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Soliton propagation for a coupled Schrödinger equation describing Rossby waves

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We study a coupled Schrödinger equation which is started from the Boussinesq equation of atmospheric gravity waves by using multiscale analysis and reduced perturbation method. For the coupled Schrödinger equation, we obtain the Manakov model of all-focusing, all-defocusing and mixed types by setting parameters value and apply the Hirota bilinear approach to provide the two-soliton and three-soliton solutions. Especially, we find that the all-defocusing type Manakov model admits bright-bright soliton solutions. Furthermore, we find that the all-defocusing type Manakov model admits bright-bright-bright soliton solutions. Therefrom, we go over how the free parameters affect the Manakov model's all-focusing type's two-soliton and three-soliton solutions' collision locations, propagation directions, and wave amplitudes. These findings are useful for setting a simulation scene in Rossby waves research. The answers we have found are helpful for studying physical properties of the equation in Rossby waves.

Keywords: Hirota bilinear method, Schrödinger equation, soliton solution, Rossby waves

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1. Introduction

In the last few years, Rossby waves have been the subject of our study.[1–3] The result based on the single-layer quasi geostrophic vortex equation[4] is an approximate result. For the actual ocean and atmosphere, because of the inhomogeneity of fluid density and the multi-physicality of mixed fluid, the two-layer fluid pattern is real existence. In addition, the two-layer fluid can better fit the actual atmosphere and ocean, because they play an important role in the ocean atmospheric fluctuation, especially in the deep-sea internal wave theory. The two-layer fluid model attracts much attention from scholars. For example, Yang *et al.* derived a set of time-space fractional coupled generalized Zakharov–Kuznetsov equations to explore the Rossby waves' property in two-layer fluids and obtained group solutions for them,[5] derived (2+1)-dimensional coupled Boussinesq equations describing Rossby waves in a two-layer cylindrical fluid and gave their exact solutions,[6] then derived coupled Kadomtsev–Petviashvili equations and gave their soliton solutions.[7] We notice that a Boussinesq equation of Rossby waves can be written as

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p'}{\partial x},\\
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=g\frac{T'}{T_0}-\frac{1}{\rho_0}\frac{\partial p'}{\partial z},\\
&\frac{\partial T'}{\partial t}+u\frac{\partial T'}{\partial x}+w\frac{\partial T'}{\partial z}+(\gamma_{\rm d}-\gamma)w=0,\\
&\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,
\end{aligned}\tag{1}$$

where x is the coordinate in latitudinal direction, z represents the coordinate in vertical direction; t donates the time; u and w represent the velocities in x and z directions, respectively; T' is the temperature perturbation. The perturbation pressure, density, zonal mean background temperature, gravitational constant, dry adiabatic lapse rate, and moist lapse rate are all donated by p', ρ_0, T_0, g, $\gamma_{\rm d}$, and γ, respectively.[8]

A coupled nonlinear Schrödinger (CNLS) equation[9] can be derived from the Boussinesq equation and can also be obtained from the literature,[10] that is,

$$\begin{aligned}
&{\rm i}\frac{\partial A_1}{\partial T}+\alpha_1\frac{\partial^2 A_1}{\partial X^2}+A_1\left(\sigma_1|A_1|^2+\Gamma_1|A_2|^2\right)=0,\\
&{\rm i}\frac{\partial A_2}{\partial T}+{\rm i}C\frac{\partial A_2}{\partial X}+\alpha_2\frac{\partial^2 A_2}{\partial X^2}\\
&+A_2\left(\Gamma_2|A_1|^2+\sigma_2|A_2|^2\right)=0,
\end{aligned}\tag{2}$$

the coefficients α_1 and α_2 are dispersion coefficients, σ_1 and Γ_2 are Landau constants, Γ_1 and σ_2 are interaction coefficients.[9] Equation (2) can be obtained through optical waves[11] or large-scale Rossby waves.[9]

Research of the CNLS equation in aspects of physics and mathematics is of continuous interest because of the science applying in many fields such as nonlinear optics,[12–15] optical communication,[16–19] Bose–Einstein condensates,[20,21] and plasma physics.[22,23]

The system (2) has received great attention by many researchers. For example, the symmetries and symmetry reductions have been obtained, and some exact solutions have been given by using the classical Lie group approach,[8] while

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there is little analysis of the solutions for the equation. In Ref. [24], two CNLS equations which are also a special case of the system (2) were discussed about the integrability properties through Painlevé analysis. At the same time, their accompanying Bäcklund transformation and the Hirota bilinear forms were build. In Ref. [25], Lou *et al.* also investigated the system's Painlevé integrability.

Introducing a transformation

$$A_2 = B\mathrm{e}^{\mathrm{i}\frac{c}{2\alpha_2}\left(-X+\frac{c}{2}T\right)}, \tag{3}$$

where $B = B(X,T)$.

Using the above transformation, let $u(X,T) = A_1(X,T)$, $v(X,T) = B(X,T)$, the system (2) is transformed into

$$\begin{aligned}
&\mathrm{i}\frac{\partial u}{\partial T} + \alpha_1\frac{\partial^2 u}{\partial X^2} + \left(\sigma_1|u|^2 + \Gamma_1|v|^2\right)u = 0,\\
&\mathrm{i}\frac{\partial v}{\partial T} + \alpha_2\frac{\partial^2 v}{\partial X^2} + \left(\Gamma_2|u|^2 + \sigma_2|v|^2\right)v = 0.
\end{aligned} \tag{4}$$

By using the inverse scattering transform approach, equation (4) is typically not integrable, with the exception of high symmetry instances $\alpha_1 = \alpha_2 = \pm 1, \sigma_1 = \Gamma_2 = \sigma_2 = \Gamma_1 = \pm 1$, a periodic travelling wave solution in terms of the Weierstrass function has been obtained.[26] Then, Makhankov found that equation (4) is completely integrable and gave the corresponding Lax pair.[27,28]

When $\alpha_1 = \alpha_2 = 1$, equation (4) reads the Manakov model.[29] When $\sigma_1 = \Gamma_2 = \sigma_2 = \Gamma_1 = 1$, equation (4) reads the focusing Manakov model. When $\sigma_1 = \Gamma_2 = \sigma_2 = \Gamma_1 = -1$, equation (4) is of the defocusing Manakov model. When $\sigma_1 = \Gamma_2 = 1, \sigma_2 = \Gamma_1 = -1$ or $\sigma_1 = \Gamma_2 = -1, \sigma_2 = \Gamma_1 = 1$, equation (4) is of the mixed Manakov model.[30,31] The focusing Manakov model has bright-bright solitons.[29] Bright-dark and dark-dark soliton solutions are supported by the defocusing Manakov model.[32] However, in the mixed case, the Manakov model admits bright-bright soliton, bright-dark soliton, and dark-dark soliton solutions.[17,18,30] Feng considered a general N-soliton solution to a vector Schrödinger equation, following that, extended to the multi-component case, of all-focusing, all-defocusing and mixed types by the KP hierarchy reduction approach.[31] Researchers have used different methods to solve the equations in the literature.[17,18,30,31] However, as far as we are aware, they have not provide the three-solution solutions and do not work on the two-soliton and three-soliton solutions' propagations. There are other methods for solving nonlinear equations,[33] but recently the Hirota bilinear method seems to be used more frequently.[34–42] Some scholars have recently focused on soliton solutions,[37–42] especially multiple solitons and their interactions.[37,41,42] As a result, we seek to provide two-soliton and three-soliton solutions and to study their propagations for Eq. (4).

The Hirota bilinear method was firstly used by Hirota for searching exact solutions to the Korteweg–de Vries equation,[43] then the method was widely used to solve various equations. Setting $\alpha_1 = \alpha_2$, $\sigma_1 = \Gamma_2$, $\sigma_2 = \Gamma_1$, $X = x$, $T = t$, then equation (4) changes to

$$\begin{aligned}
&\mathrm{i}\frac{\partial u}{\partial t} + \alpha_1\frac{\partial^2 u}{\partial x^2} + \left(\sigma_1|u|^2 + \sigma_2|v|^2\right)u = 0,\\
&\mathrm{i}\frac{\partial v}{\partial t} + \alpha_1\frac{\partial^2 v}{\partial x^2} + \left(\sigma_1|u|^2 + \sigma_2|v|^2\right)v = 0.
\end{aligned} \tag{5}$$

We try to find the soliton solutions for Eq. (5) as the Manakov model of all-focusing, all-defocusing and mixed types by using the Hirota bilinear approach. The plan of this paper is as follows: In Section 2, bilinear forms of the CNLS equation (5) is obtained by using variable transformation. In Section 3, exact two-soliton solutions and three-soliton solutions are presented with the Hirota bilinear method.[43] Finally, analysis for the solitons and conclusions are given in Sections 4 and 5.

