(\rho_t\right)$ in Eq. (29) with $\varepsilon_{\rm G}=+\infty$. The main difference between $u_{\rm G}\left(\rho_t\right)$ and the feedback control in Ref. [31] is that $u_{\rm G}\left(\rho_t\right)$ can make the system state ρ_t converge exponentially in $\Phi_{{\rm G},1}\backslash\hat{\Phi}_{{\rm G},1}$, which cannot be guaranteed in theory by the feedback control in Ref. [31].

Remark 4 In Subsection 4.2 (4.3, 4.4), we only make the system state ρ_t converge exponentially in $\Phi_{{\rm e},1}\backslash\hat{\Phi}_{{\rm e},1}$ ($\Phi_{{\rm B},1}\backslash\hat{\Phi}_{{\rm B},1}$, $\Phi_{{\rm G},1}\backslash\hat{\Phi}_{{\rm G},1}$, respectively) rather than $\Phi_{{\rm e},1}$ ($\Phi_{{\rm B},1}$, $\Phi_{{\rm G},1}$, respectively), which can be avoided if the feedback control $u_{{\rm e},1}\left(\rho_t\right)$ ($u_{{\rm B},1}\left(\rho_t\right)$, $u_{{\rm G},1}\left(\rho_t\right)$, respectively) is designed more properly. For example, the state feedback controls in Refs. [28,34] can achieve the GHZ states and local exponential stabilization of eigenstates for multi-qubit systems and N-level quantum angular momentum systems, respectively. If the feedback controls in Refs. [28,34] are applied instead of $u_{{\rm G},1}\left(\rho_t\right)$ and $u_{{\rm e},1}\left(\rho_t\right)$, respectively, and the state subspaces that the system state converge exponentially in Refs. [28,34] are regarded as $\Phi_{{\rm G},1}$ and $\Phi_{{\rm e},1}$, respectively, then the system state can be converge more faster, which means that the results of exponential stabilization for quantum systems and the unified framework proposed in this paper can be combined reasonably to obtain better control performance.

5. Numerical simulations

In this section, we take a three-qubit system with a degenerate measurement operator as an example to verify the effectiveness and rapidity of the proposed state feedback controls by stabilizing a target GHZ state.

In the simulations, the target GHZ state is chosen as

$$\rho_{\rm f} = |\phi_{\rm f}\rangle\langle\phi_{\rm f}| = \frac{1}{2}\begin{bmatrix} 1&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1 \end{bmatrix},$$

where $|\phi_{\rm f}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, while the initial state is set as $\rho_0 = {\rm diag}[0,1,0,0,0,0,0,0]$. The system parameter settings are listed as[16]

$$\begin{aligned} H_0 &= \sigma_z\otimes\sigma_z\otimes I_2 + \sigma_z\otimes I_2\otimes\sigma_z + I_2\otimes\sigma_z\otimes\sigma_z \\ &= {\rm diag}[3,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ 3], \\ H_1 &= \begin{bmatrix} 0&1&0&0&0&0&-1&0\\ 1&0&0&0&0&0&0&-1\\ 0&0&0&1&-1&0&0&0\\ 0&0&1&0&0&-1&0&0\\ 0&0&-1&0&0&1&0&0\\ 0&0&0&-1&1&0&0&0\\ -1&0&0&0&0&0&0&1\\ 0&-1&0&0&0&0&1&0 \end{bmatrix}, \\ \Gamma_{L_{\rm G}} &= 1, \quad \eta_{L_{\rm G}} = 1, \end{aligned}$$

$$H_2 = \begin{bmatrix} -2.50&-0.87&0.61&-0.35&0&0&0&0\\ -0.87&-3.50&-0.35&0.20&0&0&0&0\\ 0.61&-0.35&-2.50&-0.29&0&0&0&0\\ -0.35&0.20&-0.29&-1.50&0&0&0&0\\ 0&0&0&0&1.50&0.35&0.20&-0.87\\ 0&0&0&0&0.35&2.50&0.29&-0.61\\ 0&0&0&0&0.20&0.29&3.50&-0.35\\ 0&0&0&0&-0.87&-0.61&-0.35&2.50 \end{bmatrix},$$

$$L_{\rm G} = \frac{1}{2}(\sigma_z\otimes\sigma_z\otimes I_2 + I_2\otimes\sigma_z\otimes\sigma_z) = {\rm diag}[1,\ 0,\ -1,\ 0,\ 0,\ -1,\ 0,\ 1].$$

