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Interaction solutions for the second extended (3+1)-dimensional Jimbo–Miwa equation

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Based on the Hirota bilinear method, the second extended (3+1)-dimensional Jimbo–Miwa equation is established. By Maple symbolic calculation, lump and lump-kink soliton solutions are obtained. The interaction solutions between the lump and multi-kink soliton, and the interaction between the lump and triangular periodic soliton are derived by combining a multi-exponential function or trigonometric sine and cosine functions with quadratic functions. Furthermore, periodic-lump wave solution is derived via the ansatz including hyperbolic and trigonometric functions. Finally, 3D plots, 2D curves, density plots, and contour plots with particular choices of the suitable parameters are depicted to illustrate the dynamical features of these solutions.

Keywords: Hirota bilinear method, second extended (3+1)-dimensional Jimbo–Miwa equation, lump solution, interaction solution

PACS: 02.30.Jr, 02.70.Wz, 04.20.Jb, 04.30.Nk **DOI:** 10.1088/1674-1056/acb91c

1. Introduction

Nonlinear evolution equations (NLEEs), which can be used to model nonlinear phenomena, is an effective tool to describe the real world.[1,2] NLEEs exhibit complexity and diversity of exact solutions such as rational solution,[3–6] periodic solution,[7–9] lump solution,[10–13] and trigonometric function solution.[14,15] Lump solutions are a class of rational function solutions that are localized in all directions in space and were found by Zakharov and Manakov.[16] Also, they discovered that lump wave decays in all directions of the spaces. In recent years, lump solutions have been found from a number of equations such as Boussinesq equation,[17] the B-type Kadomtsev–Petviashvili (BKP) equation,[18,19] (3+1)-dimensional variable coefficient BKP equation[20] and the Ishimori equation.[21,22] Some shallow water wave equations have also been explored for lump solutions.[23–27]

Recently, research of lumps solutions and their interaction with solitons, triangular periodic waves and kink waves has captured considerable attention in the field of mathematical physics community.[28–32] For some NLEEs, collisions between solitons are usually completely inelastic or completely elastic. Such collisions are often accompanied by soliton fusion and fission.[33–35] Therefore, discussion of interaction between solitary waves becomes meaningful and important, and also brings a wider range of applications at the physical level.

The (3+1)-dimensional Jimbo–Miwa equation (JM)[36] is

$$u_{xxxy}+3u_{xx}u_y+3u_{xy}u_x-3u_{xz}+2u_{yt}=0, \tag{1}$$

which is the second member in the Kadomtsev–Petviashvili (KP) hierarchy firstly introduced by Jimbo and Miwa.[37] In Eq. (1), if one replaces u_{xz} with $u_{xz}+u_{yz}+u_{zz}$ or u_{yt} with $u_{xt}+u_{yt}+u_{zt}$, then we can obtain the following two extended equations:

$$u_{xxxy}+3\left(u_yu_x\right)_x-3(u_{xz}+u_{yz}+u_{zz})+2u_{yt}=0, \tag{2}$$

$$u_{xxxy}+3\left(u_yu_x\right)_x-3u_{xz}+2\left(u_{xt}+u_{yt}+u_{zt}\right)=0. \tag{3}$$

Note that both forms preserve the dimensionality and order of the Jimbo–Miwa equation. Wazwaz[38] obtained multi-soliton solutions of Eqs. (2) and (3) using the Hirota bilinear method. Sun and Chen[39] obtained the lump solutions of Eq. (3). Manafian[40] used the Hirota bilinear method to acquire several classes of solitary wave solutions in terms of a new combination of exponential function, trigonometric function, and hyperbolic functions successfully. Ali[41] obtained new solitary wave solutions of the extended (3+1)-dimensional JM equation using the Kudryashov method. Li[42] obtained abundant meromorphic exact solutions of Eqs. (2) and (3) by making use of the complex method.

The rest of the paper is organized as follows. In Section 2, the bilinear form of Eq. (3) is presented with a derivation. In Section 3, we obtain its lump waves and show a concrete solution on graphs. In Sections 4 and 5, we construct solutions by mixing lump wave with one-soliton wave and two-soliton wave, respectively. Fission and fusion phenomena exhibited by inelastic collisions are depicted through images. In Section 6, the lump-periodic solution is obtained through the ansatz including quadratic and trigonometric functions. In Section 7, we obtain the interaction solution between lump waves and triangular periodic waves with graphic analysis. In Section 8, periodic-lump wave solution is derived via

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the ansatz including hyperbolic and trigonometric functions. Finally, the conclusions are given.

2. Bilinear form

In this paper, we consider the second extended (3+1)-dimensional Jimbo–Miwa equation (3). By using the dependent variable transformation

$$u(x,y,z,t)=2(\ln f)_x. \tag{4}$$

Equation (3) can be mapped into the following bilinear form:

$$(D_x^3D_y+2(D_xD_t+D_yD_t+D_zD_t)-3D_xD_z)f\cdot f=0, \tag{5}$$

which is equivalent to

$$\begin{aligned}&f_{xxxy}f-f_{xxx}f_y-3f_{xxy}f_x+2f(f_{tx}+f_{ty}+f_{tz})\\&+3f_{xx}f_{xy}-3ff_{xz}-2(f_x+f_y+f_z)f_t+3f_xf_z=0,\end{aligned} \tag{6}$$

where $f=f(x,y,z,t)$, and the bilinear operators D_x, D_y, D_z, and D_t are defined as[43]

$$\begin{aligned}&D_x^mD_y^nD_z^pD_t^q(f\cdot g)\\&=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^n\\&\times\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right)^p\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^q\\&\times[f(x,y,z,t)g(x',y',z',t')]|_{x'=x,y'=y,z'=z,t'=t}.\end{aligned} \tag{7}$$

3. Lump solution

To seek the lump solutions of Eq. (6), we suppose f expressed in the following form:

$$\begin{aligned}&f=g^2+h^2+l,\quad l>0,\\&g=a_1x+b_1y+c_1z+d_1t+w_1,\\&h=a_2x+b_2y+c_2z+d_2t+w_2,\end{aligned} \tag{8}$$

where a_i, b_i, c_i, d_i, w_i, $i=1,2$ and l are real parameters to be determined. Plugging such a function f into Eq. (6) leads to a system of algebraic equations on the parameters. Solving the set of algebraic equations, we can find the following solutions.

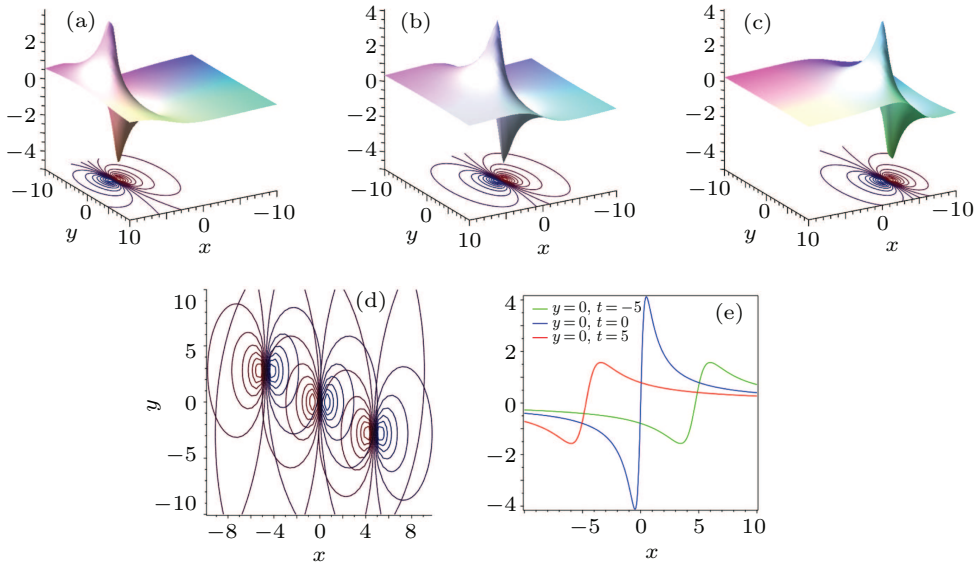


Fig. 1. The lump solution to Eq. (10) at (a) $t=-5$, (b) $t=0$, (c) $t=5$; (d) the contour plot showing the moving path with time $t=-5$, 0, 5; (e) the curve plot at different times.

