Bounds on positive operator-valued measure based coherence of superposition

Meng-Li Guo(郭梦丽)[1], Jin-Min Liang(梁津敏)[1], Bo Li(李波)[2], Shao-Ming Fei(费少明)[1,3,†], and Zhi-Xi Wang(王志玺)[1,‡]

[1] *School of Mathematical Sciences, Capital Normal University, Beijing* 100048, *China*
[2] *School of Computer and Computing Science, Hangzhou City University, Hangzhou* 310015, *China*
[3] *Max–Planck-Institute for Mathematics in the Sciences, Leipzig* 04103, *Germany*

(Received 20 November 2022; revised manuscript received 31 January 2023; accepted manuscript online 8 February 2023)

Quantum coherence is a fundamental feature of quantum physics and plays a significant role in quantum information processing. By generalizing the resource theory of coherence from von Neumann measurements to positive operator-valued measures (POVMs), POVM-based coherence measures have been proposed with respect to the relative entropy of coherence, the l_1 norm of coherence, the robustness of coherence and the Tsallis relative entropy of coherence. We derive analytically the lower and upper bounds on these POVM-based coherence of an arbitrary given superposed pure state in terms of the POVM-based coherence of the states in superposition. Our results can be used to estimate range of quantum coherence of superposed states. Detailed examples are presented to verify our analytical bounds.

Keywords: coherence, POVM-based coherence, relative entropy, l_1 norm, Tsallis relative entropy

PACS: 03.65.Aa, 03.67.–a, 03.67.Mn **DOI:** 10.1088/1674-1056/acb9f1

1. Introduction

Originated from the superposition principle of quantum mechanics, quantum coherence and entanglement are important quantum resources in quantum information processing and quantum computation.[1–3] However, in general the entanglement of a superposed pure state cannot be simply expressed as a linear summation of the entanglement of the individual states in the superposition. Linden *et al*. first investigated the relations between the entanglement of a superposed pure state and the entanglement of the states in the superposition[4] and Gour *et al*. provided the upper and lower bounds on superposed entanglement based on the Von Neumann entropy of the reduced states[5] and the entanglement measure concurrence.[6,7]

Baumgratz, Cramer, and Plenio[8] first proposed the resource theory of coherence, established a rigorous coherence quantification framework (BCP framework), and identified computable measures of coherence. Based on the BCP framework, coherence has witnessed theoretically and experimentally fruitful progress.[9–15] For a quantum system associated with a d-dimensional Hilbert space H, the BCP framework takes into account the coherence defined by an orthogonal basis $\{|j\rangle\}_{j=1}^{d}$, which we call standard coherence. A standard orthogonal basis $\{|j\rangle\}_{j=1}^{d}$ corresponds to a rank-1 projective measurement $\{|j\rangle\langle j|\}_{j=1}^{d}$. Recently, Bischof, Kampermann and Bruß[16,17] generalized the conventional framework of coherence to the case of general positive operator-valued measures (POVMs), by replacing the projective measurements with POVMs. Recently, several POVM-based coherence measures have been proposed,[16–18] such as relative entropy of POVM-based coherence $C_{\mathrm{r}}(\rho,E)$, l_1-norm of POVM-based coherence $C_{l_1}(\rho,E)$, robustness of POVM-based coherence $C_{\mathrm{rob}}(\rho,E)$, and POVM-based coherent based on Tsallis relative entropy $C_{T,\lambda}(\rho,E)$. Moreover, maximum relative entropy of coherence for quantum channels has been introduced in Ref. [20].

Similar to the case of quantum entanglement, the superposition of two incoherent states may give rise to a maximally coherent state, for instance, $|\Omega_1\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ in the computational basis $|0\rangle, |1\rangle$. While the superposition of two maximally coherent states may produce an incoherent state, for example, $|\Omega_2\rangle = (1/\sqrt{2})(|+\rangle + |-\rangle)$, where $|\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$. In Refs. [21–24] the coherence for superposition states of two mutual orthogonal states has been discussed.

In this work, we investigate the POVM-based coherence of superposition states. More specifically, given a superposition state

$$|\Omega\rangle = \sum_{k=1}^{n} \alpha_k |\phi_k\rangle, \tag{1}$$

we explore the relationship between the POVM-based coherence of the superposition state $|\Omega\rangle$ and the POVM-based coherence of the states $|\phi_k\rangle$ in the superposition. We focus on

†Corresponding author. E-mail: feishm@cnu.edu.cn
‡Corresponding author. E-mail: wangzhx@cnu.edu.cn

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four POVM-based coherence measures $C_{\rm r}(\rho,E)$, $C_{l_1}(\rho,E)$, $C_{\rm rob}(\rho,E)$, and $C_{T,\lambda}(\rho,E)$ and present the upper and lower bounds of the inequalities satisfied by these coherence measures. We illustrate our results by detailed single and two-qubit examples.