2. Bilinear forms for the coupled Schrödinger equation

To study the solitons for the CNLS equation, we will write the bilinear forms of the system. Firstly, the D-operators,[43] also called Hirota derivatives, are defined as

$$D_x^n(u,v) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^n u(x)v(y)\Big|_{y=x}, \tag{6}$$

$$\begin{aligned}
D_t^m D_x^n(u,v) \equiv{}& \frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n}u(t+s,x+y)\\
&\times v(t-s,x-y)\Big|_{s=0,y=0}.
\end{aligned} \tag{7}$$

For u and v, we employ a dependent variable transformation

$$u = \frac{g}{f}, \quad v = \frac{h}{f}, \tag{8}$$

where $g = g(x,t)$, $h = h(x,t)$, $f = f(x,t)$; g and h are the differentiable complex functions; and f is a differentiable real function. Likewise, the bilinear forms of Eq. (5) are

$$\begin{aligned}
&\left(\mathrm{i}D_t + \alpha_1 D_x^2\right)g\cdot f = 0,\\
&\left(\mathrm{i}D_t + \alpha_1 D_x^2\right)h\cdot f = 0,\\
&\alpha_1 D_x^2 f\cdot f = \sigma_1 gg^* + \sigma_2 hh^*,
\end{aligned} \tag{9}$$

where superscript * represents the conjugation of the complex function. Expand g, h, and f as the following power series in a small parameter:

$$\begin{aligned}
g &= \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \cdots,\\
h &= \varepsilon h_1 + \varepsilon^2 h_2 + \varepsilon^3 h_3 + \cdots,\\
f &= 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots,
\end{aligned} \tag{10}$$

where $g_i = g_i(x,t)$, $h_i = h_i(x,t)$, $f_i = f_i(x,t)$, $i = 1,2,\ldots$. Substituting the expansions and Eq. (10) into Eq. (9) and collecting the terms in the same power in ε, we have

$$\begin{aligned}
\varepsilon^1:\ & \left(\mathrm{i}D_t + \alpha_1 D_x^2\right) g_1 \cdot 1 = 0,\\
& \left(\mathrm{i}D_t + \alpha_1 D_x^2\right) h_1 \cdot 1 = 0,\\
& \alpha_1 D_x^2 \left(1 \cdot f_1 + f_1 \cdot 1\right) = 0, \qquad (11)\\
\varepsilon^2:\ & \left(\mathrm{i}D_t + \alpha_1 D_x^2\right) \left(g_1 \cdot f_1 + g_2 \cdot 1\right) = 0,\\
& \left(\mathrm{i}D_t + \alpha_1 D_x^2\right) \left(h_1 \cdot f_1 + h_2 \cdot 1\right) = 0,\\
& \alpha_1 D_x^2 \left(1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1\right) = \sigma_1 g_1 g_1^* + \sigma_2 h_1 h_1^*, \qquad (12)\\
& \vdots
\end{aligned}$$

The coefficients of the same powers in ε are equivalent to some linear differential equations for g_i, h_i or f_i, by appropriately selecting g_1, h_1, which satisfy the equations. We can calculate f_1, and then figure out the other functions g_i, h_i and f_i, $i = 2,3,\ldots$ in turn. The complexity of calculating the coefficients of power series increases as the power increases, especially when the coefficients of terms have a power greater than 3. To make calculations easier, we can use symbolic calculation software. Therefrom, we give the two-soliton and three-soliton solutions for the coupled system.

3. Soliton solutions of the coupled Schrödinger equation

3.1. Two-soliton solution

Assume

$$\begin{aligned}
g_1(x,t) &= m_1 \mathrm{e}^{\theta_1} + m_2 \mathrm{e}^{\theta_2},\\
h_1(x,t) &= n_1 \mathrm{e}^{\theta_1} + n_2 \mathrm{e}^{\theta_2}, \qquad (13)
\end{aligned}$$

where $\theta_i = k_i x + w_i t$, $k_i = k_{i1} + \mathrm{i}k_{i2}$, $w_i = w_{i1} + \mathrm{i}w_{i2}, k_{ij}, w_{ij} \in R$ $(i = 1,2, j = 1,2)$. Substituting $g_1(x,t)$ and $h_1(x,t)$ into Eq. (10), we can obtain $f_1(x,t) = 0$. At the same time, we can find the relationship between the parameters k_i and w_i, m_i and n_i, $w_i = \mathrm{i}\alpha_1 k_i^2$ $(i = 1,2)$, $m_1/m_2 = n_1/n_2$. Substituting $g_1(x,t)$, $h_1(x,t)$ and $f_1(x,t)$ into Eq. (12), we can obtain $g_2(x,t)$, $h_2(x,t)$ and $f_2(x,t)$. Likewise, we can get $g_n(x,t)$, $h_n(x,t)$ and $f_n(x,t)$ $(n = 3,4,\ldots)$ as follows:

$$\begin{aligned}
& g_2(x,t) = h_2(x,t) = 0,\\
& f_2(x,t) = a_{11}\mathrm{e}^{\theta_1^* + \theta_1} + a_{21}\mathrm{e}^{\theta_2 + \theta_1^*} + a_{12}\mathrm{e}^{\theta_1 + \theta_2^*} + a_{22}\mathrm{e}^{\theta_2^* + \theta_2},\\
& g_3(x,t) = b_1 \mathrm{e}^{\theta_1 + \theta_2 + \theta_1^*} + b_2 \mathrm{e}^{\theta_1 + \theta_2 + \theta_2^*},\\
& h_3(x,t) = c_1 \mathrm{e}^{\theta_1^* + \theta_1 + \theta_2} + c_2 \mathrm{e}^{\theta_2^* + \theta_1 + \theta_2}, \quad f_3(x,t) = 0,\\
& g_4(x,t) = h_4(x,t) = 0, \quad f_4(x,t) = d\,\mathrm{e}^{\theta_1 + \theta_2 + \theta_1^* + \theta_2^*},\\
& g_n(x,t) = h_n(x,t) = f_n(x,t) = 0, \quad (n = 5,6,\ldots), \qquad (14)
\end{aligned}$$

with

$$\begin{aligned}
a_{11} &= \frac{\sigma_1 m_1^2 + \sigma_2 n_1^2}{2\alpha_1 \left(k_1 + k_1^*\right)^2}, \quad a_{12} = \frac{\sigma_1 m_1 m_2 + \sigma_2 n_1 n_2}{2\alpha_1 \left(k_1 + k_2^*\right)^2}, \quad a_{21} = \frac{\sigma_1 m_1 m_2 + \sigma_2 n_1 n_2}{2\alpha_1 \left(k_2 + k_1^*\right)^2}, \quad a_{22} = \frac{\sigma_1 m_2^2 + \sigma_2 n_2^2}{2\alpha_1 \left(k_2 + k_2^*\right)^2},\\
b_1 &= \frac{(k_1 - k_2)\left[m_1 \left(\sigma_1 m_1 m_2 + \sigma_2 n_1 n_2\right)\left(k_1 + k_1^*\right) - m_2 \left(\sigma_1 m_1^2 + \sigma_2 n_1^2\right)\left(k_2 + k_1^*\right)\right]}{2\alpha_1 \left(k_1 + k_1^*\right)^2 \left(k_2 + k_1^*\right)^2},\\
b_2 &= \frac{(k_1 - k_2)\left[m_1 \left(\sigma_1 m_2^2 + \sigma_2 n_2^2\right)\left(k_1 + k_2^*\right) - m_2 \left(\sigma_1 m_1 m_2 + \sigma_2 n_1 n_2\right)\left(k_2 + k_2^*\right)\right]}{2\alpha_1 \left(k_1 + k_2^*\right)^2 \left(k_2 + k_2^*\right)^2},\\
c_1 &= \frac{(k_1 - k_2)\left[n_1 \left(\sigma_1 m_1 m_2 + \sigma_2 n_1 n_2\right)\left(k_1 + k_1^*\right) - n_2 \left(\sigma_1 m_1^2 + \sigma_2 n_1^2\right)\left(k_2 + k_1^*\right)\right]}{2\alpha_1 \left(k_1 + k_1^*\right)^2 \left(k_2 + k_1^*\right)^2},\\
c_2 &= \frac{(k_1 - k_2)\left[n_1 \left(\sigma_1 m_2^2 + \sigma_2 n_2^2\right)\left(k_1 + k_2^*\right) - n_2 \left(\sigma_1 m_1 m_2 + \sigma_2 n_1 n_2\right)\left(k_2 + k_2^*\right)\right]}{2\alpha_1 \left(k_1 + k_2^*\right)^2 \left(k_2 + k_2^*\right)^2},\\
d &= \frac{n_1^2 \left(\sigma_1 m_2^2 + \sigma_2 n_2^2\right)^2 \left(k_{11}^2 + k_{12}^2 + k_{21}^2 + k_{22}^2 - 2k_{11}k_{21} - 2k_{12}k_{22}\right)^2}{64\alpha_1^2 n_2^2 k_{11}^2 k_{21}^2 \left(k_1 + k_2^*\right)^2 \left(k_2 + k_1^*\right)^2}.
\end{aligned}$$