Case 1

$$\begin{aligned}&a_1=a_1,\quad a_2=a_2,\quad l=l,\quad b_1=\frac{2a_1d_2a_2}{3a_1^2+3a_2^2-2a_2d_2},\\&b_2=-\frac{2a_1^2d_2}{3a_1^2+3a_2^2-2a_2d_2},\\&c_1=0,\quad c_2=\frac{2d_2\left(a_1^2+a_2^2\right)}{3a_1^2+3a_2^2-2a_2d_2},\\&d_1=0,\quad d_2=d_2,\quad w_1=w_1,\quad w_2=w_2,\end{aligned} \tag{9}$$

where $3a_1^2+3a_2^2-2a_2d_2\neq0$. Then, a standardised lump solution is obtained by inserting Eq. (9) with Eq. (8) into Eq. (4)

$$u(x,y,z,t)=\frac{4(a_1A_1+a_2B_1)}{A_1^2+B_1^2+l}, \tag{10}$$

and it is observed that

$$\lim_{x^2+y^2\to\infty}u(x,y,z,t)=0,\quad \forall(z,t)\in R^2, \tag{11}$$

where

$$\begin{aligned}A_1&=a_1x+\frac{2a_1d_2a_2}{3a_1^2+3a_2^2-2a_2d_2}y+w_1,\\B_1&=a_2x-\frac{2a_1^2d_2}{3a_1^2+3a_2^2-2a_2d_2}y\\&\quad+\frac{2d_2(a_1^2+a_2^2)}{3a_1^2+3a_2^2-2a_2d_2}z+d_2t+w_2.\end{aligned} \tag{12}$$

Figure 1 shows the movement of a lump wave by selecting the parameters as $a_1=1/2$, $a_2=2$, $a_3=1$, $d_2=2$, $w_1=0$,

$w_2 = 0$. We obtain three-dimensional plots and contour plots of the lump solution to Eq. (3) as plotted in Fig. 1. The images show that the wave keeps its shape unchanged and moves in a straight line during its motion.

Case 2

$$\begin{aligned}
&a_1 = a_1, \quad a_2 = a_2, \quad b_1 = \frac{3c_1a_1 - 2a_1d_1 - 2c_1d_1}{2d_1},\\
&b_2 = -\frac{a_1\left(3c_1a_1 - 2a_1d_1 - 2c_1d_1\right)}{2a_2d_1},\\
&c_1 = c_1,\\
&c_2 = \frac{3c_1a_1^2 - 2a_1^2d_1 - 2a_1c_1d_1 + 3c_1a_2^2 - 2a_2^2d_1}{2},\\
&w_1 = w_1, \quad d_1 = d_1,\\
&d_2 = \frac{3c_1a_1^2 - 2a_1^2d_1 - 2a_1c_1d_1 + 3c_1a_2^2 - 2a_2^2d_1}{2c_1a_2},\\
&w_2 = w_2, \quad l = l,
\end{aligned} \tag{13}$$

where $a_2d_1c_1 \neq 0$.

Case 3

$$\begin{aligned}
&a_1 = a_1, \quad a_2 = a_2, \quad l = l, \quad b_1 = \frac{2a_2^4d_1 - 3(a_1 - \frac{2d_1}{3})c_2a_2^3 + 2a_1^2a_2^2d_1 - 3(a_1 - \frac{2d_1}{3})c_2(a_1^2 + a_3)a_2 - 2a_1^2a_3d_1}{2d_1a_1(a_1^2 + a_2^2 + a_3)},\\
&b_2 = \frac{3a_1c_2 - 2a_2d_1 - 2d_1c_2}{2d_1}, \quad c_1 = -\frac{a_1^4 + 2a_1^2a_2^2 + c_2a_2a_1^2 + a_2^4 + c_2a_2^3 + a_2a_3c_2}{a_1\left(a_1^2 + a_2^2 + a_3\right)},\\
&c_2 = c_2, \quad d_1 = d_1, \quad d_2 = \frac{d_1\left(a_1^4 + 2a_1^2a_2^2 + c_2a_2a_1^2 + a_2^4 + c_2a_2^3 + a_2a_3c_2\right)}{a_1c_2\left(a_1^2 + a_2^2 + a_3\right)},\\
&w_1 = w_1, \quad w_2 = \frac{\left(a_1^4 + 2a_1^2a_2^2 + c_2a_2a_1^2 + a_2^4 + c_2a_2^3 + a_2a_3c_2\right)w_1}{a_1c_2\left(a_1^2 + a_2^2 + a_3\right)},
\end{aligned} \tag{14}$$

where $a_1c_2d_1(a_1^2 + a_2^2 + a_3) \neq 0$.

Case 4

$$\begin{aligned}
&a_1 = a_1, \quad a_2 = \frac{a_1d_2}{d_1}, \quad l = l, \quad b_1 = \frac{3c_1a_1 - 2a_1d_1 - 2c_1d_1}{2d_1},\\
&b_2 = -\frac{3c_1a_1 - 2a_1d_1 - 2c_1d_1}{2d_2},\\
&c_1 = c_1, \quad c_2 = -\frac{3a_1c_1d_1 - 2a_1d_1^2 - 2a_1d_2^2 - 2c_1d_1^2}{d_2\left(3a_1 - 2d_1\right)},\\
&d_1 = d_1, \quad d_2 = d_2, \quad w_1 = w_1, \quad w_2 = w_2,
\end{aligned} \tag{15}$$

where $d_1d_2\left(3a_1 - 2d_1\right) \neq 0$.

Cases 2–4 generate three types of solutions f_i $(i = 2,3,4)$ defined by Eq. (8). And under the transformation Eq. (4), the general solution of Eq. (3) is given by

$$u = \frac{4(a_1g + a_2h)}{g^2 + h^2 + l}. \tag{16}$$

As a result, three types of lump solitons u_i $(i = 2,3,4)$ can be generated, which will not be described in detail here.

4. Collision of lump wave and a strip soliton

Here, we search for the lump-kink wave solution by using the form of combined function with an exponential function and the periodic one. Therefore, we suppose

$$f = g^2 + h^2 + l + \lambda \mathrm{e}^{(a_3x + b_3y + c_3z + d_3t)}, \tag{17}$$

where g and h are given by Eq. (8), and the parameters λ, a_3, b_3, c_3, d_3, and l are arbitrary constants. We directly substitute Eq. (17) into Eq. (6) and derive the following three sets of parameters.