2. POVM-based coherence of superposition states

In order to quantitatively describe the coherence resources contained in a given quantum state, a variety of coherence measures has been introduced from different perspectives. Before giving our main results, we first recall the definitions of $C_{\rm r}(\rho,E)$, $C_{l_1}(\rho,E)$, $C_{\rm rob}(\rho,E)$, and $C_{T,\lambda}(\rho,E)$ with respect to a positive operator-valued measure given by d measurement operators $E=\{E_j\geq 0\}_{j=1}^{d}$, $\sum_j E_j=I$ with I the identity operator.

2.1. The POVM-based coherence measures

The relative entropy coherence measure is a commonly used and well-defined coherence measure,[8] which is tightly related to the optimal distillation rate in standard coherent distillation[11] and the minimum amount of noise for complete decoherence.[9,25] Given a POVM E, the relative entropy POVM-based coherence is defined by[16,17]

$$C_{\rm r}(\rho,E)=\sum_{j=1}^{d}S\big(\sqrt{E_j}\rho\sqrt{E_j}\big)-S(\rho),$$

where $S(X)=-{\rm Tr}(X\log X)$ is the von Neumann entropy for positive semi-definite matrix X. For a pure state $|\phi\rangle$, $C_{\rm r}(\rho,E)$ can be expressed as

$$C_{\rm r}(\phi,E)=\sum_{j=1}^{d}S\big(\sqrt{E_j}|\phi\rangle\langle\phi|\sqrt{E_j}\big). \tag{2}$$

The l_1-norm POVM-based coherence $C_{l_1}(\rho,E)$ of a density matrix ρ is defined by[17,18]

$$C_{l_1}(\rho,E)=\sum_{i\neq j=1}^{d}||\sqrt{E_i}\rho\sqrt{E_j}||_{\rm tr},$$

with $||X||_{\rm tr}={\rm Tr}\sqrt{X^{\dagger}X}$ the trace norm of matrix X. For a pure state $|\phi\rangle$, the l_1-norm POVM-based coherence is given by

$$C_{l_1}(\phi,E)=\sum_{i\neq j=1}^{d}{\rm Tr}|\sqrt{E_i}|\phi\rangle\langle\phi|\sqrt{E_j}|. \tag{3}$$

The robust coherence is closely related to coherence sightings and can be used to quantify the advantages of quantum states in phase discrimination tasks. The robust POVM-based coherence $C_{\rm rob}(\rho,E)$ can be expressed as[17]

$$C_{\rm rob}(\rho,E)=\min_{\tau\in S'}\big\{s\geq 0: s\tau_{i,j}=-\sqrt{E_i}\rho\sqrt{E_j},\forall i\neq j\big\}.$$

where

$$\tau=\sum_{i,j}\tau_{i,j}\otimes|i\rangle\langle j|$$

as

$$\tau=\sum_{i,j}|\phi_i\rangle\langle\phi_j|\otimes|i\rangle\langle j|+\sum_{i}(I-|\phi_i\rangle\langle\phi_i|)\otimes|i\rangle\langle i|.$$

Note that for pure states $|\phi\rangle$, the robust POVM-based coherence $C_{\rm rob}(\rho,E)$ and the l_1-norm POVM-based coherence satisfy the following relation:

$$C_{\rm rob}(\phi,E)=C_{l_1}(\phi,E).$$

The Tsallis relative entropy of POVM-based coherence $C_{T,\lambda}(\rho,E)$ introduced in Ref. [18] is given by

$$C_{T,\lambda}(\rho,E)=\frac{1}{\lambda-1}\Big\{\sum_{j=1}^{d}{\rm Tr}\big[(\sqrt{E_j}\rho^{\lambda}\sqrt{E_j})^{1/\lambda}\big]-1\Big\} \tag{4}$$

for $\lambda\in(0,1)\cup(1,2]$.

For further use, we extend the above definition of the POVM-based coherence of a pure state to the coherence of two pure states, by analog to the case of quantum entanglement.[7] Let $|\phi\rangle$ and $|\psi\rangle$ be two pure states. We define $C_{l_1}(\phi,\psi)$ and $C_{T,\lambda}(\phi,\psi)$ as the POVM coherence of $|\phi\rangle$ and $|\psi\rangle$ by

$$\begin{aligned}C_{l_1}(\phi,\psi,E)&=\sum_{i\neq j=1}^{d}{\rm Tr}\big|\sqrt{E_i}|\phi\rangle\langle\psi|\sqrt{E_j}\big|, \qquad (5)\\ C_{T,\lambda}(\phi,\psi,E)&=\frac{1}{\lambda-1}\\ &\quad\times\Big\{\sum_{j=1}^{d}{\rm Tr}\big[\big(\sqrt{E_j}(|\phi\rangle\langle\psi|)^{\lambda}\sqrt{E_j}\big)^{1/\lambda}\big]-1\Big\}.\end{aligned}$$

Notice that according to the above definitions, we have $C(\phi,\phi,E)=C(\phi,E)$.

2.2. The upper and lower bounds of POVM-based coherence measures

In the following, we use the relative entropy POVM-based coherence to investigate the relationship between the coherence of two arbitrary pure states and the POVM-based coherence of their superposed state.

