

Interaction solutions and localized waves to the (2+1)-dimensional Hirota-Satsuma-Ito equation with variable coefficient

Xinying Yan(闫鑫颖), Jinzhou Liu(刘锦洲), and Xiangpeng Xin(辛祥鹏)
Citation: Chin. Phys. B, 2023, 32 (**7**): 070201. DOI: 10.1088/1674-1056/acb9f2
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Interaction solutions for the second extended (3+1)-dimensional Jimbo-Miwa equation

Hongcai Ma(马红彩), Xue Mao(毛雪), and Aiping Deng(邓爱平)
Chin. Phys. B, 2023, 32 (**6**): 060201. DOI: 10.1088/1674-1056/acb91c

Superposition formulas of multi-solution to a reduced (3+1)-dimensional nonlinear evolution equation

Hangbing Shao(邵杭兵) and Bilige Sudao(苏道毕力格)
Chin. Phys. B, 2023, 32 (**5**): 050204. DOI: 10.1088/1674-1056/acae7d

Propagation and modulational instability of Rossby waves in stratified fluids

Xiao-Qian Yang(杨晓倩), En-Gui Fan(范恩贵), and Ning Zhang(张宁)
Chin. Phys. B, 2022, 31 (**7**): 070202. DOI: 10.1088/1674-1056/ac4e0a

Solutions of novel soliton molecules and their interactions of (2 + 1)-dimensional potential Boiti-Leon-Manna-Pempinelli equation

Hong-Cai Ma(马红彩), Yi-Dan Gao(高一丹), and Ai-Ping Deng(邓爱平)
Chin. Phys. B, 2022, 31 (**7**): 070201. DOI: 10.1088/1674-1056/ac4cc0

High-order rational solutions and resonance solutions for a (3+1)-dimensional Kudryashov-Sinelshchikov equation

Yun-Fei Yue(岳云飞), Jin Lin(林机), and Yong Chen(陈勇)
Chin. Phys. B, 2021, 30 (**1**): 010202. DOI: 10.1088/1674-1056/abb3f3

Interaction solutions and localized waves to the (2+1)-dimensional Hirota–Satsuma–Ito equation with variable coefficient

Xinying Yan(闫鑫颖), Jinzhou Liu(刘锦洲), and Xiangpeng Xin(辛祥鹏)†

School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, *China*

(Received 8 December 2022; revised manuscript received 6 February 2023; accepted manuscript online 8 February 2023)

This article investigates the Hirota–Satsuma–Ito equation with variable coefficient using the Hirota bilinear method and the long wave limit method. The equation is proved to be Painlevé integrable by Painlevé analysis. On the basis of the bilinear form, the forms of two-soliton solutions, three-soliton solutions, and four-soliton solutions are studied specifically. The appropriate parameter values are chosen and the corresponding figures are presented. The breather waves solutions, lump solutions, periodic solutions and the interaction of breather waves solutions and soliton solutions, etc. are given. In addition, we also analyze the different effects of the parameters on the figures. The figures of the same set of parameters in different planes are presented to describe the dynamical behavior of solutions. These are important for describing water waves in nature.

Keywords: (2+1)-dimensional variable coefficient Hirota–Satsuma–Ito equation, Hirota bilinear method, long wave limit method, *N*-soliton solutions

PACS: 02.30.Jr, 02.30.Ik, 04.20.Jb, 02.30.Tb **DOI:** 10.1088/1674-1056/acb9f2

1. Introduction

Research of solutions to non-linear evolution equations (NLEEs) plays an important role in the field of nonlinearity. Therefore, search for efficient solution methods has gained more and more attention. To date, there are several effective methods that have been commonly used by mathematicians, such as Darboux transformation,[1,2] Bäcklund transformation,[3–5] Lie group method,[6–9] inverse scattering method,[10] Hirota bilinear method,[11–14] and hyperbolic method.[15] Among them, the Hirota bilinear method proposed by Hirota is a straightforward method for finding exact solutions of NLEEs. The general idea of the bilinear method is to transform the equation into a bilinear form based on the derivative operator defined by Hirota, and to solve the equation in the bilinear form by appropriately dependent variable transformation. The key to the bilinear approach is to find the appropriately dependent variable transformation. There are many kinds of solutions that can be obtained by solving NLEEs in different ways, such as soliton solutions, breather wave solutions, lump solutions, periodic solutions, trigonometric solutions, and interaction of different solutions.[16–19] Among them, soliton solutions are one of the most important research topics. *N*-soliton solutions are exact multiple wave solutions of the integrable equations and are important for describing nonlinearities in areas such as shallow water waves, plasma, and nonlinear optics.[20–22] However, in recent years, existence of interaction solutions of two different solutions of the integrable equation has attracted much attention. This is because many phenomena occurring in nature can be more easily explained by interaction solutions.[23,24] Thus, there are more and more researchers to study the interaction solutions deeply. For example, Yuan *et al.* constructed various localized interaction solutions including the general M-lumps, T-breathers and hybrid wave solutions.[25] These solutions play important roles in solving practical problems.