We obtain the two-soliton solutions to Eq. (5) as follows:

$$u = \frac{g}{f} = \frac{\varepsilon g_1 + \varepsilon^3 g_3}{1 + \varepsilon^2 f_2 + \varepsilon^4 f_4}, \quad v = \frac{h}{f} = \frac{\varepsilon h_1 + \varepsilon^3 h_3}{1 + \varepsilon^2 f_2 + \varepsilon^4 f_4}. \qquad (15)$$

Without loss of generality, setting $\varepsilon = 1$, the two-soliton solution can be expressed as

$$\begin{aligned}
u &= \frac{g}{f} = \frac{m_1 \mathrm{e}^{\theta_1} + m_2 \mathrm{e}^{\theta_2} + b_1 \mathrm{e}^{\theta_1 + \theta_2 + \theta_1^*} + b_2 \mathrm{e}^{\theta_1 + \theta_2 + \theta_2^*}}{1 + a_{11}\mathrm{e}^{\theta_1 + \theta_1^*} + a_{12}\mathrm{e}^{\theta_1 + \theta_2^*} + a_{21}\mathrm{e}^{\theta_2 + \theta_1^*} + a_{22}\mathrm{e}^{\theta_2 + \theta_2^*} + d\,\mathrm{e}^{\theta_1 + \theta_2 + \theta_1^* + \theta_2^*}},\\
v &= \frac{h}{f} = \frac{n_1 \mathrm{e}^{\theta_1} + n_2 \mathrm{e}^{\theta_2} + c_1 \mathrm{e}^{\theta_1 + \theta_2 + \theta_1^*} + c_2 \mathrm{e}^{\theta_1 + \theta_2 + \theta_2^*}}{1 + a_{11}\mathrm{e}^{\theta_1 + \theta_1^*} + a_{12}\mathrm{e}^{\theta_1 + \theta_2^*} + a_{21}\mathrm{e}^{\theta_2 + \theta_1^*} + a_{22}\mathrm{e}^{\theta_2 + \theta_2^*} + d\,\mathrm{e}^{\theta_1 + \theta_2 + \theta_1^* + \theta_2^*}}. \qquad (16)
\end{aligned}$$

3.2. Three-soliton solution

Assume

$$\begin{aligned} g_1(x,t) &= m_1 \mathrm{e}^{\theta_1} + m_2 \mathrm{e}^{\theta_2} + m_3 \mathrm{e}^{\theta_3}, \\ h_1(x,t) &= n_1 \mathrm{e}^{\theta_1} + n_2 \mathrm{e}^{\theta_2} + n_3 \mathrm{e}^{\theta_3}, \end{aligned} \tag{17}$$

where $\theta_i = k_i x + w_i t$, $k_i = k_{i1} + \mathrm{i}k_{i2}$, $w_i = w_{i1} + \mathrm{i}w_{i2}$, k_{ij}, $w_{ij} \in R$ $(i = 1,2,3;\ j = 1,2)$.

Using the method in the above section, we can find the relationship between the parameters k_i and w_i, m_i and n_i, $w_i = \mathrm{i}\alpha_1 k_i^2$ $(i = 1,2,3)$, $\frac{m_1}{n_1} = \frac{m_2}{n_2} = \frac{m_3}{n_3} = \varsigma$, and

$$\begin{aligned} g_2(x,t) &= h_2(x,t) = 0, \\ f_2(x,t) &= a_{11}\mathrm{e}^{\theta_1^*+\theta_1} + a_{12}\mathrm{e}^{\theta_2^*+\theta_1} + a_{13}\mathrm{e}^{\theta_3^*+\theta_1} \\ &\quad + a_{21}\mathrm{e}^{\theta_1^*+\theta_2} + a_{22}\mathrm{e}^{\theta_2^*+\theta_2} + a_{23}\mathrm{e}^{\theta_3^*+\theta_2} \\ &\quad + a_{31}\mathrm{e}^{\theta_1^*+\theta_3} + a_{32}\mathrm{e}^{\theta_2^*+\theta_3} + a_{33}\mathrm{e}^{\theta_3^*+\theta_3}, \\ g_3(x,t) &= b_1\mathrm{e}^{\theta_1^*+\theta_1+\theta_2} + b_2\mathrm{e}^{\theta_2^*+\theta_1+\theta_2} + b_3\mathrm{e}^{\theta_3^*+\theta_1+\theta_2} \\ &\quad + b_4\mathrm{e}^{\theta_1^*+\theta_1+\theta_3} + b_5\mathrm{e}^{\theta_2^*+\theta_1+\theta_3} + b_6\mathrm{e}^{\theta_3^*+\theta_1+\theta_3} \\ &\quad + b_7\mathrm{e}^{\theta_1^*+\theta_2+\theta_3} + b_8\mathrm{e}^{\theta_2^*+\theta_2+\theta_3} + b_9\mathrm{e}^{\theta_3^*+\theta_2+\theta_3}, \\ h_3(x,t) &= c_1\mathrm{e}^{\theta_1^*+\theta_1+\theta_2} + c_2\mathrm{e}^{\theta_2^*+\theta_1+\theta_2} + c_3\mathrm{e}^{\theta_3^*+\theta_1+\theta_2} \\ &\quad + c_4\mathrm{e}^{\theta_1^*+\theta_1+\theta_3} + c_5\mathrm{e}^{\theta_2^*+\theta_1+\theta_3} + c_6\mathrm{e}^{\theta_3^*+\theta_1+\theta_3} \\ &\quad + c_7\mathrm{e}^{\theta_1^*+\theta_2+\theta_3+} + c_8\mathrm{e}^{\theta_2^*+\theta_2+\theta_3} + c_9\mathrm{e}^{\theta_3^*+\theta_2+\theta_3}, \\ f_3(x,t) &= h_4(x,t) = g_4(x,t) = 0, \\ f_4(x,t) &= d_1\mathrm{e}^{\theta_1^*+\theta_2^*+\theta_1+\theta_2} + d_2\mathrm{e}^{\theta_1^*+\theta_3^*+\theta_1+\theta_2} + d_3\mathrm{e}^{\theta_1^*+\theta_2^*+\theta_1+\theta_3} \\ &\quad + d_4\mathrm{e}^{\theta_1^*+\theta_3^*+\theta_1+\theta_3} + d_5\mathrm{e}^{\theta_2^*+\theta_3^*+\theta_1+\theta_2} \\ &\quad + d_6\mathrm{e}^{\theta_2^*+\theta_3^*+\theta_1+\theta_3} + d_7\mathrm{e}^{\theta_1^*+\theta_2^*+\theta_2+\theta_3} \\ &\quad + d_8\mathrm{e}^{\theta_1^*+\theta_3^*+\theta_2+\theta_3} + d_9\mathrm{e}^{\theta_2^*+\theta_3^*+\theta_2+\theta_3}, \\ g_5(x,t) &= p_1\mathrm{e}^{\theta_1^*+\theta_2^*+\theta_1+\theta_2+\theta_3} + p_2\mathrm{e}^{\theta_1^*+\theta_3^*+\theta_1+\theta_2+\theta_3} \\ &\quad + p_3\mathrm{e}^{\theta_2^*+\theta_3^*+\theta_1+\theta_2+\theta_3}, \\ h_5(x,t) &= q_1\mathrm{e}^{\theta_1^*+\theta_2^*+\theta_1+\theta_2+\theta_3} + q_2\mathrm{e}^{\theta_1^*+\theta_3^*+\theta_1+\theta_2+\theta_3} \\ &\quad + q_3\mathrm{e}^{\theta_2^*+\theta_3^*+\theta_1+\theta_2+\theta_3}, \\ f_5(x,t) &= g_6(x,t) = h_6(x,t) = 0, \\ f_6(x,t) &= r\mathrm{e}^{\theta_1^*+\theta_2^*+\theta_3^*+\theta_1+\theta_2+\theta_3}, \\ g_n(x,t) &= h_n(x,t) = f_n(x,t) = 0, \quad (n = 7,8,\ldots), \end{aligned} \tag{18}$$