Before we present our two main results (Theorem 1), we briefly review the properties of the von-Neumann entropy $S(\rho)$ in Refs. [4,19] and provide a slight improvement of it. The authors in Ref. [4] have used two properties of $S(\rho)$:

$$\begin{aligned}&|\alpha_1|^2S(\rho)+|\alpha_2|^2S(\sigma)\leq S(|\alpha_1|^2\rho+|\alpha_2|^2\sigma),\\ &S(|\alpha_1|^2\rho+|\alpha_2|^2\sigma)\leq|\alpha_1|^2S(\rho)+|\alpha_2|^2S(\sigma)+h_2(|\alpha_1|^2).\end{aligned} \tag{6}$$

where

$$h_2(x)=-x\log x-(1-x)\log(1-x),\quad |\alpha_1|^2+|\alpha_2|^2=1.$$

Now, consider $|\Omega\rangle = \alpha_1|\phi\rangle + \alpha_2|\psi\rangle$, we obtain

$$S(\sqrt{E_j}|\Omega\rangle\langle\Omega|\sqrt{E_j}) \le |\alpha_1|^2 S(\sqrt{E_j}|\phi\rangle\langle\phi|\sqrt{E_j}) + |\alpha_2|^2 S(\sqrt{E_j}|\psi\rangle\langle\psi|\sqrt{E_j}) + h_2(|\alpha_1|^2).$$

Using Eq. (2), we get

$$C_{\mathrm{r}}(\Omega,E) \le |\alpha_1|^2 C_{\mathrm{r}}(\phi,E) + |\alpha_2|^2 C_{\mathrm{r}}(\psi,E) + h_2(|\alpha_1|^2). \quad (7)$$

Based on the above results, we have the following conclusion.

Theorem 1 Given two different pure states $|\phi\rangle$ and $|\psi\rangle$, denote $|\Omega\rangle = \alpha|\phi\rangle + \beta|\psi\rangle$, where α and β are complex numbers. The relative entropy POVM-based coherence $C_{\mathrm{r}}(\Omega',E)$ of the state $|\Omega'\rangle = |\Omega\rangle/||\Omega||$, with $||\Omega|| = |\langle\Omega|\Omega\rangle|$ the normalization constant, has an upper bound

$$p_1[\mu C_{\mathrm{r}}(\phi,E) + (1-\mu)C_{\mathrm{r}}(\psi,E) + h_2(\mu)], \quad (8)$$

and a lower bound $L = \max\{L_1, L_2, 0\}$ with

$$L_1 = p_2 C_{\mathrm{r}}(\phi,E) - \frac{1-\nu}{\nu} C_{\mathrm{r}}(\psi,E) - \frac{1}{\nu} h_2(\nu),$$
$$L_2 = p_3 C_{\mathrm{r}}(\psi,E) - \frac{1-\xi}{\xi} C_{\mathrm{r}}(\phi,E) - \frac{1}{\xi} h_2(\xi),$$

where $h_2(x) = -x\log x - (1-x)\log(1-x)$,

$$p_1 = \frac{(1-\mu)|\alpha|^2 + \mu|\beta|^2}{\mu(1-\mu)||\Omega||^2}, \quad \mu = \frac{|\alpha|^2}{\cos^2\theta},$$
$$p_2 = \frac{(1-\nu)|\alpha|^2}{(1-\nu)||\Omega||^2 + \nu|\beta|^2},$$
$$\nu = \frac{\sin^2\theta||\Omega||^2}{\sin^2\theta||\Omega||^2 + |\beta|^2\cos^2\theta},$$
$$p_3 = \frac{(1-\xi)|\beta|^2}{(1-\xi)||\Omega||^2 + \xi|\alpha|^2},$$
$$\xi = \frac{\sin^2\theta||\Omega||^2}{\sin^2\theta||\Omega||^2 + |\alpha|^2\cos^2\theta}$$

for $0 < \mu, \nu, \xi < 1$ and

$$\frac{|\alpha|^2}{\cos^2\theta} + \frac{|\beta|^2}{\sin^2\theta} = 1.$$

Proof We prove the theorem by introducing an auxiliary system. First we prove the upper bound (8). Consider the following bipartite state in systems A and B:

$$|\chi\rangle^{AB} = \sqrt{\mu}|\phi\rangle^A|0\rangle^B + \sqrt{1-\mu}|\psi\rangle^A|1\rangle^B.$$

According to Eq. (7), the relative entropy POVM-based coherence of $|\chi\rangle$ can be expressed as

$$C_{\mathrm{r}}(\chi,E) \le \mu C_{\mathrm{r}}(\phi,E) + (1-\mu)C_{\mathrm{r}}(\psi,E) + h_2(\mu).$$