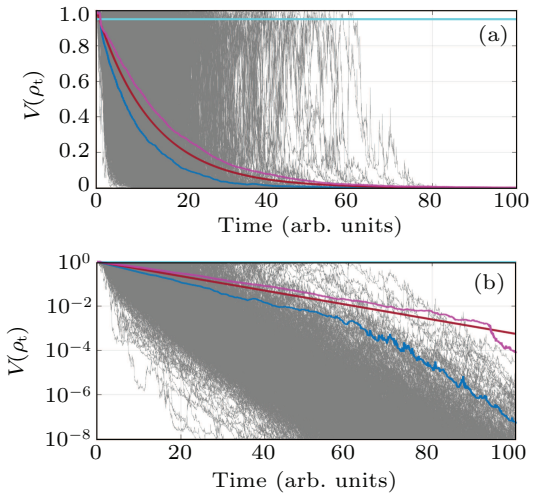


Fig. 2. Simulation results of rapidly stabilizing GHZ state by using the switching feedback control (29) and the feedback control proposed in Ref. [16]. (a) The curves of $V(\rho_t)$. (b) The semi-lon version of (a). In (a) and (b), the grey curves, blue curve and purple curve represent the sample trajectories, the mean value of 1000 arbitrary sample trajectories under the feedback control (29), and the feedback control in Ref. [16], respectively; the red curve and the cyan line represent the exponential reference with exponent $\lambda_{\rm G}$ and the boundary between $\Phi_{\rm G,1}$ and $\Phi_{\rm G,2}$, respectively.

The control parameters in Eq. (29) are set as $\lambda_{\rm G} = 0.15$, $M_{\rm G} = 1$, $s_{\rm G} = 0.5$, and $\eta_{\rm G} = 0.05$. The state space $\mathcal{S}$ is divided into $\Phi_{\rm G,1} = \{\rho \in \mathcal{S} : V_1(\rho) \le k_{\rm G}\}$ and $\Phi_{\rm G,2} = \mathcal{S}\backslash\Phi_{\rm G,1}$, where $k_{\rm G} = 0.95$. The state subspace, where the control law $\hat{u}_{\rm G,1}(\rho_t)$ works, is set as $\hat{\Phi}_{\rm G,1} = \{\rho \in \Phi_{\rm G,1} : |{\rm tr}([H_2,\rho]\rho_{\rm f})| \le \varepsilon_{\rm G}\}$ with $\varepsilon_{\rm G} = 0.05$. The simulation results of 1000 sample trajectories under the switching feedback control (29) are shown in Fig. 2. One can see that $V(\rho_t)$ converges to zero, which means that the stabilization of the target GHZ state is achieved and verifies the effectiveness of the proposed switching state feedback control. On the other hand, the convergence rate of quantum states under the switching feedback control (29) is more than $\lambda_{\rm G}$, which means the rapidity of stabilizing target state.

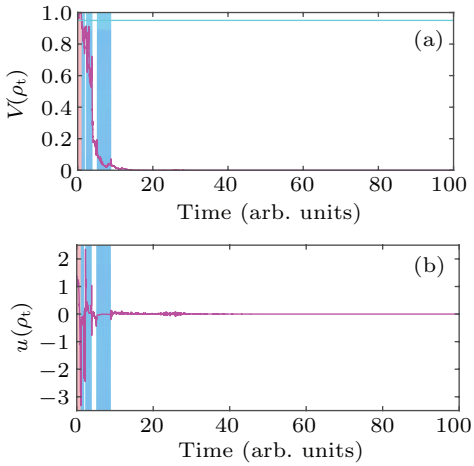


Fig. 3. Simulation results of arbitrary sample trajectory. (a) The curves of $V(\rho_t)$, (b) The curves of $u(\rho_t)$. In (a) and (b), the purple curves represent $V_{\rm G}(\rho_t)$ and $u_{\rm G}(\rho_t)$, while the blue, pink and white background present the time domain of $\rho_t \in \Phi_{\rm G,1}$,$\Phi_{\rm G,2}$ and $\hat{\Phi}_{\rm G,1}$, respectively.

Moreover, the feedback control proposed in Ref. [16], i.e., using $\tilde{u}_{\mathrm{G}}(\rho_t) = -K_{\mathrm{G}}\,\mathrm{tr}(\mathrm{i}[H_2,\rho_t]\rho_{\mathrm{f}})$ instead of $u_{\mathrm{G},1}(\rho_t)$ and $\hat{u}_{\mathrm{G},1}(\rho_t)$ in $\Phi_{\mathrm{G},1}$, is also applied to stabilize the target GHZ state. The results with $K_{\mathrm{G}} = 0.02$ are also shown in Fig. 2. One can see from Fig. 2 that $V(\rho_t)$ converges to zero faster under the switching feedback control (29) than that under the feedback control proposed in Ref. [16], which shows the advantage of the designed feedback control in this paper. A sample trajectory is presented in Fig. 3, from which one can see that the state trajectory transits the boundary between $\Phi_{\mathrm{G},1}$ and $\Phi_{\mathrm{G},2}$ no more than two times, which is consistent with the theoretical analysis.

6. Conclusions

In summary, we have studied the rapid stabilization of stochastic quantum systems. In order to rapidly stabilize the target state, we propose a unified framework for design of state feedback controls based on the Lyapunov–LaSalle-like theorem, which is suitable for the Itô stochastic differential equation. The proposed unified framework is applied to the rapid stabilization of single-qubit systems, two-qubit systems, and N-qubit systems. Moreover, we take three-qubit systems as an example to verify the effectiveness and rapidity of the designed state feedback controls based on the proposed unified framework by stabilizing GHZ states in numerical simulations.