with

$$a_{11} = \frac{\sigma_1 m_1^2 + \sigma_2 n_1^2}{2\alpha_1 \left(k_1 + k_1^*\right)^2},\ a_{12} = \frac{\sigma_1 m_1 m_2 + \sigma_2 n_1 n_2}{2\alpha_1 \left(k_1 + k_2^*\right)^2},\ a_{13} = \frac{\sigma_1 m_1 m_3 + \sigma_2 n_1 n_3}{2\alpha_1 \left(k_1 + k_3^*\right)^2},$$

$$a_{21} = \frac{\sigma_1 m_1 m_2 + \sigma_2 n_1 n_2}{2\alpha_1 \left(k_2 + k_1^*\right)^2},\ a_{22} = \frac{\sigma_1 m_2^2 + \sigma_2 n_2^2}{2\alpha_1 \left(k_2 + k_2^*\right)^2},\ a_{23} = \frac{\sigma_1 m_2 m_3 + \sigma_2 n_2 n_3}{2\alpha_1 \left(k_2 + k_3^*\right)^2},$$

$$a_{31} = \frac{\sigma_1 m_1 m_3 + \sigma_2 n_1 n_3}{2\alpha_1 \left(k_3 + k_1^*\right)^2},\ a_{32} = \frac{\sigma_1 m_2 m_3 + \sigma_2 n_2 n_3}{2\alpha_1 \left(k_3 + k_2^*\right)^2},\ a_{33} = \frac{\sigma_1 m_3^2 + \sigma_2 n_3^2}{2\alpha_1 \left(k_3 + k_3^*\right)^2},$$

$$b_1 = \frac{(k_1-k_2)^2 n_1^2 n_2 \varsigma(\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{11}^2 \left(k_2 + k_1^*\right)^2},\ b_2 = \frac{(k_1-k_2)^2 n_1 n_2^2 \varsigma(\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{21}^2 \left(k_1 + k_2^*\right)^2},\ b_3 = \frac{(k_1-k_2)^2 n_1 n_2 n_3 \varsigma(\sigma_2 + \sigma_1\varsigma^2)}{2\alpha_1 \left(k_1 + k_3^*\right)^2 \left(k_2 + k_3^*\right)^2},$$

$$b_4 = \frac{(k_1-k_3)^2 n_1^2 n_3 \varsigma(\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{11}^2 \left(k_3 + k_1^*\right)^2},\ b_5 = \frac{(k_1-k_3)^2 n_1 n_2 n_3 \varsigma(\sigma_2 + \sigma_1\varsigma^2)}{2\alpha_1 \left(k_1 + k_2^*\right)^2 \left(k_3 + k_2^*\right)^2},\ b_6 = \frac{(k_1-k_3)^2 n_1 n_3^2 \varsigma(\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{31}^2 \left(k_1 + k_3^*\right)^2},$$

$$b_7 = \frac{(k_2-k_3)^2 n_1 n_2 n_3 \varsigma(\sigma_2 + \sigma_1\varsigma^2)}{2\alpha_1 \left(k_2 + k_1^*\right)^2 \left(k_3 + k_1^*\right)^2},\ b_8 = \frac{(k_2-k_3)^2 n_2^2 n_3 \varsigma(\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{21}^2 \left(k_3 + k_2^*\right)^2},\ b_9 = \frac{(k_2-k_3)^2 n_2 n_3^2 \varsigma(\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{31}^2 \left(k_2 + k_3^*\right)^2},$$

$$c_1 = \frac{(k_1-k_2)^2 n_1^2 n_2 (\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{11}^2 \left(k_2 + k_1^*\right)^2},\ c_2 = \frac{(k_1-k_2)^2 n_1 n_2^2 (\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{21}^2 \left(k_1 + k_2^*\right)^2},\ c_3 = \frac{(k_1-k_2)^2 n_1 n_2 n_3 (\sigma_2 + \sigma_1\varsigma^2)}{2\alpha_1 \left(k_1 + k_3^*\right)^2 \left(k_2 + k_3^*\right)^2},$$

$$c_4 = \frac{(k_1-k_3)^2 n_1^2 n_3 (\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{11}^2 \left(k_3 + k_1^*\right)^2},\ c_4 = \frac{(k_1-k_3)^2 n_1 n_2 n_3 (\sigma_2 + \sigma_1\varsigma^2)}{2\alpha_1 \left(k_1 + k_2^*\right)^2 \left(k_3 + k_2^*\right)^2},\ c_6 = \frac{(k_1-k_3)^2 n_1 n_3^2 (\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{31}^2 \left(k_1 + k_3^*\right)^2},$$

$$c_7 = \frac{(k_2-k_3)^2 n_1 n_2 n_3 (\sigma_2 + \sigma_1\varsigma^2)}{2\alpha_1 \left(k_2 + k_1^*\right)^2 \left(k_3 + k_1^*\right)^2},\ c_8 = \frac{(k_2-k_3)^2 n_2^2 n_3 (\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{21}^2 \left(k_3 + k_2^*\right)^2},\ c_9 = \frac{(k_2-k_3)^2 n_2 n_3^2 (\sigma_2 + \sigma_1\varsigma^2)}{8\alpha_1 k_{31}^2 \left(k_2 + k_3^*\right)^2},$$

$$d_1 = \frac{n_1^2 n_2^2 \left(\sigma_1\varsigma^2 + \sigma_2\right)^2 \left(k_{11}^2 + k_{12}^2 + k_{21}^2 + k_{22}^2 - 2k_{11}k_{21} - 2k_{12}k_{22}\right)^2}{64\alpha_1^2 k_{11}^2 k_{21}^2 \left(k_1 + k_2^*\right)^2 \left(k_2 + k_1^*\right)^2},$$

$$d_2 = \frac{n_1^2 n_2 n_3 \left(\sigma_1\varsigma^2 + \sigma_2\right)^2 (k_1-k_2)^2 (k_1^* - k_3^*)^2}{16\alpha_1^2 k_{11}^2 \left(k_2 + k_1^*\right)^2 \left(k_1 + k_3^*\right)^2 \left(k_2 + k_3^*\right)^2},\ d_3 = \frac{n_1^2 n_2 n_3 \left(\sigma_1\varsigma^2 + \sigma_2\right)^2 (k_1-k_3)^2 (k_1^* - k_2^*)^2}{16\alpha_1^2 k_{11}^2 \left(k_3 + k_1^*\right)^2 \left(k_1 + k_2^*\right)^2 \left(k_3 + k_2^*\right)^2},$$

$$d_4 = \frac{n_1^2 n_3^2 \left(\sigma_1\varsigma^2 + \sigma_2\right)^2 \left(k_{11}^2 + k_{12}^2 + k_{31}^2 + k_{32}^2 - 2k_{11}k_{31} - 2k_{12}k_{32}\right)^2}{64\alpha_1^2 k_{11}^2 k_{31}^2 \left(k_1 + k_3^*\right)^2 \left(k_3 + k_1^*\right)^2},$$

$$d_5=\frac{n_1n_2^2n_3\left(\sigma_1\varsigma^2+\sigma_2\right)^2(k_1-k_2)^2(k_2^*-k_3^*)^2}{16\alpha_1^2k_{21}^2\left(k_2+k_3^*\right)^2\left(k_1+k_2^*\right)^2\left(k_1+k_3^*\right)^2},\quad d_6=\frac{n_1n_2n_3^2\left(\sigma_1\varsigma^2+\sigma_2\right)^2(k_1-k_3)^2(k_2^*-k_3^*)^2}{16\alpha_1^2k_{31}^2\left(k_3+k_2^*\right)^2\left(k_1+k_2^*\right)^2\left(k_1+k_3^*\right)^2},$$

$$d_7=\frac{n_1n_2^2n_3\left(\sigma_1\varsigma^2+\sigma_2\right)^2(k_2-k_3)^2(k_1^*-k_2^*)^2}{16\alpha_1^2k_{21}^2\left(k_2+k_1^*\right)^2\left(k_3+k_1^*\right)^2\left(k_3+k_2^*\right)^2},\quad d_8=\frac{n_1n_2n_3^2\left(\sigma_1\varsigma^2+\sigma_2\right)^2(k_2-k_3)^2(k_1^*-k_3^*)^2}{16\alpha_1^2k_{31}^2\left(k_2+k_1^*\right)^2\left(k_2+k_3^*\right)^2\left(k_3+k_1^*\right)^2},$$