Measuring the ancillary system B with Kraus operators

$$K_1 = |0\rangle(\cos\theta\, \mathrm{e}^{\mathrm{i}\omega_1}\langle 0| + \sin\theta\, \mathrm{e}^{\mathrm{i}\omega_2}\langle 1|),$$
$$K_2 = |1\rangle(-\sin\theta\, \mathrm{e}^{-\mathrm{i}\omega_2}\langle 0| + \cos\theta\, \mathrm{e}^{-\mathrm{i}\omega_1}\langle 1|),$$

one gets the collapsed state

$$|\chi_1\rangle|0\rangle = \left(\sqrt{\frac{\mu}{p}}\cos\theta\, \mathrm{e}^{\mathrm{i}\omega_1}|\phi\rangle + \sqrt{\frac{1-\mu}{p}}\sin\theta\, \mathrm{e}^{\mathrm{i}\omega_2}|\psi\rangle\right)|0\rangle,$$

with probability

$$p = \|\sqrt{\mu}\cos\theta\, \mathrm{e}^{\mathrm{i}\omega_1}|\phi\rangle + \sqrt{1-\mu}\sin\theta\, \mathrm{e}^{\mathrm{i}\omega_2}|\psi\rangle\|^2, \quad (9)$$

and the collapsed state

$$|\chi_2\rangle|1\rangle = \left(\sqrt{\frac{1-\mu}{1-p}}\cos\theta\, \mathrm{e}^{-\mathrm{i}\omega_1}|\psi\rangle - \sqrt{\frac{\mu}{1-p}}\sin\theta\, \mathrm{e}^{\mathrm{i}\omega_2}|\phi\rangle\right)|1\rangle,$$

with probability $1-p$. As the measurement can be seen as an incoherent operation, we obtain the following inequality:

$$pC_{\mathrm{r}}(\chi_1,E) + (1-p)C_{\mathrm{r}}(\chi_2,E) \le \mu C_{\mathrm{r}}(\phi,E) + (1-\mu)C_{\mathrm{r}}(\psi,E) + h_2(\mu).$$

Since $C_{\mathrm{r}}(\chi_2,E) \ge 0$, we have

$$C_{\mathrm{r}}(\chi_1,E) \le \frac{1}{p}[\mu C_{\mathrm{r}}(\phi,E) + (1-\mu)C_{\mathrm{r}}(\psi,E) + h_2(\mu)]. \quad (10)$$

Now we set $|\chi_1\rangle = |\Omega'\rangle$. We get

$$\frac{\alpha}{||\Omega||} = \sqrt{\frac{\mu}{p}}\cos\theta\, \mathrm{e}^{\mathrm{i}\omega_1}, \quad \frac{\beta}{||\Omega||} = \sqrt{\frac{1-\mu}{p}}\sin\theta\, \mathrm{e}^{\mathrm{i}\omega_2}.$$

It is straightforward to derive that

$$p = \frac{\|\Omega\|^2\mu(1-\mu)}{(1-\mu)|\alpha|^2 + \mu|\beta|^2}. \quad (11)$$

Substituting Eq. (11) into Eq. (10), we have the following inequality:

$$C_{\mathrm{r}}(\Omega',E) \le p_1[\mu C_{\mathrm{r}}(\phi,E) + (1-\mu)C_{\mathrm{r}}(\psi,E) + h_2(\mu)],$$

with $0 < \mu < 1$,

$$\mu = \frac{|\alpha|^2}{\cos^2\theta}, \quad p_1 = \frac{(1-\mu)|\alpha|^2 + \mu|\beta|^2}{\mu(1-\mu)\|\Omega\|^2}.$$

To prove the lower bound, we can consider the following bipartite states:

$$|\chi'\rangle^{AB} = \sqrt{\nu}|\Omega'\rangle^A|0\rangle^B + \sqrt{1-\nu}|\psi\rangle^A|1\rangle^B,$$
$$|\chi''\rangle^{AB} = \sqrt{\xi}|\Omega'\rangle^A|0\rangle^B + \sqrt{1-\xi}|\phi\rangle^A|1\rangle^B.$$

Similar to the proof of the upper bound (8), one obtains easily the lower bound L.

Concerning the l_1-norm of the POVM-based coherence measure, we have the following conclusion.

Theorem 2 Given $|\Omega\rangle = \sum_{k=1}^{n} \alpha_k |\phi_k\rangle$ with α_k complex numbers, the POVM-based coherence of the superposed sate $|\Omega'\rangle = |\Omega\rangle/||\Omega||$ has an upper bound

$$\|\Omega\|^{-2}\Big(\sum_{k=1}^{n}|\alpha_k|^2 C_{l_1}(\phi_k,E)+M\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|\Big), \tag{12}$$

and a lower bound

$$\|\Omega\|^{-2}\Big(\sum_{k=1}^{n}|\alpha_k|^2 C_{l_1}(\phi_k,E)-M\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|\Big), \tag{13}$$

where

$$M=(d-1)\sum_{i=1}^{d}\Big\|\sqrt{E_i}|\phi_k\rangle\langle\phi_{k'}|\Big\|_{\mathrm{tr}}.$$