The (2+1)-dimensional Hirota–Satsuma–Ito (HSI) equation is an integrable equation. It is obtained by the Hirota–Satsuma (HS) equation with some transformations.[26] The form of the HS equation is

$$\begin{cases} u_t = u_{xxt} + 3uu_t - 3u_x v_t - u_x, \\ v_x = -u. \end{cases} \tag{1}$$

With the transformation $u = 2(\ln f)_{xx}$, the form of the HSI equation is

$$\begin{cases} \omega_t = u_{xxt} + 3uu_t - 3u_x v_t + u_x, \\ \omega_x = -u_y, \\ v_x = -u, \end{cases} \tag{2}$$

where u, v, and ω are the functions of x, y, t. Hence, we can get the following form after some calculations:

$$u_{xx} + u_{yt} + 3u_x u_{xt} + 3u_t u_{xx} + u_{xxxt} = 0. \tag{3}$$

In Ref. [27], Liu *et al.* analyzed the HSI equation with constant coefficient using the Hirota bilinear method and obtained some interactions of different types of solutions. Kumar *et al.*[28] studied the HSI equation by Lie symmetry analysis and obtained multiple soliton solutions and interactions of soliton solutions.

Equations with variable coefficients can explain more natural phenomena than equations with constant coefficients.

†Corresponding author. E-mail: xinxiangpeng@lcu.edu.cn

© 2023 Chinese Physical Society and IOP Publishing Ltd

Therefore, in this article we study the HSI equation with variable coefficient (vcHSI). Without loss of generality, the vcHSI can be assumed as follows:

$$a(t)u_{xx}+c(t)u_{yt}+3d(t)u_xu_{xt}+3d(t)u_tu_{xx}+e(t)u_{xxxt}=0, \quad (4)$$

where $a(t)$, $c(t)$, $d(t)$, and $e(t)$ are free functions about time t. It is obvious that Eq. (4) reduces to Eq. (3) when $a(t)=c(t)=d(t)=e(t)=1$. However, the integrability of Eq. (4) is uncertain. The Painlevé analysis method is an effective way of judging the integrability of nonlinear differential equations and finding exact solutions.[29,30] We now perform the Painlevé analysis on Eq. (4) to find integrability conditions. The solution of Eq. (4) can be assumed as follows:

$$u=\sum_{j=0}^{\infty}u_j\varphi^{j-\alpha}, \quad (5)$$

where u_j and φ are functions about x, y, t.

Substituting $u=u_0\varphi^{-\alpha}$ into Eq. (4) yields $\alpha=1$ by homogeneous balance. Hence, $u=u_0/\varphi$ when $j=0$. After some calculations, we can reach

$$u_0=\frac{2e(t)\varphi_x}{d(t)}. \quad (6)$$

Substituting $\alpha=1$ into Eq. (5) yields

$$u\cong\frac{2e(t)\varphi_x}{d(t)\varphi}+u_j\varphi^{j-1}. \quad (7)$$

The resonance points can be determined at $j=-1,1,4,6$ after substituting Eq. (6) into Eq. (4) and extracting the coefficient of φ^{-5}. Here, u_2, u_3, and u_5 can be expressed by letting the coefficients of φ^{-3}, φ^{-2}, and φ^0 appear.

Because u_1, u_4, and u_6 are free functions when $a(t)=c(t)=d(t)=e(t)$ or $c(t)=d(t)=e(t)$, the integrability conditions are $a(t)=c(t)=d(t)=e(t)$ or $c(t)=d(t)=e(t)$.

Here, we investigate the following vcHSI equation:

$$a(t)u_{xx}+u_{yt}+3u_xu_{xt}+3u_tu_{xx}+u_{xxxt}=0. \quad (8)$$

It is not difficult to find that Eq. (8) is equal to Eq. (4) when $c(t)=d(t)=e(t)=1$.

This article is arranged as follows: In Section 2, the dependent variable transformation and the bilinear operators are introduced, and the bilinear form of the vcHSI equation is obtained with the Hirota bilinear method. In Section 3, the expressions of two-soliton solutions, three-soliton solutions, and four-soliton solutions are studied on the basis of bilinear form. We choose the appropriate parameter values and draw the corresponding figures. In addition, the dynamical behavior of solutions in different planes is analyzed. In Section 4, the long wave limit method is introduced. This method is used to obtain other solutions including lump solution, the interaction of one linear wave and multi-soliton, and the interaction of second-lump solution and kink solution, etc. The corresponding figures are presented to describe the evolution of solutions. In Section 5, the conclusions of this paper can be found.

2. The bilinear form of the vcHSI equation

First of all, we introduce the dependent variable transformation[26]

$$u=2(\ln f)_x=\frac{2f_x}{f}. \quad (9)$$

Substituting Eq. (9) into Eq. (8), we can obtain the bilinear form with bilinear operators as follows:

$$\left(a(t)D_x^2+D_yD_t+D_x^3D_t\right)(f\cdot f)=0, \quad (10)$$

where f is the function of x, y, t; D_yD_t, $D_x{}^2$, and $D_x{}^3D_t$ are bilinear operators as follows:[11,12]

$$\begin{aligned}D_x{}^mD_y{}^nD_t{}^q(g\cdot h)=&(\partial_x-\partial_{x'})^m\left(\partial_y-\partial_{y'}\right)^n(\partial_t-\partial_{t'})^q\\&\times g(x,y,t)h\left(x',y',t'\right)\big|_{x'=x,y'=y,t'=t}. \quad (11)\end{aligned}$$

We now describe the process of assuming f, using $N=1$ as an example. Firstly, we assume

$$u=\exp\left[k_1(x+p_1y+w_1(t))+\gamma_1^0\right], \quad (12)$$

where k_1, p_1, and γ_1^0 are free constants, $w_1(t)$ is the function of t, after substituting Eq. (12) into linear terms in Eq. (8) we can reach[31]

$$w_1(t)=\int\frac{-a(t)}{k_1{}^2+p_1}\mathrm{d}t. \quad (13)$$

Based on the Hirota bilinear method we can assume

$$f=1+\exp(\gamma_1), \quad (14)$$

where $\gamma_1=k_1(x+p_1y+w_1(t))+\gamma_1^0$. Hence, if f is the solution to Eq. (10), the u we obtained by transformation (9) is the single soliton solution to Eq. (8).