$$d_9=\frac{n_2^2n_3^2\left(\sigma_1\varsigma^2+\sigma_2\right)^2\left(k_{21}^2+k_{22}^2+k_{31}^2+k_{32}^2-2k_{21}k_{31}-2k_{22}k_{32}\right)^2}{64\alpha_1^2k_{21}^2k_{31}^2\left(k_2+k_3^*\right)^2\left(k_3+k_2^*\right)^2},$$

$$p_1=\frac{(k_1-k_3)^2n_1^2n_2^2n_3\varsigma\left(\sigma_1\varsigma^2+\sigma_2\right)^2\left(k_{11}^2+k_{12}^2+k_{21}^2+k_{22}^2-2k_{11}k_{21}-2k_{12}k_{22}\right)^2(k_2-k_3)^2}{64\alpha_1^2k_{11}^2k_{21}^2\left(k_1+k_2^*\right)^2\left(k_2+k_1^*\right)^2\left(k_3+k_1^*\right)^2\left(k_3+k_2^*\right)^2},$$

$$p_2=\frac{(k_1-k_2)^2n_1^2n_2n_3^2\varsigma\left(\sigma_1\varsigma^2+\sigma_2\right)^2\left(k_{11}^2+k_{12}^2+k_{31}^2+k_{32}^2-2k_{11}k_{31}-2k_{12}k_{32}\right)^2(k_2-k_3)^2}{64\alpha_1^2k_{11}^2k_{31}^2\left(k_1+k_3^*\right)^2\left(k_3+k_1^*\right)^2\left(k_2+k_1^*\right)^2\left(k_2+k_3^*\right)^2},$$

$$p_3=\frac{(k_1-k_2)^2n_1n_2^2n_3^2\varsigma\left(\sigma_1\varsigma^2+\sigma_2\right)^2\left(k_{31}^2+k_{32}^2+k_{21}^2+k_{22}^2-2k_{31}k_{21}-2k_{32}k_{22}\right)^2(k_1-k_3)^2}{64\alpha_1^2k_{21}^2k_{31}^2\left(k_3+k_2^*\right)^2\left(k_2+k_3^*\right)^2\left(k_1+k_2^*\right)^2\left(k_1+k_3^*\right)^2},$$

$$q_1=\frac{n_1^2n_2^2n_3\left(\sigma_1\varsigma^2+\sigma_2\right)^2\left(k_{11}^2+k_{12}^2+k_{21}^2+k_{22}^2-2k_{11}k_{21}-2k_{12}k_{22}\right)^2(k_1-k_3)^2(k_2-k_3)^2}{64\alpha_1^2k_{11}^2k_{21}^2\left(k_1+k_2^*\right)^2\left(k_2+k_1^*\right)^2\left(k_3+k_1^*\right)^2\left(k_3+k_2^*\right)^2},$$

$$q_2=\frac{n_1^2n_2n_3^2\left(\sigma_1\varsigma^2+\sigma_2\right)^2\left(k_{11}^2+k_{12}^2+k_{31}^2+k_{32}^2-2k_{11}k_{31}-2k_{12}k_{32}\right)^2(k_1-k_2)^2(k_2-k_3)^2}{64\alpha_1^2k_{11}^2k_{31}^2\left(k_1+k_3^*\right)^2\left(k_3+k_1^*\right)^2\left(k_2+k_1^*\right)^2\left(k_2+k_3^*\right)^2},$$

$$q_3=\frac{n_1n_2^2n_3^2\left(\sigma_1\varsigma^2+\sigma_2\right)^2\left(k_{31}^2+k_{32}^2+k_{21}^2+k_{22}^2-2k_{31}k_{21}-2k_{32}k_{22}\right)^2(k_1-k_2)^2(k_1-k_3)^2}{64\alpha_1^2k_{21}^2k_{31}^2\left(k_3+k_2^*\right)^2\left(k_2+k_3^*\right)^2\left(k_1+k_2^*\right)^2\left(k_1+k_3^*\right)^2},$$

$$r=\frac{n_1^2n_2^2n_3^2\left(\sigma_1\zeta^2+\sigma_2\right)^3\left(k_{31}^2+k_{32}^2+k_{21}^2+k_{22}^2-2k_{31}k_{21}-2k_{32}k_{22}\right)^2}{512\alpha_1^3k_{11}^2k_{21}^2k_{31}^2\left(k_3+k_2^*\right)^2\left(k_2+k_3^*\right)^2}\times\frac{\left(k_{11}^2+k_{12}^2+k_{21}^2+k_{22}^2-2k_{11}k_{21}-2k_{12}k_{22}\right)^2\left(k_{11}^2+k_{12}^2+k_{31}^2+k_{32}^2-2k_{11}k_{31}-2k_{12}k_{32}\right)^2}{\left(k_1+k_2^*\right)^2\left(k_2+k_1^*\right)^2\left(k_1+k_3^*\right)^2\left(k_3+k_1^*\right)^2}.$$

We can write the following three-soliton solutions for Eq. (5):

$$u=\frac{g}{f}=\frac{\varepsilon g_1+\varepsilon^3g_3+\varepsilon^5g_5}{1+\varepsilon^2f_2+\varepsilon^4f_4+\varepsilon^6f_6},$$
$$v=\frac{h}{f}=\frac{\varepsilon h_1+\varepsilon^3h_3+\varepsilon^5h_5}{1+\varepsilon^2f_2+\varepsilon^4f_4+\varepsilon^6f_6}. \tag{19}$$

Without loss of generality, setting $\varepsilon=1$, we can write the three-soliton solutions as

$$u=\frac{g}{f}=\frac{g_1+g_3+g_5}{1+f_2+f_4+f_6},$$
$$v=\frac{h}{f}=\frac{h_1+h_3+h_5}{1+f_2+f_4+f_6}. \tag{20}$$

4. Discussion

4.1. Discussion of the two-soliton solutions

Figure 1 presents the interaction diagram of two-soliton solutions (16) with the parameters $k_1=1+0.2\mathrm{i}$, $k_2=0.6+\mathrm{i}$, $\alpha_1=n_1=1$, $n_2=1/2$, $m_1=2$. Figures 1(a), 1(b), and 1(c) show the two-soliton solutions for Eq. (5), which reflect the all-focusing Manakov model, all-defocusing Manakov model, and mixed types, respectively. Our findings corroborate the view in Ref. [29] that the all-focusing type Manakov model admits bright-bright solitons solutions and the view in Ref. [18] that mixed type ones admit bright-bright solitons solutions. Furthermore, we find that the all-defocusing type Manakov model admits bright-bright solitons solutions. Compared to Figs. 1(a) and 1(b), we find that the waves amplitude of the two-soliton becomes larger when we change the value of parameter σ_2 from 2 to -2 and the other parameters remain unchanged. Nevertheless, the direction of propagation of waves remains unchanged.

Figure 2 shows the interaction diagram of two-soliton solutions (16) with the same parameters as in Fig. 1(a), but Fig. 2(a) with $k_1=2+0.2\mathrm{i}$, Fig. 2(b) with $k_2=-2+\mathrm{i}$. Comparing the interaction between two solitons in Figs. 1(a), 2(a) and 2(b), we find that the amplitudes of the solitons can be controlled by changing the value of the real parts of k_j $(j=1,2)$, that is, the amplitudes of the solitons increase with the absolute value of the real parts of k_j. When the value of the real parts of k_1 increases from 1 to 2, one of the two-soliton's amplitude increases. When the value of the real parts of k_2 decreases from 0.6 to -2, the other one's amplitude increases

either.

Comparing Figs. 1(a), 3(a) and 3(b), and lots of other diagrams drawn during the study, we can find that the propagation directions of solitons are determined by the imaginary part of k_j $(j = 1,2)$. When the imaginary part of k_j is equal to zero, the propagation direction of the soliton is perpendicular to the x-axis. When the imaginary part of k_j is greater than zero, the soliton's propagation path forms an acute angle with the positive x-axis, the angle decreases with the increase of the imaginary part of k_j, and finally tends to be parallel to the x-axis. When the imaginary part of k_j is less than zero, the soliton's propagation path forms an obtuse angle with the positive x-axis, the angle increases with the decrease of the imaginary part of k_j, and finally tends to be parallel to the x-axis. The change in the above angle seems to be positive correlation with the inverse cotangent function.