Proof From the definition of the l_1-norm POVM-based coherence (3), we have

$$\begin{aligned}
&\|\Omega\|^2 C_{l_1}(\Omega',E)\\
=&\sum_{i\neq j=1}^{d}\mathrm{Tr}\Big|\sqrt{E_i}\sum_{kk'=1}^{n}\alpha_k\alpha_{k'}|\phi_k\rangle\langle\phi_{k'}|\sqrt{E_j}\Big|\\
\leq&\sum_{k=1}^{n}|\alpha_k|^2\sum_{i\neq j=1}^{d}\mathrm{Tr}\Big|\sqrt{E_i}|\phi_k\rangle\langle\phi_k|\sqrt{E_j}\Big|\\
&+\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|\sum_{i\neq j=1}^{d}\mathrm{Tr}\Big|\sqrt{E_i}|\phi_k\rangle\langle\phi_{k'}|\sqrt{E_j}\Big|\\
=&\sum_{k=1}^{n}|\alpha_k|^2 C_{l_1}(\phi_k,E)+\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|C_{l_1}(\phi_k,\phi_{k'},E)\\
\leq&\sum_{k=1}^{n}|\alpha_k|^2 C_{l_1}(\phi_k,E)+\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|M,
\end{aligned}$$

where the first inequality is due to $|a+b|\leq|a|+|b|$, the second inequality is based on the Theorem 2 in Ref. [26] with $M=(d-1)\sum_{i=1}^{d}\Big\|\sqrt{E_i}|\phi_k\rangle\langle\phi_{k'}|\Big\|_{\mathrm{tr}}$.

Next, by using the inverse triangle inequality, $|a+b|\geq||a|-|b||\geq|a|-|b|$, similar to the proof of the upper bound we can easily get Eq. (13).

The robustness of coherence[17,27] plays an important role in the characterization of quantum states in phase discrimination. For pure states, the robustness of POVM-based coherence is equivalent to the l_1-norm of POVM-based coherence. Therefore, Theorem 2 also gives rise to the bounds for the robustness of POVM-based coherence.

Concerning the POVM-based coherence of Tsallis relative entropy $C_{T,\lambda}(\rho,E)$, we present the following lemma.

Lemma 1 For $\lambda\in(0,1)\cup(1,2]$, we have

$$C_{T,\lambda}(\rho,E)\leq-\ln_\lambda\left(\frac{1}{[d\sum_{j=1}^{d}\mathrm{Tr}(\sqrt{E_j}\rho^2\sqrt{E_j})]^{1/\lambda}}\right),$$

where $\ln_\lambda x=(x^{1-\lambda}-1)/(1-\lambda)$.

Proof Due to Eq. (4), for $\lambda\in(0,1)\cup(1,2]$ according to Refs. [28,29], we obtain

$$C_{T,\lambda}(\rho,E)\leq\frac{[d^{\lambda-1}\sum_{j=1}^{d}\mathrm{Tr}(\sqrt{E_j}\rho^\lambda\sqrt{E_j})]^{1/\lambda}-1}{\lambda-1}. \tag{14}$$

Applying the Jensen's inequality, we have

$$\begin{aligned}
&\frac{\left[\sum_{j=1}^{d}\mathrm{Tr}\left(\sqrt{E_j}\rho^\lambda\sqrt{E_j}\right)\right]^{1/\lambda}}{\lambda-1} &(15)\\
\leq&\frac{\left[\left(\sum_{j=1}^{d}\mathrm{Tr}\left(\sqrt{E_j}\rho^2\sqrt{E_j}\right)\right)^{1/\lambda}\right]^{\lambda-1}}{\lambda-1}. &(16)
\end{aligned}$$

Combining Eqs. (15) and (14), we complete the proof.

Based on the above results, we have the following conclusion.

Theorem 3 Given a state $|\Omega\rangle=\sum_{k=1}^{n}\alpha_k|\phi_k\rangle$ with α_k the complex numbers. The POVM-based coherence of the normalized superposed state $|\Omega'\rangle=\frac{|\Omega\rangle}{\|\Omega\|}$ has an upper bound

$$\begin{aligned}
&\|\Omega\|^{-2}\left(\sum_{k=1}^{n}|\alpha_k|^2 C_{T,\lambda}(\phi_k,E)-\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|\ln_\lambda X\right)\\
&+\frac{1}{\|\Omega\|^2(\lambda-1)}\left(\sum_{k=1}^{n}|\alpha_k|^2+\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|-\|\Omega\|^2\right)
\end{aligned}$$

for $\lambda\in(0,1)\cup(1,2]$, where

$$X=1/\left(d\sum_{j=1}^{d}\mathrm{Tr}[\sqrt{E_j}(|\phi_k\rangle\langle\phi_{k'}|)^2\sqrt{E_j}]\right)^{1/\lambda}, \tag{17}$$

and a lower bound $L=\max\{L_1,0\}$ for $\lambda\in(1,2]$, where

$$\|\Omega\|^2 L_1=\sum_{k=1}^{n}|\alpha_k|^2 C_{T,\lambda}(\phi_k,E)+N\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|+\frac{1}{(\lambda-1)}\left(\sum_{k=1}^{n}|\alpha_k|^2+\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|-\|\Omega\|^2\right),$$

$$N=\frac{1}{\lambda-1}\left\{\sum_{j=1}^{d}\mathrm{Tr}\left[\left(\sqrt{E_j}\right)^{1/\lambda}|\phi_k\rangle\langle\phi_{k'}|\left(\sqrt{E_j}\right)^{1/\lambda}\right]-1\right\}.$$