In the near future, we will consider the further work as follows: (i) design of state feedback controls for the global exponential stabilization of the general stochastic quantum systems instead of some special quantum systems;[34] (ii) robustness analysis of the state feedback strategies proposed in this paper when the physical parameters or the initial state are uncertain, as the work in Ref. [17]; (iii) exponential stabilization of stochastic delay quantum systems, specially the case of time-delay control, i.e., $u_t = u(\rho_{t-\tau})$ with constant delay time τ or $u_t = u\left(\rho_{t-\tau(t)}\right)$ with time-varying delay time $\tau(t)$.

Acknowledgements

Project supported in part by the National Natural Science Foundation of China (Grant No. 72071183), and Research Project Supported by Shanxi Scholarship Council of China (Grant No. 2020-114).

References

[1] D'Alessandro D and Dahleh M 2001 *IEEE Trans. Automat. Control* **46** 866
[2] Shabani A and Jacobs K 2008 *Phys. Rev. Lett.* **101** 230403
[3] van Damme L, Ansel Q, Glaser S J and Sugny D 2017 *Phys. Rev. A* **95** 063403
[4] Kuang S and Cong S 2008 *Automatica* **44** 98
[5] Kuang S, Dong D and Petersen I R 2017 *Automatica* **81** 164
[6] Wen J, Shi Y and Lu X 2018 *J. Franklin Institute* **355** 2562
[7] Wang Y, Hu C, Shi Z, Huang B, Song J and Xia Y 2019 *Ann. Phys.* **531** 1900006
[8] Belavkin V P 1983 *Avtomat. i Telemekh.* **44** 178
[9] Belavkin V P 1992 *J. Multivariate Anal.* **42** 171
[10] Mirrahimi M and van Handel R 2007 *SIAM J. Control Optim.* **46** 445
[11] Ge S S, Vu T L and Huang C C 2012 *Automatica* **48** 1031
[12] Cong S, Wen J, Kuang S and Meng F 2016 *Sci. China Inform. Sci.* **59** 112502
[13] Vu T L, Ge S S and Lee T H *2015 54th IEEE Conference on Decision and Control*, December 15–18, 2015, Osaka, Japan, p. 7749
[14] Zhou J, Kuang S and Cong S 2017 *J. Syst. Sci. Complex.* **30** 347
[15] Liu Y, Kuang S and Cong S 2017 *IEEE Trans. Cybern.* **47** 3827
[16] Kuang S, Li G, Liu Y, Sun X and Cong S 2022 *IEEE Trans. Cybern.* **52** 11213
[17] Liang W, Amini N and Mason P 2021 *SIAM J. Control Optim.* **59** 669
[18] Cardona G, Sarlette A and Rouchon P 2020 *Automatica* **112** 108719
[19] Sayrin C, Dotsenko I, Zhou X, Peaudecerf B, Rybarczyk T, Gleyzes S, Rouchon P, Mirrahimi M, Amini H, Brune M, Raimond J and Haroche S 2011 *Nature* **477** 73
[20] Wen J, Shi Y, Pang X, Jia J and Zeng J 2020 *J. Franklin Institute* **357** 7515
[21] Yamamoto N, Tsumura K and Hara S 2007 *Automatica* **43** 981
[22] van Handel R, Stockton J K and Mabuchi H 2005 *J. Opt. B: Quant. Semiclassic. Opt.* **7** S179
[23] Mancini S and Bose S 2004 *Phys. Rev. A* **70** 022307
[24] Dotsenko I, Mirrahimi M, Brune M, Haroche S, Raimond J M and Rouchon P 2009 *Phys. Rev. A* **80** 013805
[25] Amini N, Somaraju R A, Dotsenko I, Sayrin C, Mirrahimi M and Rouchon P 2013 *Automatica* **49** 2683
[26] Wang J and Wiseman H M 2001 *Phys. Rev. A* **64** 063810
[27] Ticozzi F, Schirmer S G and Wang X 2010 *IEEE Trans. Automat. Control* **55** 2901
[28] Liang W, Amini N and Mason P 2022 *IEEE Trans. Automat. Control* **67** 2918
[29] Wen J, Shi Y, Jia J and Zeng J 2021 *Results Phys.* **22** 103929
[30] Sun X, Kuang S and Dong D *2017 IEEE International Conference on Systems, Man, and Cybernetics*, October 5–8, 2017, Banff, Canada, p. 310
[31] Li G, Liu Y, Kuang S and Xiang C 2022 *J. Franklin Institute* **359** 2073
[32] Deng H, Krstic M and Williams R 2001 *IEEE Trans. Automat. Control* **46** 1237
[33] van Handel R, Stockton J K and Mabuchi H 2005 *IEEE Trans. Automat. Control* **50** 768
[34] Liang W, Amini N and Mason P 2019 *SIAM J. Control Optim.* **57** 3939