Case 1

$$\begin{aligned}
&a_1 = -\frac{c_1\left(a_3^4 + 6a_3^2 + 9\right)}{a_3^2\left(a_3^2 + 6\right)}, \quad a_2 = 0, \quad a_3 = a_3, \quad l = \frac{6\left(a_3^2 + 3\right)^2\left(a_3^4 + 6a_3^2 + 9\right)c_1^2}{\left(a_3^2 + 6\right)^3a_3^6},\\
&b_1 = \frac{6c_1}{a_3^2\left(a_3^2 + 6\right)}, \quad b_2 = \frac{3a_3^8c_1^2 + 36a_3^6c_1^2 + 162a_3^4c_1^2 - 4a_3^4d_2^2 + 324a_3^2c_1^2 - 24a_3^2d_2^2 + 243c_1^2}{2(a_3^2 + 3)^2(a_3^2 + 6)a_3^2d_2}, \quad b_3 = -\frac{2a_3}{a_3^2 + 3},\\
&c_1 = c_1, \quad c_2 = \frac{2d_2}{a_3^4 + 6a_3^2 + 9}, \quad c_3 = -\frac{a_3^3}{a_3^2 + 3}, \quad d_1 = 0, \quad d_2 = d_2, \quad d_3 = -\frac{a_3^3}{2}, \quad w_1 = w_1, \quad w_2 = w_2,
\end{aligned} \tag{18}$$

where a_3, c_1, d_2, w_1, w_2 are real constants and conditions are given as

$$a_3 \neq 0, \quad d_2 \neq 0, \quad l > 0, \tag{19}$$

to guarantee the well-definedness of f. Then the exact interaction solution of u is expressed as follows:

$$u = \frac{4a_1A_2 + 4a_2B_2 + 2a_3\lambda\,\mathrm{e}^{C_2}}{A_2^2 + B_2^2 + l + \lambda\mathrm{e}^{C_2}}, \tag{20}$$

where

$$
\begin{aligned}
A_2 &= -\frac{c_1\left(a_3^4+6a_3^2+9\right)}{a_3^2\left(a_3^2+6\right)}x+\frac{6c_1}{a_3^2\left(a_3^2+6\right)}y+c_1z+w_1,\\
B_2 &= \frac{3a_3^8c_1^2+36a_3^6c_1^2+162a_3^4c_1^2-4a_3^4d_2^2+324a_3^2c_1^2-24a_3^2d_2^2+243c_1^2}{2(a_3^2+3)^2(a_3^2+6)a_3^2d_2}y+\frac{2d_2}{a_3^4+6a_3^2+9}z+d_2t+w_2,\\
C_2 &= a_3x-\frac{2a_3}{a_3^2+3}y-\frac{a_3^3}{a_3^2+3}z-\frac{a_3^3}{2}t.
\end{aligned}
\tag{21}
$$

When the selection of parameters are $\lambda=2$, $a_3=2$, $c_1=5$, $d_2=3$, $w_1=-5$, $w_2=0$. The interaction between a lump and a soliton reads

$$
u_1=\frac{2(\frac{2401x}{32}-\frac{147y}{16}-\frac{245z}{4}+\frac{245}{4}+4\mathrm{e}^{2x-\frac{4y}{7}-\frac{8z}{7}-4t})}{(-\frac{49x}{8}+\frac{3y}{4}+5z-5)^2+(\frac{11909y}{784}+\frac{6z}{49}+3t)^2+\frac{7203}{1280}+2\mathrm{e}^{2x-\frac{4y}{7}-\frac{8z}{7}-4t}}.
\tag{22}
$$

When the selection of parameters are $\lambda=2$, $a_3=-2$, $c_1=-5$, $d_2=3$, $w_1=-5$, $w_2=0$. The interaction between a lump and a soliton reads

$$
u_2=\frac{2(\frac{2401x}{32}-\frac{147y}{16}-\frac{245z}{4}-\frac{245}{4}-4\mathrm{e}^{-2x+\frac{4y}{7}+\frac{8z}{7}+4t})}{(\frac{49x}{8}-\frac{3y}{4}-5z-5)^2+(\frac{11909y}{784}+\frac{6z}{49}+3t)^2+\frac{7203}{1280}+2\mathrm{e}^{-2x+\frac{4y}{7}+\frac{8z}{7}+4t}}.
\tag{23}
$$

The interaction between a lump and a soliton is shown for Eq. (20) in Figs. 2 and 3 with different time evolution. In these figures, we can see that lump-kink solution reflects the fusion and fission phenomenon of lump and kink waves. Since both the numerator and the denominator contain $\mathrm{e}^{\pm(2x-\frac{4y}{7}-\frac{8z}{7}-4t)}$, so u will approach to a constant when (x,y) is fixed as t goes to ∞. From the structure of the solution, we find that

$$
\begin{aligned}
&\text{(i) if}\quad d_3<0,\quad \lim_{t\to+\infty}u(x,y,z,t)=0,\\
&\lim_{t\to-\infty}u(x,y,z,t)=2a_3,
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&\text{(ii) if}\quad d_3>0,\quad \lim_{t\to+\infty}u(x,y,z,t)=2a_3,\\
&\lim_{t\to-\infty}u(x,y,z,t)=0.
\end{aligned}
\tag{25}
$$

In Fig. 2, the fission phenomenon of the lump-type soliton is shown from the algebraic-exponential solution (22). At time $t=-5$, only a kink wave exists. With time t, the lump-type soliton is separated from the mixed type solitary wave. Finally, the lump and kink waves will move independently.

As seen in Fig. 3, we can clearly observe this fusion phenomenon. The lump wave first appears on one side of the kink wave at time $t=-3$. Then, it begins to move toward the kink wave. When $t=10$, the lump-type soliton is almost completely absorbed by stripe soliton. Finally, only the kinky wave exists.

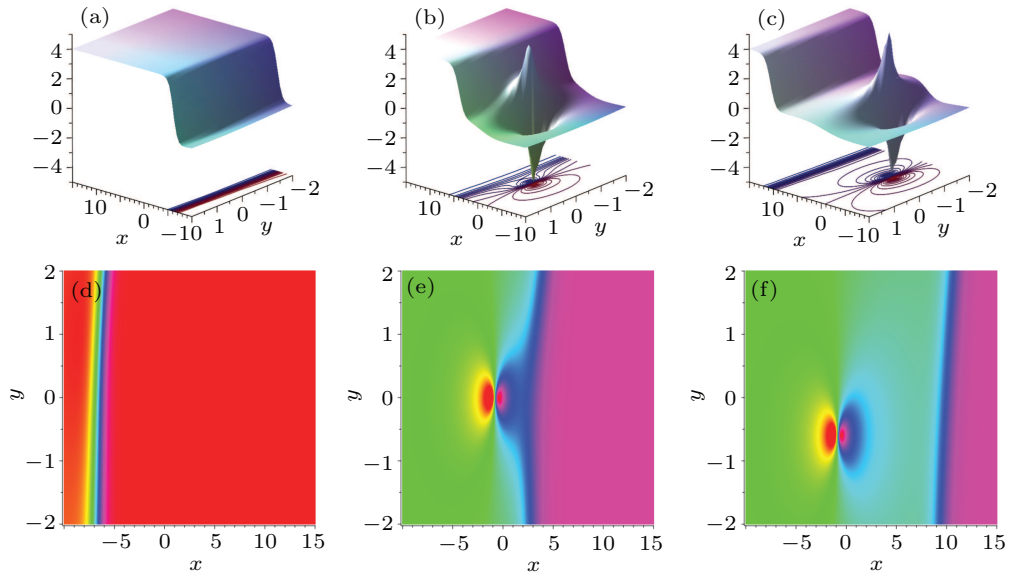


Fig. 2. The spatial structure of fission of Eq. (22) with $z=0$. [(a), (d)] Three-dimensional contour and density plot with time $t=-5$. [(b), (e)] Three-dimensional contour and density plot with time $t=0$. [(c), (f)] Three-dimensional contour and density plot with time $t=3$.