Proof Let $\{|i\rangle_{i=1}^{m}\}$ be a set basic vectors such that $|\phi_k\rangle=\sum_{i=1}^{m}b_i^k|i\rangle$. We have $|\Omega\rangle=\sum_{k=1}^{n}\sum_{i}^{m}\alpha_k b_i^k|i\rangle$ and

$$\|\Omega\|^2 C_{T,\lambda}(\Omega',E)=\frac{1}{\lambda-1}\left\{\sum_{j=1}^{d}\mathrm{Tr}\left[\sqrt{E_j}\Big[\sum_{kk'=1}^{n}\sum_{ii'=1}^{m}\alpha_k b_i^k\alpha_{k'}b_{i'}^{k'}|i\rangle\langle i'|\Big]^\lambda\sqrt{E_j}\right]^{1/\lambda}-\|\Omega\|^2\right\}$$

$$
\begin{aligned}
&= \frac{1}{\lambda-1}\left\{\sum_{k=1}^{n}|\alpha_k|^2\sum_{j=1}^{d}\mathrm{Tr}\left[\sqrt{E_j}\Big[\sum_{ii'=1}^{m}|b_i^k b_{i'}^k|\cdot|i\rangle\langle i'|\Big]^{\lambda}\sqrt{E_j}\right]^{1/\lambda}-\|\Omega\|^2\right.\\
&\quad\left.+\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_k'|\sum_{j=1}^{d}\mathrm{Tr}\left[\sqrt{E_j}\Big[\sum_{ii'=1}^{m}|b_i^k b_{i'}^{k'}|\cdot|i\rangle\langle i'|\Big]^{\lambda}\sqrt{E_j}\right]^{1/\lambda}\right\}\\
&= \sum_{k=1}^{n}|\alpha_k|^2 C_{T,\lambda}(\phi_k,E)+\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|C_{T,\lambda}(\phi_k,\phi_{k'},E)\\
&\quad+\frac{1}{\lambda-1}\Big(\sum_{k=1}^{n}|\alpha_k|^2+\sum_{k\neq k'=1}^{n}|\alpha_k\alpha_{k'}|-\|\Omega\|^2\Big).
\end{aligned}
$$

Based on Lemma 1, we obtain the upper bound in theorem.

Next, by using the Araki–Lieb–Thirring Inequality,[30] $\mathrm{Tr}(A^rB^rA^r)^q \geq \mathrm{Tr}(ABA)^{rq}$ for $r\geq 1$ and $q\geq 0$, we have

$$
C_{T,\lambda}(\phi_k,\phi_{k'},E)=\frac{1}{\lambda-1}\left\{\sum_{j=1}^{d}\mathrm{Tr}\Big[(E_j^{1/2\lambda})^{\lambda}(|\phi_k\rangle\langle\phi_k'|)^{\lambda}(E_j^{1/2\lambda})^{\lambda}\Big]^{1/\lambda}-1\right\}\geq\frac{1}{\lambda-1}\left\{\sum_{j=1}^{d}\mathrm{Tr}\Big[\big(\sqrt{E_j}\big)^{1/\lambda}|\phi_k\rangle\langle\phi_k'|\big(\sqrt{E_j}\big)^{1/\lambda}\Big]-1\right\},
$$

where $\lambda\in(1,2]$. Then, we obtain the lower bound in theorem.

3. Numerical results

In this section, we demonstrate numerically our results by investigating different superposition states. We consider a single-qubit case for the l_1 coherence measure and two-qubit case for the relative entropy and the Tsallis relative entropy coherence measures. In both cases, we plot the relations between exact value, upper bound, and lower bound. Crucially, a POVM of n-qubit system is expressed as 4^n linear independent positive operators $\{E_i=A_iA_i^{\dagger}\}_{i=0}^{4^n-1}$, which can be obtained according to the Naimark theorem.[31,32] Specially, each measurement operator A_i embedded in a larger unitary can be found via a projective measurement in the standard computational basis of a larger Hilbert space.

Consider a POVM with respect to a single-qubit system $\{E_i=A_iA_i^{\dagger}\}_{i=0}^{3}$, which can be realized by performing a two-qubit unitary $U=\sum_{ijkl=0}^{3}u_{ij}^{kl}|ij\rangle\langle kl|$ on the qubit system and an ancillary system. Implementing U on an initial state $\rho_a\otimes|0\rangle\langle 0|_b$, one measures the two qubits system in the standard computational basis $\{|m\rangle\}_{m=0}^{3}$. Denote (q_a,q_b) the measurement outcome with $q_a,q_b\in\{0,1\}$. The corresponding probability of each outcome is given by