3. The N-soliton solutions of the vcHSI equation

In this section, we introduce the different expressions for solutions to the NLEEs with different values of N on the basis of bilinear form.[32]

When $N=2$, using the same method on the basis of the single soliton solution,[31] we can obtain

$$w_2(t)=\int\frac{-a(t)}{k_2{}^2+p_2}\mathrm{d}t. \quad (15)$$

With the help of hirota bilinear method, the expression for f is assumed as follows:

$$f=1+\exp(\gamma_1)+\exp(\gamma_2)+b_{12}\exp(\gamma_1+\gamma_2), \quad (16)$$

where $\gamma_i=k_i(x+p_iy+w_i(t))+\gamma_i^0,(i=1,2)$, with k_i, p_i, γ_i^0 and b_{12} being free constants.

Substituting the solution that is obtained by taking Eq. (16) together with Eq. (15) into Eq. (9) into Eq. (8), we will get a system. After solving the system, we can know

$$b_{12}=\frac{k_1\left(k_1\left({k_1}^2-3k_1k_2+4{k_2}^2+2p_1+p_2\right)-3k_2\left({k_2}^2+p_1+p_2\right)\right)+{k_2}^2\left({k_2}^2+p_1+2p_2\right)+\left(p_1-p_2\right)^2}{k_1\left(k_1\left({k_1}^2+3k_1k_2+4{k_2}^2+2p_1+p_2\right)+3k_2\left({k_2}^2+p_1+p_2\right)\right)+{k_2}^2\left({k_2}^2+p_1+2p_2\right)+\left(p_1-p_2\right)^2}. \tag{17}$$

Substituting Eq. (17) into the solution and taking the parameters as $a(t)=t$, $k_1=-1$, $k_2=1$, $p_1=-3I$, $p_2=3I$, $\gamma_1^0=\gamma_2^0=0$, we can get breather waves in the plane. Figure 1 shows the dynamical behavior of the breather waves in (x,y) plane. It is clear that the position of the breather waves changes along x-axis with time from Figs. 1(a)–1(c). However, the amplitude and the waveform of the breather waves keep the same. Figures 1(d)–1(f) are their corresponding density plots.

Without loss of generality, we have chosen another set of parameters to draw the 3D-plots and show them in Fig. 2. When $p_1=1-2I$, $p_2=1+2I$, $\gamma_1^0=\gamma_2^0=0$, the coefficient function $a(t)$ is $\coth(t)$. We can obtain a set of interactions of soliton solutions and period solutions. Figures 2(a)–2(c) show the change of u for different k_1 and k_2. In Fig. 2(a), $k_1=1$ and $k_2=-1$. In Fig. 2(b), $k_1=2$ and $k_2=-2$. In Fig. 2(c), $k_1=3$ and $k_2=-3$.

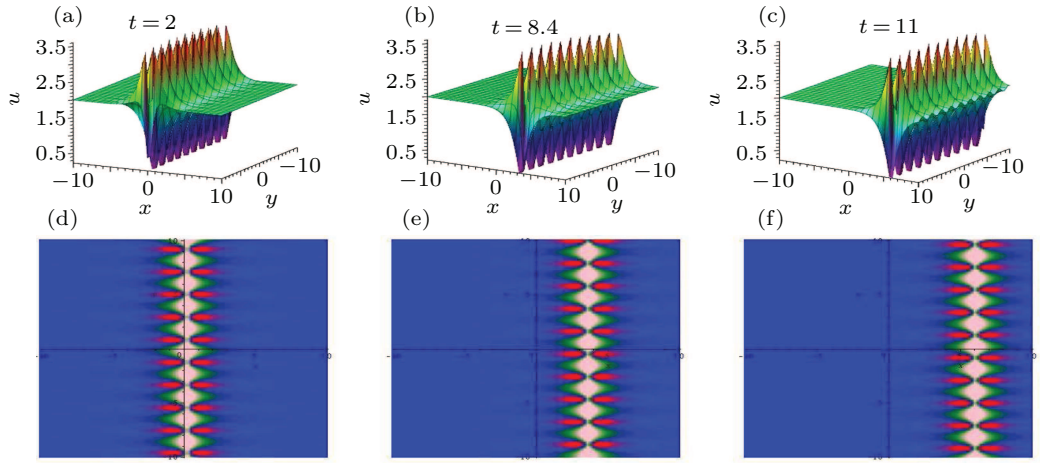


Fig. 1. The breather waves in the (x,y) plane and their density plots.