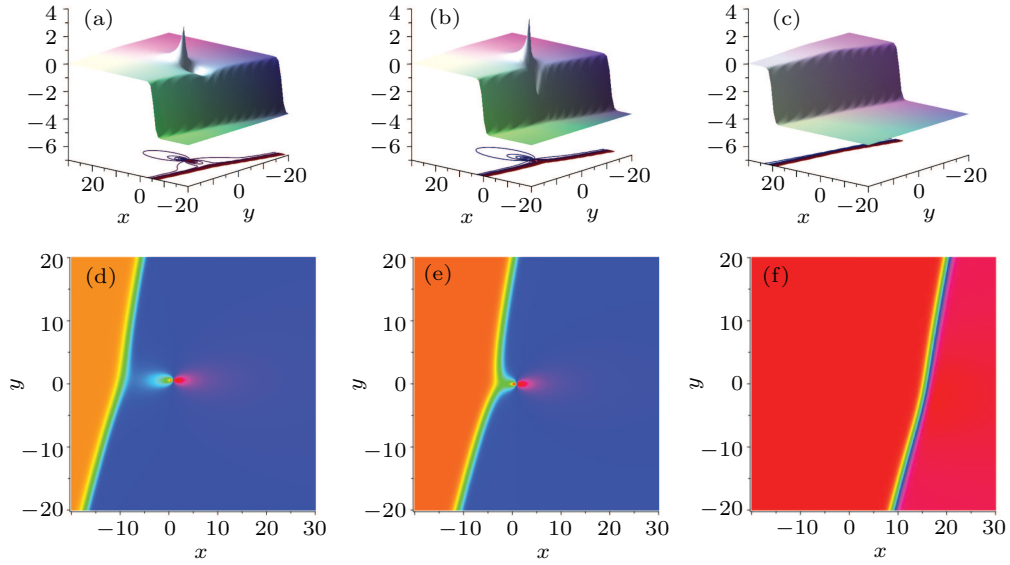


Fig. 3. The spatial structure of fusion of Eq. (23) with $z = 0$. [(a), (d)] Three-dimensional contour and density plot with $t = -3$. [(b), (e)] Three-dimensional contour and density plot with $t = 0$. [(c), (f)] Three-dimensional contour and density plot with $t = 10$.

Case 2

$$
\begin{aligned}
&a_1 = a_1, \quad a_2 = 0, \quad a_3 = a_3,\\
&l = -\frac{\left(27a_1^2a_3^2 + 4a_3^2d_2^2 + 81a_1^2 - 36d_2^2\right)a_1^2}{4d_2^2a_3^2\left(a_3^2+9\right)},\\
&b_1 = \frac{a_1(27a_1^2a_3^2 + 4a_3^2d_2^2 + 81a_1^2 - 36d_2^2)}{(a_3^2+3)^2(9a_1^2+4d_2^2)},\\
&b_2 = -\frac{4d_2(-4a_3^2d_2^2 + 81a_1^2)}{3(a_3^2+3)^2(9a_1^2+4d_2^2)}, \quad b_3 = -\frac{3a_3}{a_3^2+3},\\
&c_1 = -\frac{4a_1a_3^2d_2^2\left(a_3^2+9\right)}{\left(a_3^2+3\right)^2\left(9a_1^2+4d_2^2\right)}, \quad d_1 = 0, \quad d_2 = d_2,\\
&d_3 = \frac{3a_3}{2}, \quad c_2 = \frac{2d_2(9a_1^2a_3^4 + 81a_1^2a_3^2 - 8a_3^2d_2^2 + 162a_1^2)}{3(a_3^2+3)^2(9a_1^2+4d_2^2)},\\
&c_3 = -\frac{\left(27a_1^2a_3^2 + 20a_3^2d_2^2 - 162a_1^2\right)a_3}{3\left(a_3^2+3\right)\left(9a_1^2+4d_2^2\right)},\\
&w_1 = w_1, \quad w_2 = w_2,
\end{aligned}
\tag{26}
$$

where a_1, a_3, d_2, w_1, and w_2 are real constants, and the conditions satisfy $a_1 \neq 0$, $a_3 \neq 0$, $d_2 \neq 0$, and $l > 0$.

Case 3

$$
\begin{aligned}
&a_1 = a_1, \quad a_2 = a_2, \quad a_3 = a_3, \quad l = \frac{-\left(a_3^2+3\right)^2a_2^4 - a_1^2a_3^2\left(a_3^2+6\right)a_2^2 + 9a_1^4}{\left(2a_3^2+9\right)a_3^2a_1^2},\\
&b_1 = \frac{3\left(a_3^2+3\right)^3a_1a_2^2 - 15a_1^3\left(a_3^2+\frac{9}{5}\right)}{\left(a_3^2+3\right)^2\left(2a_1^2a_3^2 - a_2^2a_3^2 + 3a_1^2 - 3a_2^2\right)}, \quad b_2 = \frac{\left(a_3^2+3\right)^3a_2^4 - 2\left(a_3^4+6a_3^2+\frac{27}{2}\right)\left(a_3^2+3\right)a_1^2a_2^2 + 6a_1^4a_3^2}{a_2\left(a_3^2+3\right)^2\left(2a_1^2a_3^2 - a_2^2a_3^2 + 3a_1^2 - 3a_2^2\right)},\\
&b_3 = -\frac{a_3\left(a_2^2a_3^4 + 4a_1^2a_3^2 + 6a_2^2a_3^2 + 9a_1^2 + 9a_2^2\right)}{\left(a_3^2+3\right)\left(2a_1^2a_3^2 - a_2^2a_3^2 + 3a_1^2 - 3a_2^2\right)}, \quad c_1 = -\frac{a_3^2a_1\left(2a_1^2a_3^2 + 2a_2^2a_3^2 + 9a_1^2 + 9a_2^2\right)}{\left(a_3^2+3\right)\left(2a_1^2a_3^2 - a_2^2a_3^2 + 3a_1^2 - 3a_2^2\right)},\\
&c_2 = 0, \quad c_3 = -\frac{a_3^3}{a_3^2+3}, \quad d_1 = 0, \quad d_2 = \frac{a_2\left(2a_3^2+9\right)\left(a_1^2+a_2^2\right)\left(a_3^2+3\right)a_3^2}{\left(-2a_3^4 - 18a_3^2 - 36\right)a_2^2 + 4a_1^2a_3^2}, \quad d_3 = -\frac{a_3^3}{2}, \quad w_1 = w_1, \quad w_2 = w_2,
\end{aligned}
\tag{27}
$$

where a_1, a_2, a_3, w_1, and w_2 are real constants and the conditions satisfy $a_1a_2a_3 \neq 0$, $2a_1^2a_3^2 - a_2^2a_3^2 + 3a_1^2 - 3a_2^2 \neq 0$, $\left(-2a_3^4 - 18a_3^2 - 36\right)a_2^2 + 4a_1^2a_3^2 \neq 0$, and $l > 0$.