$$
\begin{aligned}
P_{(q_a,q_b)}&=\langle q_aq_b|U(\rho_a\otimes|0\rangle\langle 0|_b)U^{\dagger}|q_aq_b\rangle\\
&=\sum_{kl=0}^{3}u_{q_aq_b}^{kl}\langle kl|(\rho_a\otimes|0\rangle\langle 0|_b)\sum_{kl=0}^{3}(u_{q_aq_b}^{kl})^*|kl\rangle\\
&=\sum_{k=0}^{3}u_{q_aq_b}^{k0}\langle k|\rho_a\sum_{k=0}^{3}(u_{q_aq_b}^{k0})^*|k\rangle\\
&=\mathrm{Tr}[|E_{q_aq_b}\rangle\langle E_{q_aq_b}|\rho_a],
\end{aligned}
\tag{18}
$$

where $|E_{(q_aq_b)}\rangle=\sum_{k=0}^{3}u_{q_aq_b}^{k0}|k\rangle$. Hence, the measurement operators of each POVM is given by $E_i=E_{(q_aq_b)}=|E_{(q_aq_b)}\rangle\langle E_{(q_aq_b)}|$ for $i=0,\ldots,3$, where i has a binary representation $i=2q_a+q_b$. In particular, let us we consider the following two-qubit unitary U,

$$
U(\boldsymbol{\theta})=\mathrm{CNOT}\cdot\Big[R_y(\theta_1)\otimes R_y(\theta_2)\Big],
\tag{19}
$$

parameterized by $\boldsymbol{\theta}=(\theta_1,\theta_2)$, where $R_y(\theta_j)=\mathrm{e}^{-\mathrm{i}\theta_j\sigma_y/2}$, $\mathrm{i}^2=-1$, and CNOT is the CNOT-gate. Setting $\boldsymbol{\theta}=(0.301723,0.011681)$ we have a POVM given by

$$
\begin{aligned}
&|E_0\rangle=(u_{00}^{00},u_{00}^{10})^{\dagger},\ \ |E_1\rangle=(u_{01}^{00},u_{01}^{10})^{\dagger},\\
&|E_2\rangle=(u_{10}^{00},u_{10}^{10})^{\dagger},\ \ |E_3\rangle=(u_{11}^{00},u_{11}^{10})^{\dagger}.
\end{aligned}
$$

Given two 1-qubit states $|\phi\rangle=\mathrm{e}^{-\mathrm{i}\theta_j0.432/2}|0\rangle$ and $|\psi\rangle=\mathrm{e}^{-\mathrm{i}\theta_j0.618/2}|0\rangle$, we calculate the exact value, upper bound, and lower bound of l_1 coherence of the superposition state

$$
|\Omega\rangle=\alpha|\phi\rangle+\beta|\psi\rangle,\ \ |\Omega'\rangle=\frac{|\Omega\rangle}{\|\Omega\|},
\tag{20}
$$

in which the coefficients α,β are random scalar drawn from the uniform distribution in the interval $(0,1)$. We run our procedure 10 times by randomly choosing the coefficients α,β. Figure 1(b) shows the upper and lower bounds of $C_{l_1}(|\Omega'\rangle\langle\Omega'|,E)$.

For two-qubit case, consider POVMs with 16 measurement operators, $\{E_i=A_iA_i^{\dagger}\}_{i=0}^{15}$, which can be realized by performing a unitary

$$
V=\sum_{\boldsymbol{ij}=0}^{3}v_{i_1i_2i_3i_4}^{j_1j_2j_3j_4}|\boldsymbol{i}\rangle\langle\boldsymbol{j}|
$$

on a four-qubit system, where the index $\boldsymbol{i}$ has binary representation $\boldsymbol{i}=i_1i_2i_3i_4$. Similar to the single qubit case, it is easily verified that each POVM has a form $E_i=|E_{(q_aq_bq_cq_d)}\rangle\langle E_{(q_aq_bq_cq_d)}|$ where $(q_aq_bq_cq_d)$ denote the measurement outcomes, $i=2^3q_a+2^2q_b+2q_c+q_d$, and

$$
|E_{(q_aq_bq_cq_d)}\rangle=\sum_{j_1j_2=0}^{1}v_{q_aq_bq_cq_d}^{j_1j_200}|j_1j_2\rangle.
\tag{21}
$$

Let us consider

$$V(\gamma)=\prod_{i,j=0}^{2}\mathrm{CNOT}_{i,i+1}\cdot\left[R_y(\gamma_1)\otimes R_y(\gamma_2)\otimes R_y(\gamma_3)\otimes R_y(\gamma_4)\right] \tag{22}$$

given by the parameter vector

$$\begin{aligned}\gamma&=(\gamma_1,\gamma_2,\gamma_3,\gamma_4)\\&=(0.30173,0.01168,0.53991,0.09537,0.14651).\end{aligned}$$

Given two 2-qubit states

$$|\phi_1\rangle=U(\theta_1)|00\rangle,\ |\psi_1\rangle=U(\theta_2)|00\rangle,\ |\psi_2\rangle=U(\theta_3)|00\rangle,$$

where parameters

$$\begin{aligned}&\theta_1=(0.4827,0.3760),\quad \theta_2=(0.9394,0.2212),\\&\theta_3=(0.1557,0.8190).\end{aligned}$$