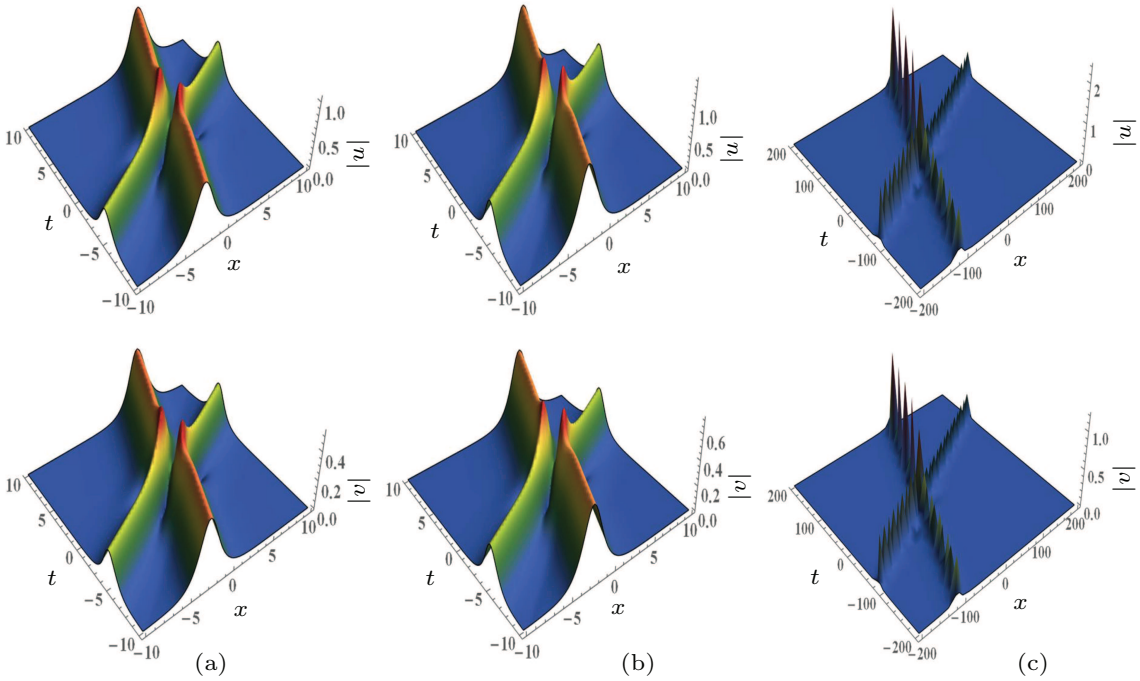


Fig. 1. Two-soliton interaction diagram via solutions (16), with $k_1 = 1+0.2\mathrm{i}$, $k_2 = 0.6+\mathrm{i}$, $\alpha_1 = n_1 = 1$, $n_2 = 1/2$, $m_1 = 2$, but (a) with $\sigma_1 = 2$, $\sigma_2 = 2$, (b) with $\sigma_1 = 2$, $\sigma_2 = -2$, (c) with $k_1 = 0.08+0.2\mathrm{i}$, $k_2 = 0.08+\mathrm{i}$, $\sigma_1 = -2$, $\sigma_2 = -2$.

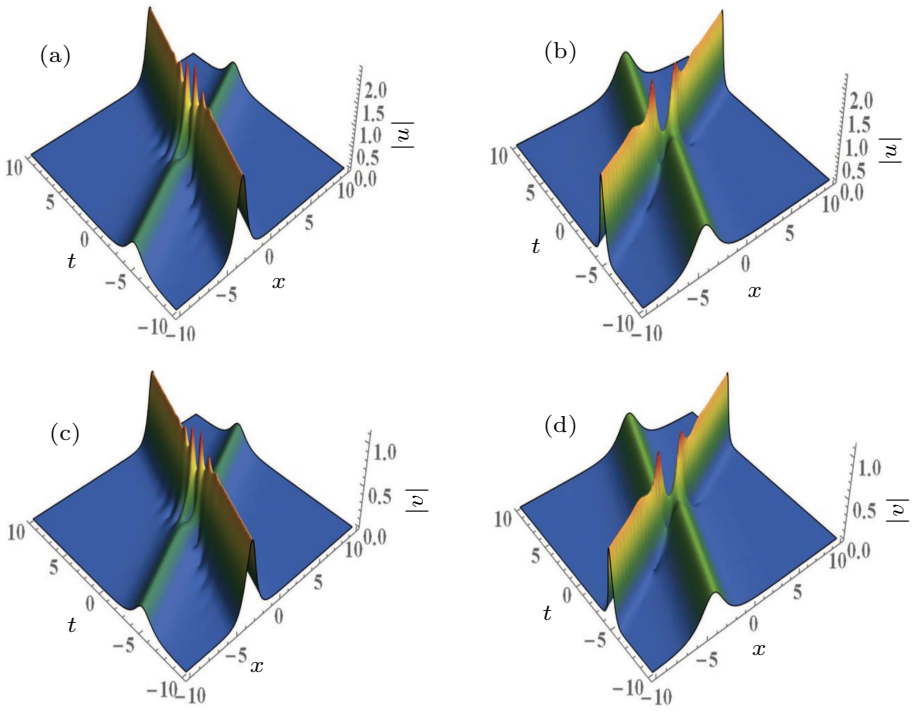


Fig. 2. Two-soliton interaction diagram shown in solutions (16) with the same parameters as in Fig. 1(a), but [(a), (c)] with $k_1 = 2+0.2\mathrm{i}$ and [(b), (d)] with $k_2 = -2+\mathrm{i}$.

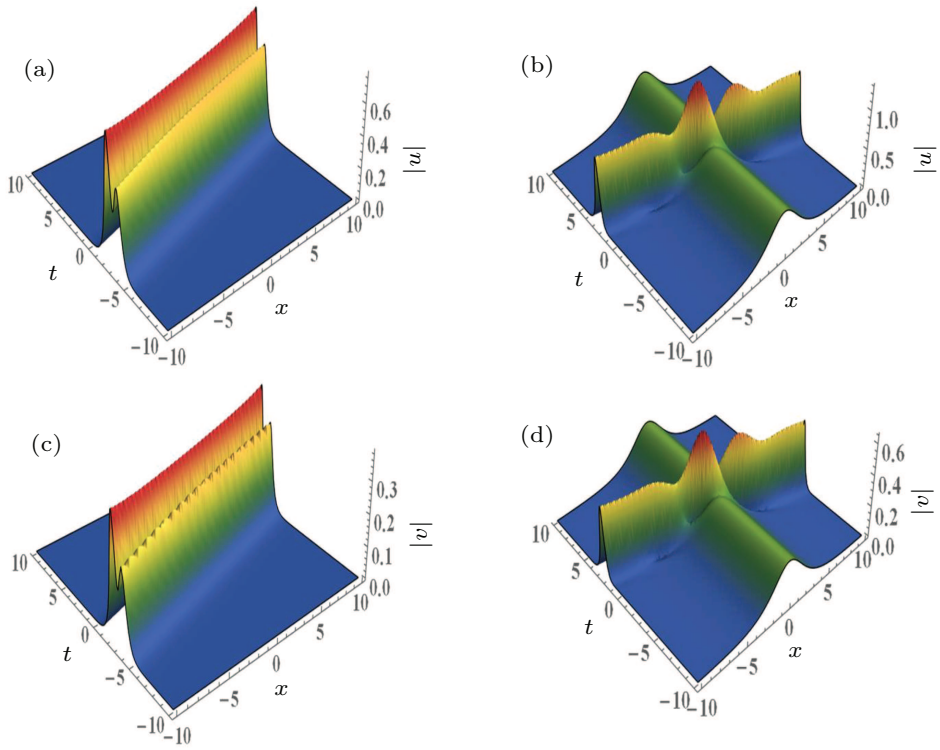


Fig. 3. Two-soliton interaction diagram in solutions (16) with the same parameters as in Fig. 1(a), but [(a), (c)] with $k_1 = 1+2\mathrm{i}$, $k_2 = 0.6+2\mathrm{i}$, and [(b), (d)] with $k_1 = 1-2\mathrm{i}$, $k_2 = 0.6$.