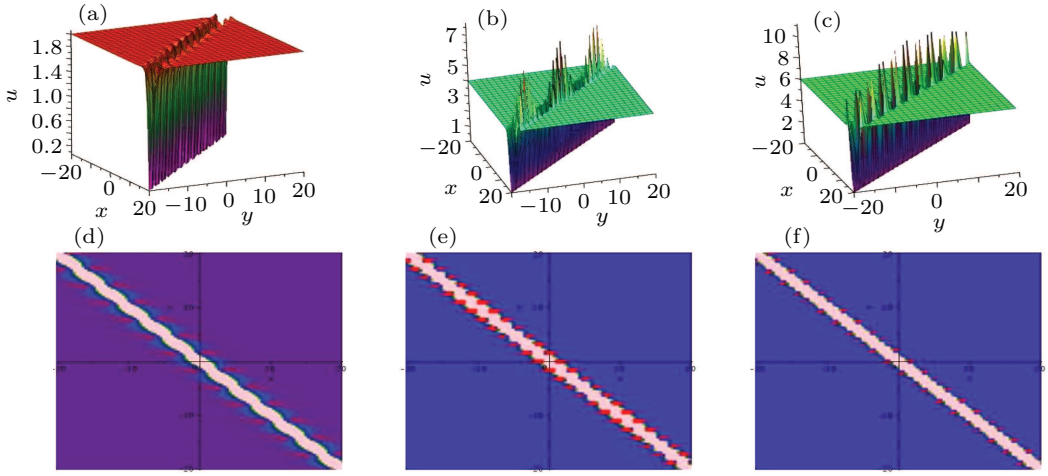


Fig. 2. The interaction of breather waves and soliton solutions in the (x,y) plane and their density plots.

When $N=3$, we can employ the same method to obtain

$$f=1+\exp(\gamma_1)+\exp(\gamma_2)+\exp(\gamma_3)+b_{12}\exp(\gamma_1+\gamma_2)+b_{13}\exp(\gamma_1+\gamma_3)+b_{23}\exp(\gamma_2+\gamma_3)+b_{12}b_{13}b_{23}\exp(\gamma_1+\gamma_2+\gamma_3), \tag{18}$$

where

$$\gamma_i=k_i\left(x+p_iy+w_i(t)\right)+\gamma_i^0,\qquad w_i(t)=\int\frac{-a(t)}{{k_i}^2+p_i}\mathrm{d}t,\qquad i=1,2,3,$$

$$b_{13}=\frac{k_1\left(k_1\left({k_1}^2-3k_1k_3+4{k_3}^2+2p_1+p_3\right)-3k_3\left({k_3}^2+p_1+p_3\right)\right)+{k_3}^2\left({k_3}^2+p_1+2p_3\right)+\left(p_1-p_3\right)^2}{k_1\left(k_1\left({k_1}^2+3k_1k_3+4{k_3}^2+2p_1+p_3\right)+3k_3\left({k_3}^2+p_1+p_3\right)\right)+{k_3}^2\left({k_3}^2+p_1+2p_3\right)+\left(p_1-p_3\right)^2},$$

$$b_{23}=\frac{k_2\left(k_2\left({k_2}^2-3k_2k_3+4{k_3}^2+2p_2+p_3\right)-3k_3\left({k_3}^2+p_2+p_3\right)\right)+{k_3}^2\left({k_3}^2+p_2+2p_3\right)+\left(p_2-p_3\right)^2}{k_2\left(k_2\left({k_2}^2+3k_2k_3+4{k_3}^2+2p_2+p_3\right)+3k_3\left({k_3}^2+p_2+p_3\right)\right)+{k_3}^2\left({k_3}^2+p_2+2p_3\right)+\left(p_2-p_3\right)^2}, \tag{19}$$

where k_i, p_i, and γ_i^0 are free constants.

Substituting Eq. (18) into Eq. (9) and taking the parameters as $t=1$, $k_1=2+I$, $k_2=2-I$, $k_3=2+I$, $p_1=-I$, $p_2=I$, $p_3=-I$, and $\gamma_1^0=\gamma_2^0=\gamma_3^0=0$, we obtain the results as shown in Fig. 3. Figures 3(a), 3(b), and 3(c) show all the interactions of breather waves, kink solutions, and excitation solutions, and they show the 3D plots when $a(t)$ is taken as different functions. It is obvious that the amplitude of breathers becomes smaller from Fig. 3(a) to Fig. 3(c).

Next, we choose parameters $k_1=1+2I$, $k_2=1-2I$, $k_3=1+2I$, $p_1=1-I$, $p_2=1+I$, $p_3=1-I$, $\gamma_1^0=\gamma_2^0=\gamma_3^0=0$, and present the corresponding figures on different planes as Fig. 4. The coefficient function $a(t)$ is taken as $\cos t$. In the (x,y) plane, t is taken as 0, the solution is the interaction of kink solution and breather waves, as shown in Fig. 4(a). In the (y,t) plane, x is taken as 0, the solution is the kink solution, as shown in Fig. 4(b). In the (x,t) plane, y is taken as 0, the solution is the interaction of kink solution and breather waves, as shown in Fig. 4(c).