5. Interaction between a lump and two-soliton solution

Similarly, by adding an exponential function to the above formula of f, we can obtain the interaction solution between a lump wave and the two-kink soliton. Take f as the combination of positive quadratic function and two exponential functions, the new solution has the following form:

$$
\begin{aligned}
f \;=\; & g^2 + h^2 + l + \theta \cdot \mathrm{e}^{(a_3x + b_3y + c_3z + d_3t)}\\
& + \omega \cdot \mathrm{e}^{-(a_3x + b_3y + c_3z + d_3t)},
\end{aligned}
\tag{28}
$$

where g and h are given by Eq. (8), and the θ, ω, a_3, b_3, c_3, d_3, and l are arbitrary constants. Substituting Eq. (28) into Eq. (6) and collecting the coefficients, we derive the following set of parameters:

$$
l = l, \quad a_1 = \varepsilon a_3, \quad a_2 = 0, \quad a_3 = a_3,
$$

$$b_1=-\varepsilon\frac{a_3^3}{3\left(a_3^2+3\right)},\quad b_2=b_2,\quad b_3=\frac{a_3^3}{3\left(a_3^2+3\right)},$$
$$c_1=0,\quad c_2=0,\quad c_3=-\frac{a_3^3}{a_3^2+3},\quad d_1=0,\quad d_2=0,$$
$$d_3=-\frac{a_3^3}{2},\quad w_1=w_1,\quad w_2=w_2, \tag{29}$$

where a_3, b_2, l, w_1, and w_2 are real free parameters, and $\varepsilon=\pm1$, which needs to satisfy $\theta>0$ and $\omega>0$. Taking Eqs. (29) and (28) into the transformation $u=2(lnf)_x$, we can obtain the solution of u as follows:

$$u=\frac{4a_1A_3+4a_2B_3+2a_3(\theta\,\mathrm{e}^{C_3}-\omega\,\mathrm{e}^{-C_3})}{A_3^2+B_3^2+l+\theta\,\mathrm{e}^{C_3}+\omega\,\mathrm{e}^{-C_3}}, \tag{30}$$

where

$$A_3=\varepsilon a_3x-\varepsilon\frac{a_3^3}{3\left(a_3^2+3\right)}y+w_1,$$
$$B_3=b_2y+w_2,$$
$$C_3=a_3x+\frac{a_3^3}{3\left(a_3^2+3\right)}y-\frac{a_3^3}{a_3^2+3}z-\frac{a_3^3}{2}t. \tag{31}$$

To illustrate the interaction solution between a lump wave and two-kink soliton solutions, we select the parameters $\theta=1/20$, $\omega=1/20$, $l=0$, $a_3=-1/2$, $b_2=4$, $w_1=0$, $w_2=0$, and $\varepsilon=1$. The interaction between a lump and two-soliton solution reads

$$u=\frac{2[\frac{1}{2}x-\frac{1}{78}y-\frac{1}{40}(\mathrm{e}^{-\frac{1}{2}x-\frac{1}{78}y+\frac{1}{26}z+\frac{1}{16}t}+\mathrm{e}^{\frac{1}{2}x+\frac{1}{78}y-\frac{1}{26}z-\frac{1}{16}t})]}{(-\frac{1}{2}x+\frac{1}{78}y)^2+16y^2+\frac{1}{20}(\mathrm{e}^{-\frac{1}{2}x-\frac{1}{78}y+\frac{1}{26}z+\frac{1}{16}t}+\mathrm{e}^{\frac{1}{2}x+\frac{1}{78}y-\frac{1}{26}z-\frac{1}{16}t})}. \tag{32}$$

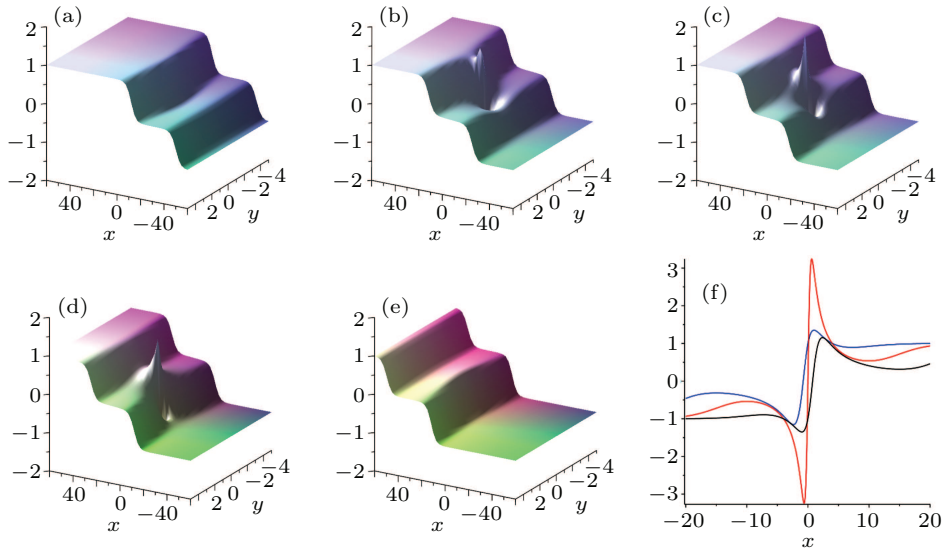


Fig. 4. Evolution plots of Eq. (32) with $z=0$ at (a) $t=-240$, (b) $t=-50$, (c) $t=0$, (d) $t=50$, (e) $t=240$, (f) the height of relevant waves in $y=0$ plane at $t=-50$ (blue), $t=0$ (red), and $t=50$ (black).

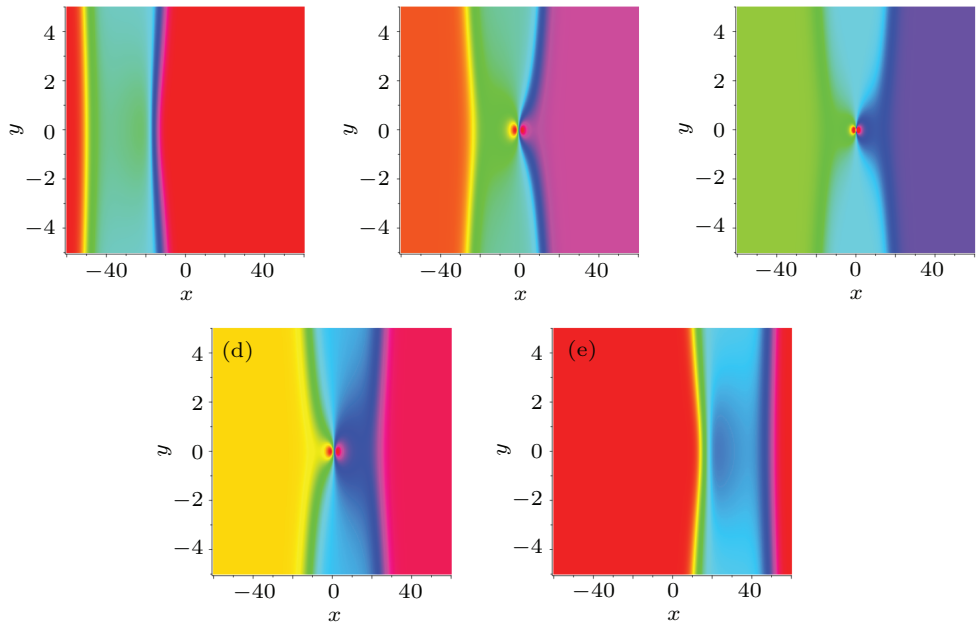


Fig. 5. Density plots of Eq. (32) with $z=0$ at (a) $t=-240$, (b) $t=-50$, (c) $t=0$, (d) $t=50$, (e) $t=240$.

From the equation, we can see that $d_1 = d_2 = 0$, so the rational function part of u is independent of t. This means that the lump soliton of the combinatorial solution is not moving with time. Noticeably, $d_3 \neq 0$ causes the collision phenomenon. The interaction between a lump and two-soliton u is presented in Fig. 4, in which (a) and (e) illustrate that there are two stripe solitons with invariable propagation, but (a) displays that the lump soliton has not appeared, (e) indicates that the lump soliton has disappeared. From (b)–(d), we can see that the lump wave starts to appear and slowly separates from the kinky wave. When $t = 0$, its amplitude reaches its maximum. With time, the lump wave moves to the other side of the kinky wave, while the amplitude gradually decreases. Eventually, Fig. 4(e) indicates that the lump wave is phased out and the amplitude is reduced to 0. Figure 4(f) shows the height in $y = 0$ at different times. Figure 5 gives the corresponding density plots.

