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A quantum algorithm for Toeplitz matrix-vector multiplication

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Toeplitz matrix-vector multiplication is widely used in various fields, including optimal control, systolic finite field multipliers, multidimensional convolution, etc. In this paper, we first present a non-asymptotic quantum algorithm for Toeplitz matrix-vector multiplication with time complexity $\mathcal{O}(\kappa \mathrm{polylog} n)$, where κ and $2n$ are the condition number and the dimension of the circulant matrix extended from the Toeplitz matrix, respectively. For the case with an unknown generating function, we also give a corresponding non-asymptotic quantum version that eliminates the dependency on the L_1-norm ρ of the displacement of the structured matrices. Due to the good use of the special properties of Toeplitz matrices, the proposed quantum algorithms are sufficiently accurate and efficient compared to the existing quantum algorithms under certain circumstances.

Keywords: quantum algorithm, Toeplitz matrix-vector multiplication, circulant matrix

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1. Introduction

Quantum computing has demonstrated its superior performance in solving certain computing problems, such as factoring large numbers[1] and unstructured database searching.[2] In the era of big data, machine learning methods have become increasingly important, but their computational complexity remains high. Due to the high efficiency of quantum computing, increasing numbers of researchers have devoted themselves to the new research field of quantum machine learning (QML) in recent years, hoping to accelerate classical machine learning algorithms with the help of the superposition and entanglement in quantum systems. Representative QML algorithms include positive semidefinite programming,[3] solving linear equations,[4] clustering,[5,6] topology analysis,[7] principal component analysis,[8,9] support vector machines,[10] association rule mining,[11] dimensionality reduction,[12,13] recommendation systems,[14,39] visual tracking,[15,16] the advanced encryption standard,[17] etc.

Toeplitz matrix-vector multiplication is widely used in the fields of optimal control,[18] systolic finite field multipliers,[19] and multidimensional convolution,[20] and thus it has become a hot topic.[18–29] Generally speaking, the Toeplitz matrix is obtained by discretizing a continuous function, and its dimension is connected to the discretization grid parameter. In other words, the Toeplitz matrix can be generated where its diagonal elements are the Fourier transform coefficients of the generating function. Moreover, in practice, the dimensionality of Toeplitz matrices is usually very large; thus, processing them is computationally expensive. Therefore, it is necessary to explore more efficient algorithms to compute matrix-vector products. Wan *et al.*[30,31] proposed an asymptotic quantum algorithm and a block-encoding-based algorithm, respectively. Although their tasks are to solve the Toeplitz system, they can also be naturally applied to the multiplication of the Toeplitz matrix and vector. However, these algorithms are either inaccurate or depend on the L_1-norm ρ of the displacement of the structured matrices (this parameter may be a large value). To solve this problem, in this paper, we will present a non-asymptotic and efficient (independent of ρ) quantum algorithm for Toeplitz matrix-vector multiplication.

The rest of this paper is organized as follows. In Section 2, we review the classical Toeplitz matrix-vector multiplication. Then, we describe the details of the proposed quantum algorithm in Sections 3 and 4. In Section 5, the error and runtime analyses of the proposed quantum algorithm are given. In Section 6, we make a comparison between the proposed quantum algorithm and other existing algorithms. The conclusion is given in Section 7.

2. Classical Toeplitz matrix-vector multiplication

A Toeplitz matrix is an $n \times n$ matrix $T = [t_{k,j}; k, j = 0, 1, \ldots, n-1]$, where $t_{k,j} = t_{k-j}$.

More clearly, a Toeplitz matrix is of the form

$$T = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-(n-2)} & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-(n-2)} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & \ddots & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{1}$$

namely, all elements along each diagonal direction in a Toeplitz matrix are constants. In practice, the Toeplitz matrices are usually obtained by discretizing continuous functions;[32] namely, the diagonal elements $\{t_k\}_{-(n-1)}^{n-1}$ of T

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are the Fourier transform coefficients of a function $f \in C_{2\pi}^{+}$, i.e.,

$$t_k = \frac{1}{2\pi}\int_0^{2\pi} f(\iota)\,\mathrm{e}^{-\mathrm{i}k\iota}\mathrm{d}\iota, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

where $C_{2\pi}^{+}$ denotes the set of all 2π-periodic strictly positive continuous real-valued functions defined on $[0, 2\pi]$.[33] The function f is called the generating function of the sequence of the Toeplitz matrix T.[34] The Toeplitz matrix T will be Hermitian if f is a real-valued function. And when T is Hermitian, its eigenvalues are represented as σ_k, satisfying

$$f_{\min} \leq \sigma_k \leq f_{\max}, \tag{3}$$

where $f_{\min}$ and $f_{\max}$ represent the minimum and maximum values of f, respectively. Note that the generating function f is strictly positive and thus the Toeplitz matrix T is nonsingular.[35] For a further description of Toeplitz matrices, see Ref. [36].

Definition 1 Vector $b = [b_0, b_1, \ldots, b_{n-1}]^{\mathrm{T}} \in \mathbb{C}^n$ is the discrete Fourier transform (DFT) of $a = [a_0, a_1, \cdots, a_{n-1}]^{\mathrm{T}} \in \mathbb{C}^n$, where

$$b_j = \sum_{k=0}^{n-1} x_k \omega^{jk}, \tag{4}$$

and $\omega = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{n}}$.