When $N=4$, the expression for f is

$$f=\sum_{\beta=0,1}\exp\left(\sum_{i=1}^{4}\beta_i\gamma_i+\sum_{1\le i<j}^{4}\beta_i\beta_j\ln b_{ij}\right), \tag{20}$$

where

$$\gamma_i=k_i(x+p_iy+w_i(t))+\gamma_i^0,\quad w_i(t)=\int\frac{-a(t)}{{k_i}^2+p_i}\mathrm{d}t,\quad i=1,2,3,4,$$

$$b_{ij}=\frac{k_i\left(k_i\left({k_i}^2-3k_ik_j+4{k_j}^2+2p_i+p_j\right)-3k_j\left({k_j}^2+p_i+p_j\right)\right)+{k_j}^2\left({k_j}^2+p_i+2p_j\right)+(p_i-p_j)^2}{k_i\left(k_i\left({k_i}^2+3k_ik_j+4{k_j}^2+2p_i+p_j\right)+3k_j\left({k_j}^2+p_i+p_j\right)\right)+{k_j}^2\left({k_j}^2+p_i+2p_j\right)+(p_i-p_j)^2},\quad 1\le i<j\le 4, \tag{21}$$

with k_i, p_i, and γ_i^0 being free constants.

Substituting Eq. (20) into Eq. (9) and taking the parameters as $k_1=1$, $k_2=2$, $k_3=3$, $k_4=4$, $p_1=-2$, $p_2=1$, $p_3=-1$, $p_4=1$, $\gamma_1^0=\gamma_2^0=\gamma_3^0=\gamma_4^0=0$, for the coefficient function $a(t)$ taken as $\sinh t$, we obtain the profiles at different times, as shown in Fig. 5. It reflects the change in multiple solitons.

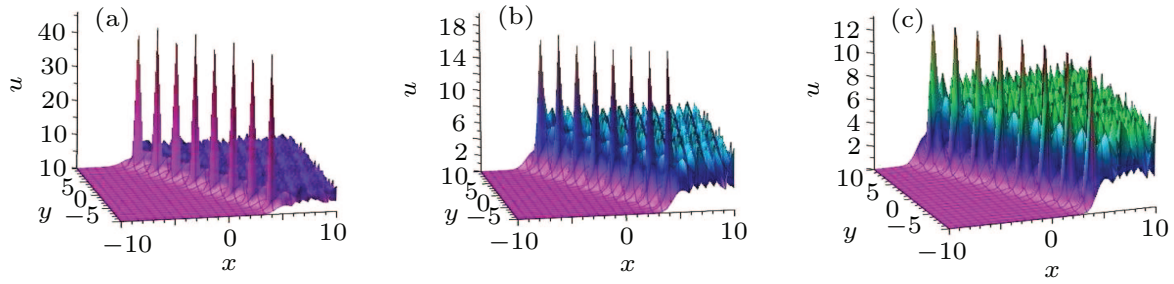


Fig. 3. The interaction of breather waves, kink solutions and excitation solutions in the (x,y) plane at (a) $a(t)=\cosh t$, (b) $a(t)=\cos t$, (c) $a(t)=t$.

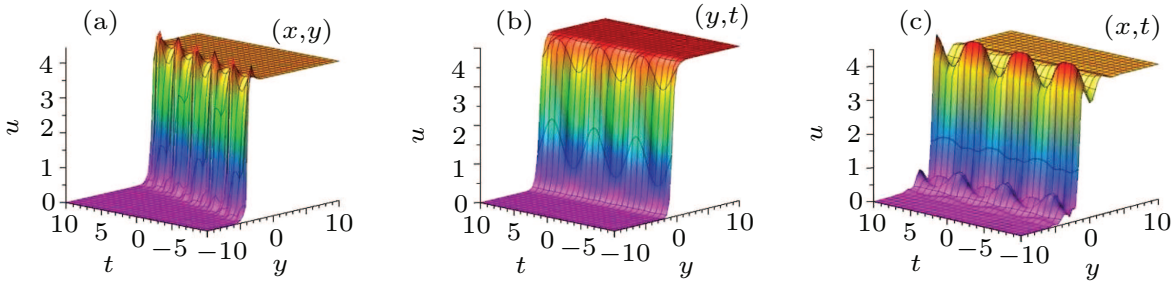


Fig. 4. The interaction of breather waves and kink solutions in different planes.

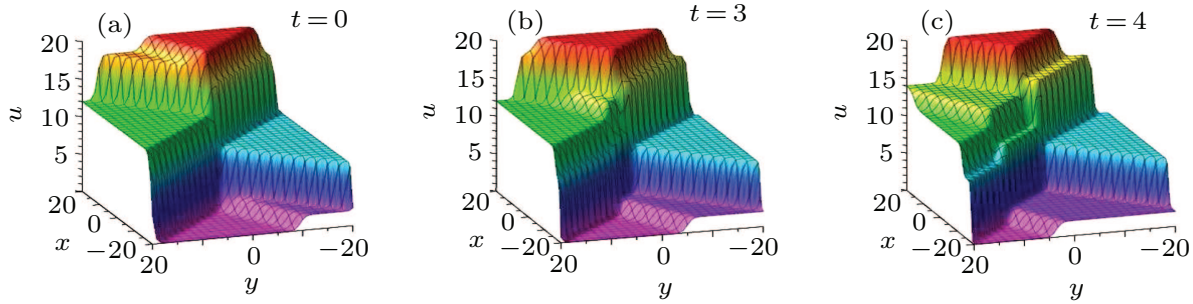


Fig. 5. The multi-soliton solutions in (x,y) plane.