6. Lump-periodic solution

In this section, we consider the interaction between lump waves and triangular periodic waves, then Eq. (3) can be assumed as follows:

$$f = g^2 + h^2 + l + \lambda \cos(a_3 x + b_3 y + c_3 z + d_3 t), \tag{33}$$

where g and h are given by Eq. (8), and the parameters λ, a_3, b_3, c_3, d_3, and l are arbitrary constants. We obtain the following sets of parameters by removing coefficients of $\cos(a_3 x + b_3 y + c_3 z + d_3 t)$, $\sin(a_3 x + b_3 y + c_3 z + d_3 t)$, x, y, z, and t after substituting Eq. (33) into Eq. (6):

$$\begin{aligned}
&a_1 = a_1, \quad l = -\frac{36a_1^6a_3^3d_3^2 + 12a_1^4a_3^5d_2^2 - 9a_1^2a_3^7d_3^2 + 3a_3^9d_2^2 - 8a_3^8d_2^2d_3 - 72a_1^6d_3^3 + 72a_1^4a_3^2d_2^2d_3 + 18a_1^2a_3^4d_3^3 + 18a_3^6d_2^2d_3}{4d_2^2\left(3a_3^3 - 4a_3^2d_3 + 18d_3\right)a_3^4a_1^2},\\
&a_2 = 0, \quad a_3 = a_3, \quad b_1 = \frac{36a_1^3d_3^2\left(a_3^3 - 2d_3\right) + 12a_1a_3^2d_2^2\left(a_3^3 + 6d_3\right)}{A}, \quad b_2 = \frac{8a_3d_2\left(9a_1^2a_3^3d_3 - 6a_1^2a_3^2d_3^2 + 2a_3^4d_2^2 + 18a_1^2d_3^2\right)}{A},\\
&b_3 = -\frac{4\left(9a_1^2a_3^3d_3^2 - 3a_3^5d_2^2 + 8a_3^4d_2^2d_3 - 18a_1^2d_3^3 - 18a_3^2d_2^2d_3\right)a_3}{A}, \quad c_1 = -\frac{4a_1a_3^4d_2^2\left(3a_3^3 - 4a_3^2d_3 + 18d_3\right)}{A},\\
&c_2 = \frac{2d_2a_3\left(9a_1^2a_3^6 - 12a_1^2a_3^5d_3 + 18a_1^2a_3^3d_3 + 24a_1^2a_3^2d_3^2 - 8a_3^4d_2^2 - 72a_1^2d_3^2\right)}{A}, \quad w_1 = w_1, \quad w_2 = w_2,\\
&c_3 = \frac{-4a_3^8d_2^2 + d_3\left(18a_1^2 + 16d_2^2\right)a_3^7 - 40a_3^5d_2^2d_3 + 36a_1^2a_3^4d_3^2 - 144a_1^2a_3d_3^3}{A}, \quad d_1 = 0, \quad d_2 = d_2, \quad d_3 = d_3,
\end{aligned} \tag{34}$$

where $A = 27a_1^2a_3^7 - 36a_1^2a_3^6d_3 + 12a_3^7d_2^2 - 16a_3^6d_2^2d_3 + 54a_1^2a_3^4d_3 + 36a_1^2a_3^3d_3^2 - 36a_3^5d_2^2 + 72a_3^4d_2^2d_3 - 216a_1^2a_3d_3^2 + 72a_1^2d_3^3 - 72a_3^2d_2^2d_3$. The parameters satisfy $A \neq 0$, $3a_3^3 - 4a_3^2d_3 + 18d_3 \neq 0$, $a_3 \neq 0$, and $l > |\lambda|$. Putting the above values into Eq. (33) and using the transformation $u = 2(\ln f)_x$, we obtain

$$u = \frac{4a_1A_4 + 4a_2B_4 - 2a_3\lambda \sin C_4}{A_4^2 + B_4^2 + l + \lambda \cos C_4}, \tag{35}$$

where

$$\begin{aligned}
A_4 &= a_1x + \frac{36a_1^3d_3^2\left(a_3^3 - 2d_3\right) + 12a_1a_3^2d_2^2\left(a_3^3 + 6d_3\right)}{A}y - \frac{4a_1a_3^4d_2^2\left(3a_3^3 - 4a_3^2d_3 + 18d_3\right)}{A}z + w_1,\\
B_4 &= \frac{8a_3d_2\left(9a_1^2a_3^3d_3 - 6a_1^2a_3^2d_3^2 + 2a_3^4d_2^2 + 18a_1^2d_3^2\right)}{A}y\\
&\quad + \frac{2d_2a_3\left(9a_1^2a_3^6 - 12a_1^2a_3^5d_3 + 18a_1^2a_3^3d_3 + 24a_1^2a_3^2d_3^2 - 8a_3^4d_2^2 - 72a_1^2d_3^2\right)}{A}z + d_2t + w_2,\\
C_4 &= a_3x - \frac{4\left(9a_1^2a_3^3d_3^2 - 3a_3^5d_2^2 + 8a_3^4d_2^2d_3 - 18a_1^2d_3^3 - 18a_3^2d_2^2d_3\right)a_3}{A}y\\
&\quad + \frac{-4a_3^8d_2^2 + d_3\left(18a_1^2 + 16d_2^2\right)a_3^7 - 40a_3^5d_2^2d_3 + 36a_1^2a_3^4d_3^2 - 144a_1^2a_3d_3^3}{A}z + d_3t.
\end{aligned} \tag{36}$$

It is easy to see that $a_3 \neq 0$, $b_3 \neq 0$, and some interesting phenomenon can also be observed. With the increasing coefficient λ, the trigonometric functions dominate the values of coefficients in quadratic functions. The 3D contours and density plots for the interaction between a lump and triangular periodic waves of Eq. (35) are depicted in Fig. 6. The specific parameters are chosen as $\lambda = 1$, $a_1 = -1/2$, $a_3 = 2$, $d_2 = 3$, $d_3 = 3$, $w_1 = 0$, $w_2 = 0$, and $w_3 = 0$.

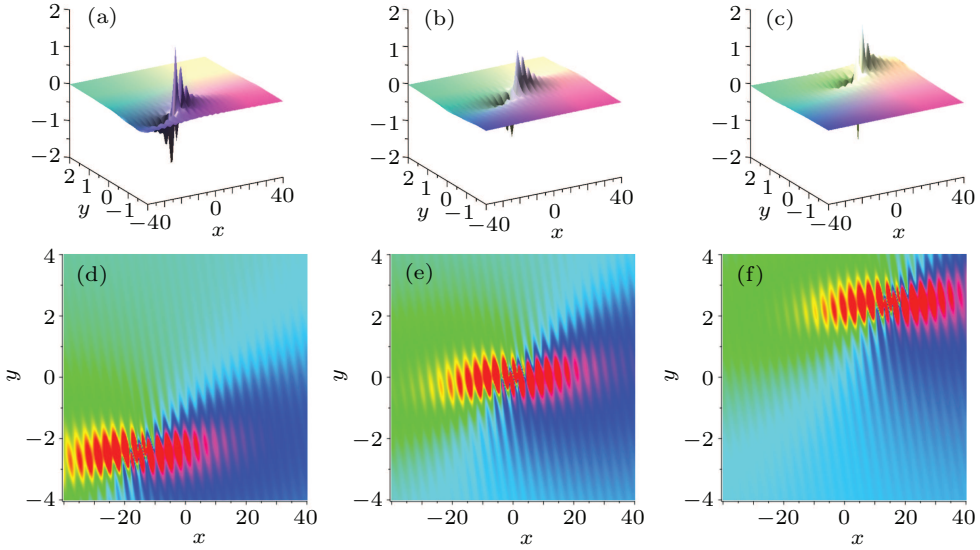


Fig. 6. Profile of Eq. (35) with $z=0$. [(a), (d)] Three-dimensional contour and density plot with time $t=-7$. [(b), (e)] Three-dimensional contour and density plot with $t=0$. [(c), (f)] Three-dimensional contour and density plot with $t=7$.