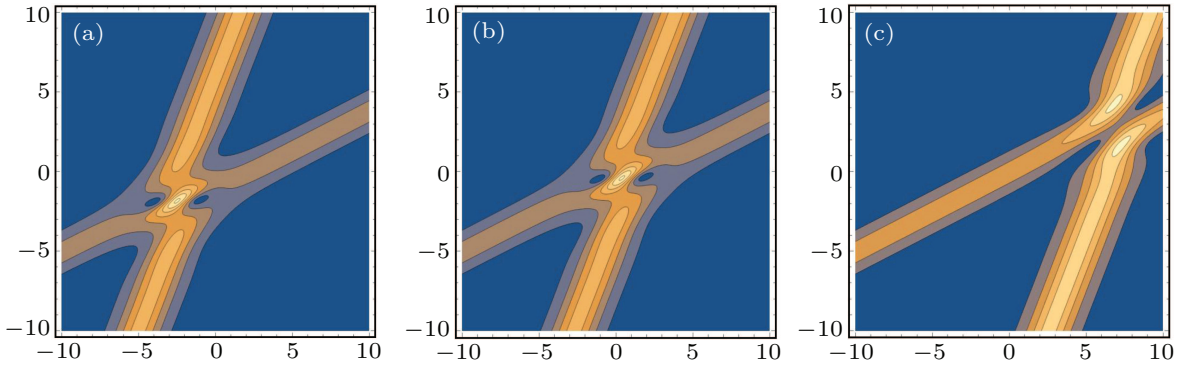


Fig. 4. The contour plot of the absolute value of the function u in expression (16) with the same parameters as given in Fig. 1(a), but (a) with $n_1 = -10$, $n_2 = 0.5$, (b) with $n_1 = -1$, $n_2 = 0.5$, and (c) with $n_1 = 0.005$, $n_2 = 0.5$.

Figure 4 shows the contour plot of the absolute value of the function u in expression (16) with the same parameters as given in Fig. 1(a), but Fig. 4(a) with $n_1 = -10$, $n_2 = 0.5$, Fig. 4(b) with $n_1 = -1$, $n_2 = 0.5$, Fig. 4(c) with $n_1 = 0.005$, $n_2 = 0.5$. We find that the absolute value of the ratio n_1/n_2 affects the location of the collision of two solitons, where the energies of the two solitons converge. Fixing n_2 and decreasing the absolute value of n_1, we find that one of the two-soliton shifts to the right, and the other one does not shift; the change of the contour plot of the absolute value of the function v in expression (16) is the same as the function u. When the absolute value of the ratio n_1/n_2 decreases, the location of the collision of two solitons moves to right upper the coordinate surface. In addition, the value of m_1 has a similar effect on the ratio n_1/n_2.

4.2. Discussion of the three-soliton solutions

Figure 5 portrays the interaction diagram of three-soliton solutions with the parameters $k_1 = 0.6+0.2\mathrm{i}$, $k_2 = 0.6+\mathrm{i}$, $k_3 = 0.6-0.5\mathrm{i}$, $\alpha_1 = \varsigma = n_1 = n_2 = n_3 = 1$, $\sigma_1 = \sigma_2 = -2$. By observing Fig. 5, the three-soliton solutions for Eq. (4), which reads the Manakov model of all-defocusing type, we can find that the all-defocusing type Manakov model admits bright-bright-bright solitons solutions. This conclusion can be regarded as a generalization of the two-soliton case. A graph of $|u|$ is shown in Fig. 5(a), where the collision causes the amplitude of one soliton to be suppressed and the amplitude of the other two solitons to be enhanced. It is interesting to note that the graph of $|v|$ shown in Fig. 5(b) also shows the same kinds of changes. The similar phenomena of this form of intensity redistribution have surfaced in the potential application

to communication signal amplification.[18]

Figure 6 portrays the interaction diagram of three-soliton solutions (20) with the parameters $\sigma_1 = 2$, $\sigma_2 = 2$, $\alpha_1 = \varsigma = n_1 = n_2 = n_3 = 1$, $k_3 = 1.2 - 0.5\mathrm{i}$, but Fig. 6(a) with $k_1 = 1 + 0.2\mathrm{i}$, $k_2 = 0.6 + \mathrm{i}$, Fig. 6(b) with $k_1 = 2.5 + 0.2\mathrm{i}$, $k_2 = 0.6 + \mathrm{i}$, Fig. 6(c) with $k_1 = 1 + 0.2\mathrm{i}$, $k_2 = 2.5 + \mathrm{i}$. By observing the three-soliton solutions shown in Figs. 6(a)–6(c), in comparisons of the real parts of wave numbers k_1 and k_2 shown in Figs. 6(a) and 6(b), Figs. 6(a) and 6(c), it can be found that the amplitudes of the solitons increase with the absolute value of the real parts of k_j $(j = 1, 2)$, the result is the same as the case of two solitons. It means that the wave amplitude of the three-soliton solution is also controlled by the wave number of the real parts of k_j.

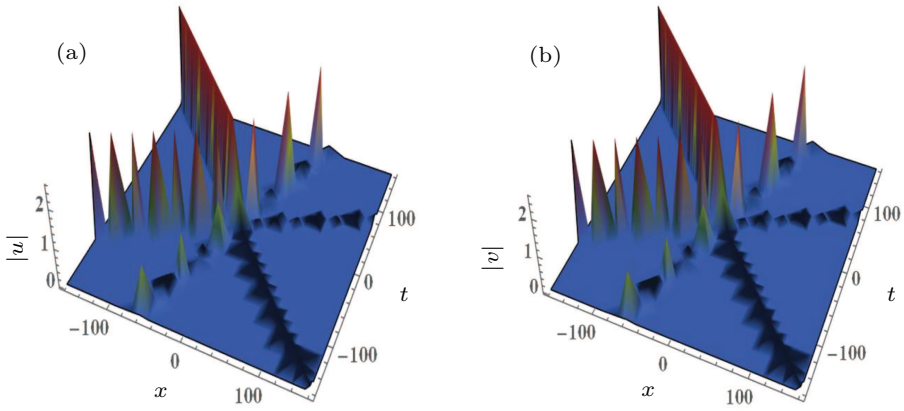


Fig. 5. Three-soliton interaction diagram via solutions (20), with $k_1 = 0.6 + 0.2\mathrm{i}$, $k_2 = 0.6 + \mathrm{i}$, $k_3 = 0.6 - 0.5\mathrm{i}$, $\alpha_1 = \varsigma = n_1 = n_2 = n_3 = 1$, $\sigma_1 = \sigma_2 = -2$.

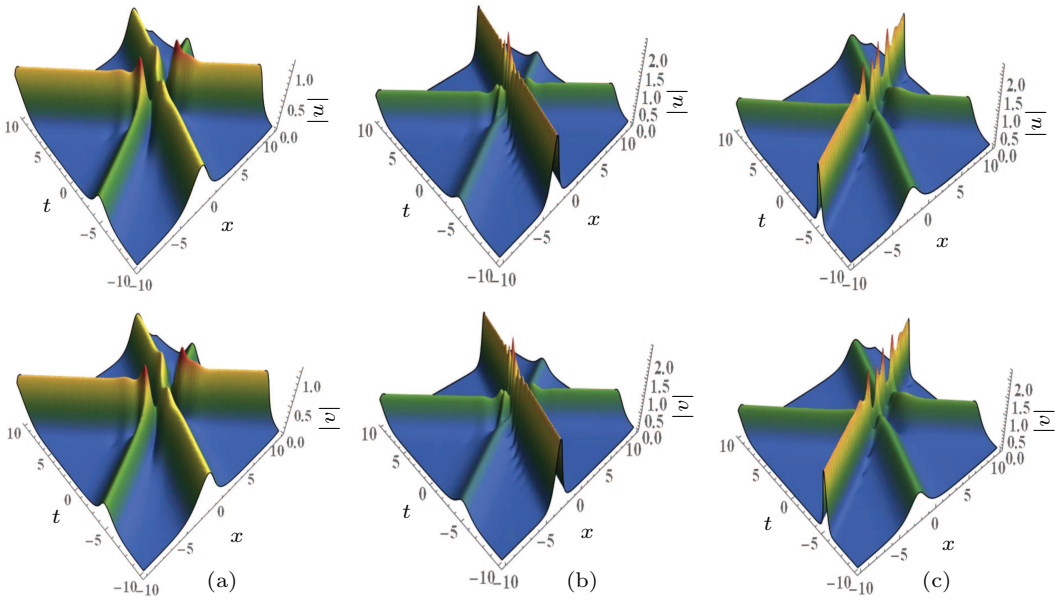


Fig. 6. Three-soliton interaction diagram via solutions (20), for $\sigma_1 = 2$, $\sigma_2 = 2$, $\alpha_1 = \varsigma = n_1 = n_2 = n_3 = 1$, $k_3 = 1.2 - 0.5\mathrm{i}$: (a) with $k_1 = 1 + 0.2\mathrm{i}$, $k_2 = 0.6 + \mathrm{i}$, (b) with $k_1 = 2.5 + 0.2\mathrm{i}$, $k_2 = 0.6 + \mathrm{i}$, (c) with $k_1 = 1 + 0.2\mathrm{i}$, $k_2 = 2.5 + \mathrm{i}$.