4. The lump solutions of the vcHSI equation

In this section, the long wave limit method is used to solve the solutions. The solutions obtained using this method is generally the lump solutions. The basic idea to this method is taking the limit of the soliton solutions.[32,33] The long wave limit method is now widely used to find multi-lump solutions. Next, we describe the process of solving solutions.

When $N=2$, we have known the expressions for f, $w_i(t)$, and b_{12} in the above section. Now, we take the parameters as $k_1=\varepsilon\cdot l_1$, $k_2=\varepsilon\cdot l_2$, $\gamma_1^0=I\cdot\pi$, $\gamma_2^0=-I\cdot\pi$, and substitute them into the expressions for b_{12} and $w_i(t)$. After expanding them and using Taylor's formula we can obtain

$$f=(\theta_1\theta_2+\theta_0)\cdot l_1 l_2, \tag{22}$$

where $\theta_i=x+p_iy-p_i\int a(t)\mathrm{d}t$ with $i=1,2$, $\theta_0=\frac{-6(p_1+p_2)}{(p_1-p_2)^2}$. Substituting Eq. (22) into Eq. (9), the solution is obtained.

We choose proper parameters as $p_1=-4-I$, $p_2=4-I$, take the coefficient function $a(t)$ as $\ln(t)$, and put them into the solution we obtained. Figure 6 shows the corresponding results. There are two solitons with different amplitudes that collide with each other. Figures 6(a)–6(c) show the results taken at different times. In addition, the amplitude of the two waves decreases with time.

When $N=3$, this method is used to determine the expression for f. After substituting the parameters as the same as $N=2$, we can reach

$$\begin{aligned} f &= (\theta_1\theta_2+a_{12})l_1l_2 \\ &\quad +(a_{13}a_{23}+a_{13}\theta_2+a_{23}\theta_1+\theta_1\theta_2+a_{12})l_1l_2\mathrm{e}^{\gamma_3}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} a_{ij}&=\frac{-6p_i-6p_j}{p_i^2-2p_ip_j+p_j^2}, \quad 1\le i<j\le 2,\\ \theta_i&=x+p_iy-p_i\int a(t)\mathrm{d}t, \quad i=1,2,\\ a_{i3}&=\frac{-6k_3\left(k_3^2+p_i+p_3\right)}{k_3^2\left(k_3^2+p_i+2p_3\right)+\left(p_i-p_3\right)^2}, \quad i=1,2,\\ \gamma_3&=k_3\left(x+p_3y-\int\frac{a(t)}{k_3^2+p_3}\mathrm{d}t\right)+\gamma_3^0, \end{aligned} \tag{24}$$

with l_1 and l_2 being free constants.

We can draw the results as shown in Fig. 7, after substituting Eq. (23) into Eq. (9). The parameters are taken as $t=1$, $p_1=1-2I$, $p_2=1+2I$, $p_3=0$, $\gamma_3^0=0$, and the coefficient function $a(t)$ is chosen as $t^2/5$. When $k_3=0$, the solution is second-order lump solution. It is clear that the amplitude of one lump wave remains unchanged, and the amplitude of the other lump wave increases with k_3. Finally, when $k_3=1.3$, the solution is the interaction of kink solution and lump solutions in different planes.

Now, we choose a set of parameters as $k_3=-2$, $p_1=-I$, $p_2=3$, $p_3=0$, $\gamma_3^0=0$, and take the coefficient function as $\sin t$. The corresponding results are presented in Fig. 8. Figures 8(a)–8(c) show the emergence of periodic waves at different x. Figures 8(d)–8(f) show the annihilation of periodic waves at different x. When $x=0$, the amplitude of the periodic wave reaches its maximum.

When $N=4$, we use this method to solve f. After some calculations, we can determine the expression for f as follows:

$$\begin{aligned} f &= \left(\theta_1\theta_2\theta_3\theta_4+a_{12}\theta_3\theta_4+a_{13}\theta_2\theta_4\right.\\ &\quad +a_{14}\theta_2\theta_3+a_{23}\theta_1\theta_4+a_{24}\theta_1\theta_3+a_{34}\theta_1\theta_2\\ &\quad \left.+a_{12}a_{34}+a_{13}a_{24}+a_{14}a_{23}\right)\cdot l_1l_2l_3l_4\varepsilon^4, \end{aligned} \tag{25}$$

where $\theta_i=x+p_iy-p_i\int a(t)\mathrm{d}t$ with $i=1,2,3,4$, $a_{ij}=\frac{-6p_i-6p_j}{p_i^2-2p_ip_j+p_j^2}$ with $1\le i<j\le 4$; $l_i\,(i=1,2,3,4)$ are free constants. For appropriate parameters, we show the results in Figs. 9 and 10.