We calculate the exact values, upper bounds, and lower bounds of relative entropy coherence of the superposition state

$$\begin{aligned}|\Omega_1\rangle&=\alpha|\phi_1\rangle+\beta|\psi_1\rangle,\\|\Omega_1'\rangle&=\frac{|\Omega_1\rangle}{\||\Omega_1\rangle\|},\end{aligned} \tag{23}$$

and Tsallis relative entropy coherence of the superposition state

$$\begin{aligned}|\Omega_2\rangle&=\alpha|\phi_1\rangle+\beta|\psi_2\rangle,\\|\Omega_2'\rangle&=\frac{|\Omega_2\rangle}{\||\Omega_2\rangle\|},\end{aligned} \tag{24}$$

in which the coefficients α,β are random scalar drawn from the uniform distribution in the interval $(0,1)$. We run our procedure 10 times by randomly choosing the coefficients α,β. Figure 1(a) shows the bounds of the relative entropy of POVM-based coherence. Only upper bound of the Tsallis relative entropy is shown in Fig. 2(a) when $\lambda=0.3$. Figure 2(b) plots the upper and lower bounds of the Tsallis relative entropy of POVM coherence when the parameter $\lambda=1.5$. Clearly, the numerical results are good agree with our theoretical analysis.

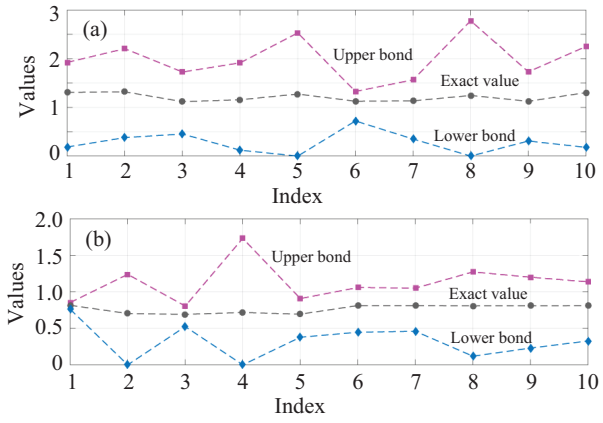


Fig. 1. (a) The relative entropy of POVM-based coherence. (b) The l_1 norm of POVM-based coherence.

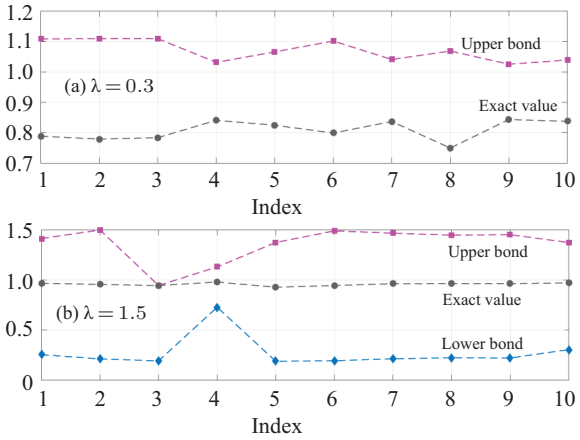


Fig. 2. The Tsallis relative entropy of POVM coherence.

4. Conclusion

The superposition principle of quantum states is fundamental in quantum mechanics. Generally, quantum objects exhibit both the wave and particle nature. In particular, when the POVM E is taken to be the von Neumann projective measurement, the coherence $C_{l_1}(\rho,E)$ stands for the wave property of quantum objects. In this case the coherence $C_{l_1}(\rho,E)$ and the distinguishability of which-path information (a measure of particle-property) in multi-path interference satisfy a trade-off relation,[33] which has been verified in a generalized multi-path delayed-choice experiment on a large-scale quantum nanophotonic chip.[34]

Based on the POVM-based coherence measures $C_{\mathrm{r}}(\rho,E)$, $C_{l_1}(\rho,E)$, $C_{\mathrm{rob}}(\rho,E)$, and $C_{T,\lambda}(\rho,E)$, we haver studied coherence of superposition for arbitrary dimensional pure states. We have explored the relationship between the POVM-based coherence of a superposed state and the POVM-based coherence of the states in the superposition, and derived analytically the lower and upper bounds. To illustrate our results, we have presented detailed examples related to the POVM-based l_1 norm coherence for the one-qubit case and the POVM-based relative entropy coherence for the two-qubit case. Our results can be used to estimate the range of quantum coherence of superposed states, and may highlight investigations on superposition of coherence for quantum channels.[20]

Acknowledgments

Project supported by the National Natural Science Foundation of China (Grant Nos. 12075159, 12171044, and 12175147), the Natural Science Foundation of Beijing (Grant No. Z190005), the Academician Innovation Platform of Hainan Province, Shenzhen Institute for Quantum Science and Engineering, and Southern University of Science and Technology (Grant No. SIQSE202001).

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