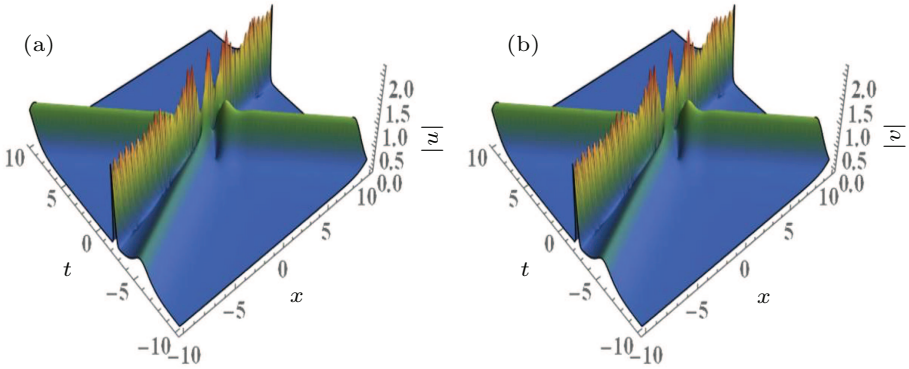


Fig. 7. Three-soliton interaction diagram in solutions (20) with the same parameters as given in Fig. 6(b), but with $k_1 = 2.5 + 2.5\mathrm{i}$.

When the imaginary parts of one wave number k_j $(j = 1,2,3)$ are changed and the other parameters remain unchanged, we can control the wave to rotate. This means that the propagation directions of solitons are determined by the imaginary parts of k_j $(j = 1,2,3)$. Compared Fig. 6(b) and Fig. 7, when we change the imaginary part of the wave number k_1, from 0.6 to 2.5, the imaginary part value of k_1 is greater than zero, the soliton's propagation path forms an acute angle with the positive x-axis, the angle decreases with the increase of the imaginary part of k_1, and the other laws in the case of two solitons are also true.

Figure 8 portrays the interaction diagram of three-soliton solutions (20) with the parameters $k_1 = 1+0.2\mathrm{i}$, $k_2 = 0.6+\mathrm{i}$, $k_3 = 1.2-0.5\mathrm{i}$, $\alpha_1 = \varsigma = 1$, $\sigma_1 = 2$, $\sigma_2 = 2$, but Fig. 8(a) with $n_1 = n_2 = n_3 = 1$, Fig. 8(b) with $n_1 = n_3 = 1$, $n_2 = 0.01$, and Fig. 8(c) with $n_1 = n_3 = 1$, $n_2 = 100$. In Figs. 8(a)–8(c), we fix $n_1 = n_3 = 1$ and change the absolute value of n_2, and it is found that one of the three solitons shifts and the other two solitons do not shift. In comparison of Figs. 8(a) and 8(b), one of the three solitons shifts to the right with decrease of the absolute value of n_2. In comparison of Figs. 8(a) and 8(c), one of the three solitons shifts to the left with increase of the absolute value of n_2. The change of the function v in expression (20) is the same as the function u. We can see that the value of the ratio n_i/n_j affects the location of the collision of three solitons, and the law is also the same as in the case of two solitons.

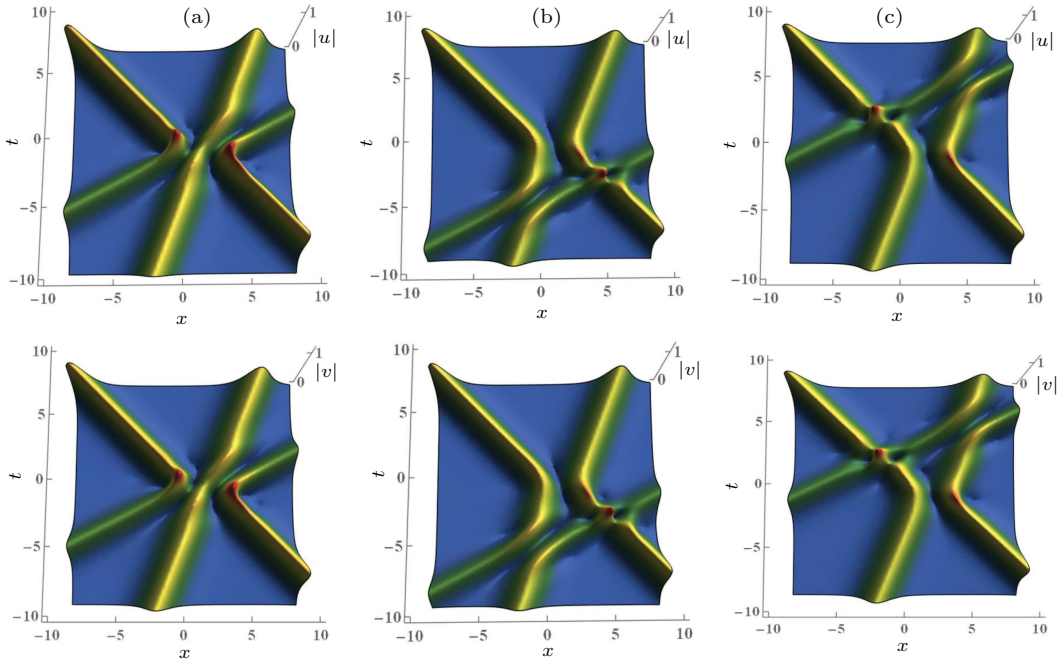


Fig. 8. Three-soliton interaction diagram of solutions (20), for $k_1 = 1+0.2\mathrm{i}$, $k_2 = 0.6+\mathrm{i}$, $k_3 = 1.2-0.5\mathrm{i}$, $\alpha_1 = \varsigma = 1$, $\sigma_1 = 2$, $\sigma_2 = 2$: (a) with $n_1 = n_2 = n_3 = 1$, (b) with $n_1 = n_3 = 1$, $n_2 = 0.01$, and (c) with $n_1 = n_3 = 1$, $n_2 = 100$.

5. Conclusions

In summary, we have studied a coupled Schrödinger equation that comes from the Boussinesq equation of atmospheric gravity waves by using multiscale analysis and reduced perturbation method to describe Rossby waves. By setting values of parameters, we obtain the Manakov model of all-focusing, all-defocusing and mixed types, and utilize the Hirota bilinear method to give the two-soliton and three-soliton solutions. We find that the all-defocusing type Manakov model admits bright-bright soliton solutions. This finding has not been reported in the literature to our best known-ledge. We also find that the all-defocusing type Manakov model admits bright-bright-bright soliton solutions, which can be regarded as a generalization of the two-soliton solutions.

Then, we discuss the influence of free parameters on the wave amplitude, the propagation direction and the location of the collision in the two-soliton and three-soliton solutions for the all-focusing type Manakov model. The following conclusions are drawn:

(i) The amplitudes of the solitons increase with the absolute value of the real parts of k_j $(j = 1,2,3)$ for the two-soliton and the three-soliton solutions.

(ii) The propagation direction of solitons can be controlled by the imaginary part of k_j $(j = 1,2,3)$ for the two-soliton and the three-soliton solutions.

(iii) The absolute value of the ratio n_1/n_2 changes the location of the collision in the two-soliton and three-soliton solutions.

Solitons maintain their original amplitudes, velocities, and shapes in elastic collisions, with the exception of a little

phase shift following the collision.[44] From this point of view, the collisions in Figs. 1(a), 1(b), 2, 3, 6, 7, and 8 are elastic, and the collisions in Figs. 1(c) and 5 are inelastic.

It is crucial to keep in mind that the Hirota bilinear method does not work when $g_n(x,t) = h_n(x,t) = f_n(x,t) = 0,\ (n = 5,6,\ldots)$ [or $g_n(x,t) = h_n(x,t) = f_n(x,t) = 0,\ (n = 7,8,\ldots)$] are not zero in the case of solving two-soliton (or three-soliton) solutions. The result shows that the soliton propagation for the coupled Schrödinger equation describing Rossby waves we studied is affected by the above factors but may not limited to them. There should be more cases necessarily to be investigated. Whether the intensity redistribution phenomena we observed in Fig. 5 can occur in other N-soliton cases requires further investigation. We will investigate whether N-soliton solutions have similar properties in another work. These conclusions are useful for setting a simulation scene in research of Rossby waves.

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