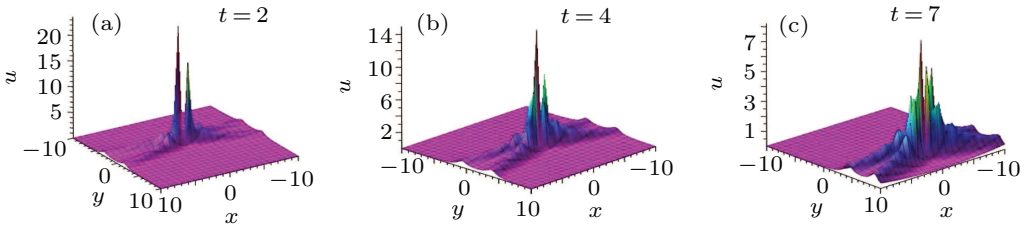


Fig. 6. The dynamical behavior of second-order lump solutions with t in (x,y) plane.

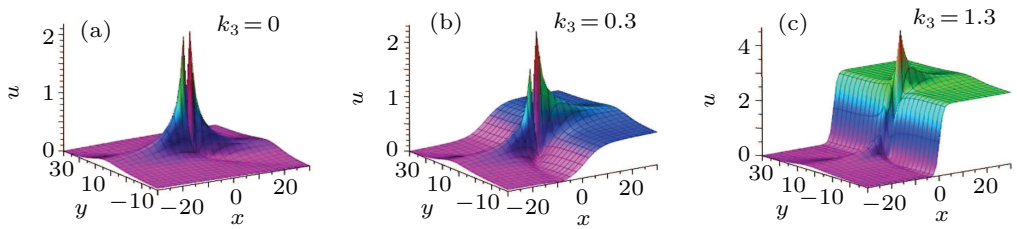


Fig. 7. The dynamical behavior of second-order lump solutions with k_3 in (x,y) plane.

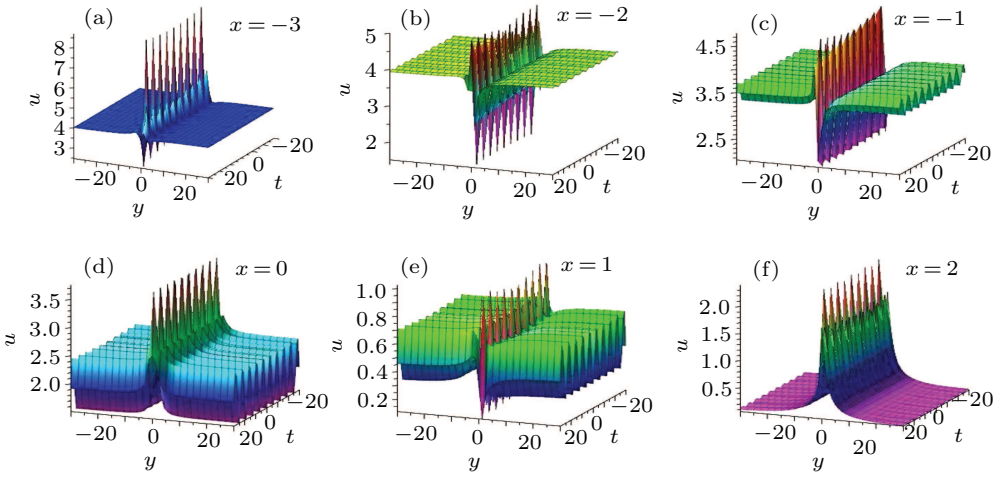


Fig. 8. The process of periodic wave emergence and annihilation in (y,t) plane.

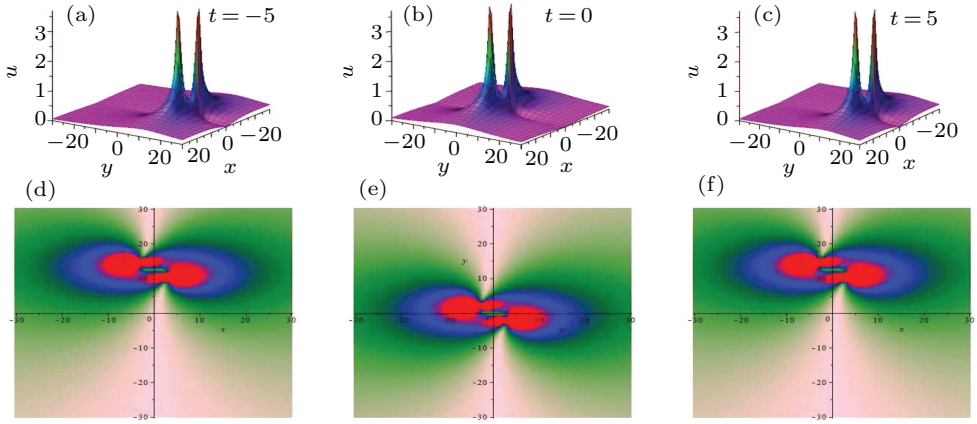


Fig. 9. The changes of position of second-order lump solution and their density plots.

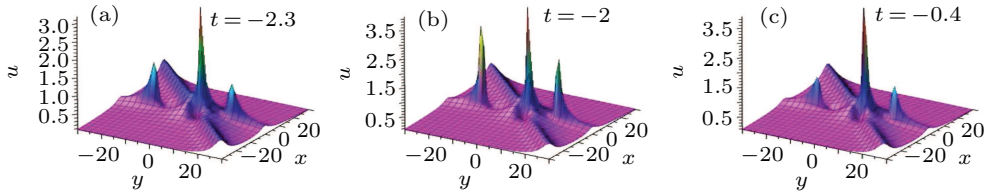


Fig. 10. The interaction of one linear wave and multi-soliton solution in (x,y) plane.