7. Interaction solution between lump waves and triangular periodic waves

To attain interaction between lump waves and triangular periodic waves, we apply the form of a quadratic function with sine and cosine functions:

$$\begin{aligned} f \;=\; & g^2+h^2+l+\theta\cos(a_3x+b_3y+c_3z+d_3t) \\ & +\omega\sin(a_3x+b_3y+c_3z+d_3t), \end{aligned} \tag{37}$$

where g and h are given by Eq. (8), and the parameters θ, ω, a_3, b_3, c_3, d_3, and l are arbitrary constants. By substituting Eq. (37) into Eq. (6) and gathering the coefficients of x, y, z, t, cos, and sin, the parameters are determined as follows:

Case 1

$$\begin{aligned} & l=l, \quad a_1=0, \quad a_2=0, \quad a_3=a_3, \\ & b_1=-\frac{(3a_3-2d_3)\,c_1}{a_3^3-2d_3}, \quad b_2=-\frac{c_2\,(3a_3-2d_3)}{a_3^3-2d_3}, \\ & b_3=0, \quad c_1=c_1, \quad c_2=c_2, \quad c_3=\frac{2a_3d_3}{3a_3-2d_3}, \\ & d_1=0, \quad d_2=0, \quad d_3=d_3, \end{aligned}$$

$$w_1=-\frac{c_2w_2}{c_1}, \quad w_2=w_2, \tag{38}$$

under the condition $a_3^3-2d_3\neq 0, 3a_3-2d_3\neq 0$, then the function f is positive and well defined. The interaction between the lump and triangular periodic solution is expressed as

$$u=\frac{4a_1A_5+4a_2B_5-2a_3\theta\sin C_6+2a_3\omega\cos C_6}{A_5^2+B_5^2+l+\theta\cos C_5+\omega\sin C_5}, \tag{39}$$

where

$$\begin{aligned} A_5 &= -\frac{(3a_3-2d_3)\,c_1}{a_3^3-2d_3}y+c_1z-\frac{c_2w_2}{c_1}, \\ B_5 &= -\frac{c_2\,(3a_3-2d_3)}{a_3^3-2d_3}y+c_2z+w_2, \\ C_5 &= a_3x+\frac{2a_3d_3}{3a_3-2d_3}z+d_3t. \end{aligned} \tag{40}$$

In order to analyze the dynamic behaviors of the solution (39), we let $\theta=2$, $\omega=2$, $l=1$, $a_3=-5$, $c_1=2$, $c_2=2$, $d_3=3$, and $w_2=2$. Then we can reach

$$u=\frac{2\left[10\sin\left(3t-5x+\frac{10z}{7}\right)-10\cos\left(3t-5x+\frac{10z}{7}\right)\right]}{\left(-\frac{42y}{131}+2z-2\right)^2+\left(-\frac{42y}{131}+2z+2\right)^2+1+2\cos\left(3t-5x+\frac{10z}{7}\right)+2\sin\left(3t-5x+\frac{10z}{7}\right)}. \tag{41}$$

The interaction between the lump and triangular periodic waves of u is presented in Fig. 7, in (x,y), (x,z) and (y,z) planes, with three-dimensional contours and density plots.

Case 2

$$\begin{aligned} & b_1=-\frac{(3a_3-2d_3)\,c_1}{a_3^3-2d_3}, \quad b_2=b_2, \quad b_3=0, \quad c_1=c_1, \\ & c_2=-\frac{b_2\left(a_3^3-2d_3\right)}{3a_3-2d_3}, \quad c_3=\frac{2a_3d_3}{3a_3-2d_3}, \\ & d_1=0, \quad d_2=0, \quad d_3=d_3, \quad w_1=0, \quad w_2=0, \end{aligned} \tag{42}$$

$$l=l, \quad a_1=0, \quad a_2=0, \quad a_3=a_3,$$

where $a_3^3-2d_3\neq 0$, and $3a_3-2d_3\neq 0$.

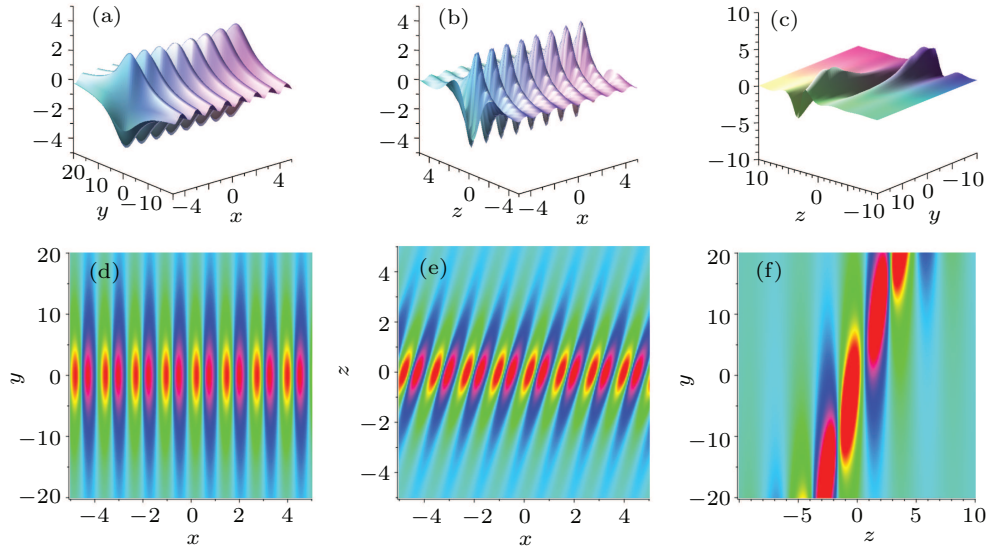


Fig. 7. Profile of Eq. (39). [(a), (d)] Three-dimensional contour and density plot at (x,y) plane with $t=0$ and $z=0$. [(b), (e)] Three-dimensional contour and density plot at (x,z) plane with $t=0$ and $y=0$. [(c), (f)] Three-dimensional contour and density plot at (y,z) plane with $t=0$ and $x=0$.

8. Periodic-lump wave solution

In this section, we construct the periodic lump solution for Eq. (3). More specifically, we assume f in Eq. (6) as the ansatz of a linear combination of trigonometric and hyperbolic functions:

$$\begin{aligned} f &= \theta\cosh(a_1x+b_1y+c_1z+d_1t+w_1) \\ &\quad +\omega\cos(a_2x+b_2y+c_2z+d_2t+w_2), \end{aligned} \tag{43}$$

where a_i, b_i, c_i, d_i, w_i, $(i=1,2)$, θ, and ω are real parameters to be determined. Substituting Eq. (43) into Eq. (6) and collecting all the coefficients about $\cosh(a_1x+b_1y+c_1z+d_1t+w_1)$, $\sinh(a_1x+b_1y+c_1z+d_1t+w_1)$, $\cos(a_2x+b_2y+c_2z+d_2t+w_2)$, and $\sin(a_2x+b_2y+c_2z+d_2t+w_2)$, we can obtain a set of determining equations for the above parameters. Solving these equations, we obtain the following solutions:

$$\begin{aligned}
\omega &= \omega,\quad \theta=\theta,\quad a_1=a_1,\quad a_2=a_2,\quad b_1=\frac{3\omega^2a_1^2a_2b_2+3\omega^2a_2^3b_2-3\omega^2a_1^2+3\theta^2a_1^2}{-a_1B},\quad b_2=b_2,\\
c_1 &= \{(-\omega^2+4\theta^2)b_2a_1^6+[-3b_2(\omega^2+4\theta^2)a_2^2+(6\omega^2+3\theta^2)a_2+(-6\omega^2+15\theta^2)b_2-2\omega^2d_2+8\theta^2d_2]a_1^4+[-15\omega^2a_2^4b_2+(6\omega^2+3\theta^2)a_2^3\\
&\quad +[(18\omega^2-9\theta^2)b_2+6\omega^2d_2]a_2^2+(-6\omega^2b_2d_2-9\omega^2+9\theta^2)a_2+(-9\omega^2+9\theta^2)b_2]a_1^2+3\omega^2a_2^3b_2(a_2^3-2d_2)\}\left[a_1B(3a_2-2d_2)\right]^{-1},\\
c_2 &= \{-12\omega^2a_2^5b_2+[((4\omega^2-4\theta^2)a_1^2+12\omega^2-3\theta^2)b_2+6\omega^2d_2]a_2^3+(6\omega^2b_2d_2-9\theta^2a_1^2)a_2^2+[(12\theta^2a_1^4+9\theta^2a_1^2)b_2-2[(\omega^2-4\theta^2)a_1^2\\
&\quad +3\omega^2-3\theta^2]d_2]a_2-2[(\omega^2-4\theta^2)a_1^2+3\omega^2-3\theta^2]d_2b_2-9\theta^2a_1^4\}\cdot\left[B(3a_2-2d_2)\right]^{-1},\\
d_1 &= \frac{3a_1}{2},\quad d_2=d_2,\quad w_1=w_1,\quad w_2=w_2,
\end{aligned} \tag{44}$$

where $B=\left(-\omega^2a_1^2+3\omega^2a_2^2+4\theta^2a_1^2-3\omega^2+3\theta^2\right)$, and parameters ω, θ, a_1, a_2, b_2, d_2, w_1, and w_2 are free real constants. The solution should satisfy $a_1B(3a_2-2d_2)\neq 0$, and $|\theta|>|\omega|$ needs to be imposed for the non-singular solution. Substituting these results into Eq. (43), we obtain the periodic lump solution as follow:

$$u=\frac{2a_1\theta\sinh A_6-2a_2\omega\sin B_6}{\theta\cosh A_6+\omega\cos B_6}, \tag{45}$$

where

$$\begin{aligned}
A_6 &= a_1x+\frac{3\omega^2a_1^2a_2b_2+3\omega^2a_2^3b_2-3\omega^2a_1^2+3\theta^2a_1^2}{-a_1B}y+\{(-\omega^2+4\theta^2)b_2a_1^6-[3b_2(\omega^2+4\theta^2)a_2^2-(6\omega^2+3\theta^2)a_2+(6\omega^2-15\theta^2)b_2\\
&\quad +2\omega^2d_2-8\theta^2d_2]a_1^4+[-15\omega^2a_2^4b_2+(6\omega^2+3\theta^2)a_2^3+((18\omega^2-9\theta^2)b_2+6\omega^2d_2)a_2^2-(6\omega^2b_2d_2+9\omega^2-9\theta^2)a_2\\
&\quad -(9\omega^2-9\theta^2)b_2]a_1^2+3\omega^2a_2^3b_2(a_2^3-2d_2)\}z\cdot\left[a_1B(3a_2-2d_2)\right]^{-1}+\frac{3a_1}{2}t+w_1,\\
B_6 &= a_2x+b_2y+\{-12\omega^2a_2^5b_2+[((4\omega^2-4\theta^2)a_1^2+12\omega^2-3\theta^2)b_2+6\omega^2d_2]a_2^3+(6\omega^2b_2d_2-9\theta^2a_1^2)a_2^2+\left[(12\theta^2a_1^4+9\theta^2a_1^2)b_2\right.\\
&\quad \left.-2[(\omega^2-4\theta^2)a_1^2+3\omega^2-3\theta^2]d_2\right]a_2-2[(\omega^2-4\theta^2)a_1^2+3\omega^2-3\theta^2]d_2b_2-9\theta^2a_1^4\}z\cdot\left[B(3a_2-2d_2)\right]^{-1}+d_2t+w_2.
\end{aligned} \tag{46}$$

In order to draw the graph and to analyze the dynamic behaviors of periodic lump solution, we choose the parameters $\theta=2$,

$\omega = 1$, $a_1 = 1$, $a_2 = 1/2$, $b_2 = 1$, $d_2 = 1$, $w_1 = 0$, $w_2 = 0$, and obtain

$$u = \frac{4\sinh(x - \frac{29}{66}y - \frac{927}{88}z + \frac{3t}{2}) - \sin(\frac{1}{2}x + y - \frac{185}{33}z + t)}{2\cosh(x - \frac{29}{66}y - \frac{927}{88}z + \frac{3}{2}t) + \cos(\frac{1}{2}x + y - \frac{185}{33}z + t)}. \quad (47)$$

Figure 8 shows the energy distribution of the periodic lump solution (47). Figures 8(a), 8(b), and 8(c) exhibit the periodic lump solution in the (x,y), (x,z), and (y,z) planes, respectively. Figures 8(d), 8(e), and 8(f) are the corresponding density plots, respectively.

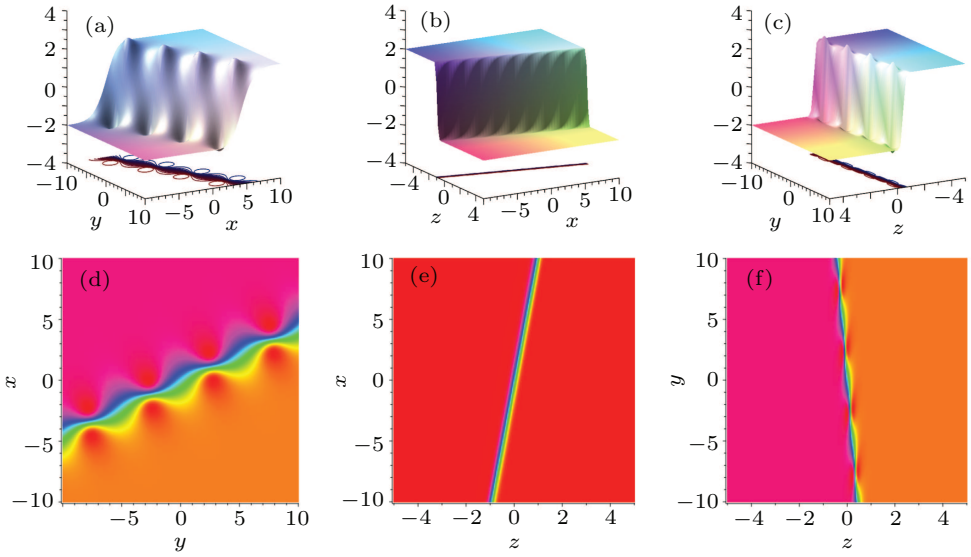


Fig. 8. Perspective views of Eq. (47) at (a) $t = 0$, $z = 0$, (b) $t = 0$, $y = 0$, and (c) $t = 0$, $x = 0$. (d) The density plot corresponding to (a). (e) The density plot corresponding to (b). (f) The density plot corresponding to (c).

9. Conclusions

In summary, we have obtained some new interaction solutions of the second extended (3+1)-dimensional Jimbo–Miwa equation by using the alansatz form. By mixing a quadratic function with one-exponential or two-exponential functions, interaction between lump and kinky soliton solutions are found. Then, we find the fission and fusion phenomena of the interaction solutions by selecting special parameters. Interaction of lump solutions and triangular periodic solutions is obtained by setting the test functions in the form of linear combinations of quadratic and trigonometric functions. New types of interaction solutions are found by introducing a new set consisting of hyperbolic and trigonometric functions. The detailed dynamic analysis of the solution is given in graphical ways.

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