In Fig. 9, we choose $p_1 = I$, $p_2 = -I$, $p_3 = 2I$, $p_4 = -2I$, and take the coefficient function $a(t)$ as t. We can find that the position of the second-order lump solutions has changed with time. The position of the solution at $t = 5$ is same as the position of the solution at $t = -5$. In addition, the amplitude of the second-order lump keeps the same. In Fig. 10, we choose $p_1 = 1$, $p_2 = -I$, $p_3 = 5I$, $p_4 = -5I$, and take the coefficient function $a(t)$ as $\cot t$. These solutions are interaction of one linear wave and multi-soliton solutions. The amplitude of multi-soliton changes with t.

5. Conclusions

In summary, we have investigated the Hirota–Satsuma–Ito equation with variable coefficient. We use the WTC method to prove the equation is integrable. We solve its bilinear form with the help of the Hirota bilinear method. The N-soliton solutions are constructed on the basis of bilinear form. We specifically study the form of the solutions for $N = 2,3,4$. In addition, we introduce the long wave limit method. The form of the solutions for $N = 2,3,4$ is studied using this method. By selecting appropriate parameters we obtain multiple types of solutions including breather waves solutions, lump solutions, periodic solutions, and the interaction of breathers and periodic solutions, etc. Their corresponding figures are presented to describe the evolution of solutions.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 11505090), Research Award Foundation for Outstanding Young Scientists of Shandong

Province (Grant No. BS2015SF009), the Doctoral Foundation of Liaocheng University (Grant No. 318051413), Liaocheng University Level Science and Technology Research Fund (Grant No. 318012018), and Discipline with Strong Characteristics of Liaocheng University–Intelligent Science and Technology (Grant No. 319462208).

References

[1] Xin X P, Liu Y T, Xia Y R and Liu H Z 2021 *Appl. Math. Lett.* **119** 107209
[2] Yang D Y, Tian B, Qu Q X, Chen R Z and Chen S S 2021 *Chaos Solitons Fractals* **150** 110487
[3] Chen S J, Ma W X and Lü X 2020 *Commun. Nonlinear. Sci.* **83** 105135
[4] Gao L N, Zi Y Y, Yin Y H, Ma W X and Lü X 2017 *Nonlinear Dyn.* **89** 2233
[5] Liu P, Cheng J, Ren B and Yang J R 2020 *Chin. Phys. B* **29** 020201
[6] Kumar S, Kumar D and Kumar A 2021 *Chaos Solitons Fractals* **142** 110507
[7] Sadat R, Saleh R, Kassem M and Mousa M M 2020 *Chaos Solitons Fractals* **140** 110134
[8] Ren R C and Zhang S L 2020 *Comput. Appl. Math.* **39** 250
[9] Wu J W, He J T and Lin J 2022 *Eur. Phys. J. Plus* **137** 1
[10] Ma W X 2021 *Proc. Am. Math. Soc.* **149** 251
[11] Ma W X and Fan E G 2011 *Comput. Math. Appl.* **61** 950
[12] Ma W X and Zhou Y 2018 *J. Differ. Equ.* **264** 2633
[13] Yin Y H, Chen S J and Lü X 2020 *Chin. Phys. B* **29** 120502
[14] Wang H F and Zhang Y F 2020 *Chin. Phys. B* **29** 040501
[15] Shang Y D 2008 *Chaos Solitons Fractals* **36** 762
[16] Lü X and Chen S J 2021 *Nonlinear Dyn.* **103** 947
[17] Fang T and Wang Y H 2018 *Comput. Math. Appl.* **76** 1476
[18] Wang H, Tian S F, Zhang T T, Chen Y and Fang Y 2019 *Comput. Appl. Math.* **38** 164
[19] Pu J C, Li J and Chen Y 2021 *Chin. Phys. B* **30** 060202
[20] Zhao X H 2021 *Appl. Math. Lett.* **121** 107383
[21] Ma Y L, Wazwaz A M and Li B Q 2021 *Nonlinear Dyn.* **104** 1581
[22] Zhang Z, Yang X Y, Li W T and Li B 2019 *Chin. Phys. B* **28** 110201
[23] Liu J G and Zhu W H 2021 *Nonlinear Dyn.* **103** 1841
[24] Chen Q Q, Qi Z Q, Chen J C and Li B 2021 *Results Phys.* **27** 104480
[25] Yuan P S, Qi J X, Li Z L and An L H 2021 *Chin. Phys. B* **30** 040503
[26] Ma W X 2019 *Front. Math. China* **14** 619
[27] Liu Y Q, Wen X Y and Wang D S 2019 *Comput. Math. Appl.* **77** 947
[28] Kumar S, Nisar K S and Kumar A 2021 *Results Phys.* **28** 104621
[29] Zhou T Y, Tian B, Chen Y Q and Shen Y 2022 *Nonlinear Dyn.* **108** 2417
[30] Wu J W, Cai Y J and Lin J 2022 *Chin. Phys. B* **31** 030201
[31] Wazwaz A M 2020 *Waves Random Complex Media* **30** 776
[32] Huang L L, Yue Y F and Chen Y 2018 *Comput. Math. Appl.* **76** 831
[33] Zhang S J and Bao T 2021 *Nonlinear Dyn.* **106** 2465