To be specific,

$$b = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{(n-1)2} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} a = Fa. \tag{5}$$

The circulant matrix is a special kind of Toeplitz matrix with the form

$$C(c) = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & \ddots & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}, \tag{6}$$

where $c = [c_0, c_1, \ldots, c_{n-1}]^{\mathrm{T}}$. Specifically, each column of the circulant matrix is a downward circular shift of the column preceding it, which means that the circulant matrix $C(c)$ can be determined by its first column c.

Obviously, any circulant is a matrix polynomial (namely, the associated polynomial) in the cyclic permutation matrix L, i.e.,

$$C(c) = \sum_{k=0}^{n-1} c_k L^k, \tag{7}$$

where L is given by

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{8}$$

From this we can get

$$FL = WF, \tag{9}$$

where

$$W = \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1}). \tag{10}$$

So,

$$L^k = \left(F^{-1}WF\right)^k = F^{-1}W^kF = \frac{1}{n}F^*W^kF = \bar{F}^*W^k\bar{F}, \tag{11}$$

where $\bar{F} = \frac{1}{\sqrt{n}}F$ is a unitary matrix. Further, we have

$$C(c) = \sum_{k=0}^{n-1} c_k L^k = \sum_{k=0}^{n-1} c_k \bar{F}^*W^k\bar{F} = \bar{F}^*\Lambda\bar{F}, \tag{12}$$

where

$$\Lambda = \sum_{k=0}^{n-1} c_k W^k = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}), \tag{13}$$

is a diagonal matrix, and $\lambda_j = \sum_{k=0}^{n-1} c_k \omega^{kj}$, $j = 0, 1, 2 \ldots, n-1$. It is easy to verify that the equality holds:

$$[\lambda_0, \lambda_1, \ldots, \lambda_{n-1}]^{\mathrm{T}} = Fc. \tag{14}$$

Obviously, Eq. (12) is an eigen-decomposition of the circulant matrix, and it is also a unitary decomposition. Here, λ_j are the eigenvalues of the matrix $C(c)$, and can be calculated by Eq. (14), i.e., by performing a fast Fourier transform (FFT) on the vector c.

Therefore, for any vector x, the product of the circulant matrix $C(c)$ and x can be calculated as follows:

$$C(c)x = \bar{F}^*\Lambda\bar{F}x = F^{-1}\left(\Lambda\left(Fx\right)\right) = F^{-1}\left(\left(Fc\right)\odot\left(Fx\right)\right), \tag{15}$$

where $\odot$ denotes element-wise multiplication.

To compute the Toeplitz matrix-vector product $Tx = y$, T is first embedded into a large circulant matrix of size $2n$:

$$C(c) = \begin{pmatrix} T & S \\ S & T \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{16}$$

where

$$S = \begin{bmatrix} 0 & t_{n-1} & \cdots & t_2 & t_1 \\ t_{-(n-1)} & 0 & t_{n-1} & \cdots & t_2 \\ \vdots & t_{-(n-1)} & 0 & \ddots & \vdots \\ t_2 & \ddots & \ddots & \ddots & t_{n-1} \\ t_{-1} & t_{-2} & \cdots & t_{-(n-1)} & 0 \end{bmatrix}, \tag{17}$$

$$c = \left[t_0, t_1, \ldots, t_{n-1}, 0, t_{-(n-1)}, \ldots, t_{-2}, t_{-1}\right]^{\mathrm{T}}. \tag{18}$$

In fact, we have exactly[36]

$$\begin{aligned}\lambda_j &= \sum_{k=0}^{2n-1} c_k \mathrm{e}^{-\frac{2\pi \mathrm{i} jk}{2n}} \\ &= \sum_{k=0}^{m-1} t_k \mathrm{e}^{-\frac{2\pi \mathrm{i} jk}{2n}} + \sum_{k=m+1}^{2n-1} t_{n-k} \mathrm{e}^{-\frac{2\pi \mathrm{i} jk}{2n}} \\ &= \sum_{k=-m}^{m} t_k \mathrm{e}^{-\frac{2\pi \mathrm{i} jk}{2n}} = f\left(\frac{2\pi j}{2n}\right).\end{aligned} \tag{19}$$

Specifically, the eigenvalues of $C(c)$ are simply the values of $f(\iota)$ with ι uniformly spaced between 0 and 2π. Defining $2\pi k/n = \iota_k$, for any power s, we have[36]

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=0}^{2n-1} \lambda_k^s = \frac{1}{2\pi} \int_0^{2\pi} f(\iota)^s \mathrm{d}\iota, \tag{20}$$

where the continuity of $f(\iota)$ guarantees the existence of the limit of Eq. (20) as a Riemann integral.

Finally, the product of T and any vector x can be calculated by multiplying the circulant matrix $C(c)$ with the extended vector $\begin{pmatrix} x \\ 0 \end{pmatrix}$, i.e.,

$$\begin{pmatrix} Tx \\ 0 \end{pmatrix} = \begin{pmatrix} I_n & 0 \end{pmatrix} C(t) \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}. \tag{21}$$

3. Quantum Toeplitz matrix-vector multiplication

In this section, we design a quantum algorithm for Toeplitz matrix-vector multiplication. Specifically, we bypass the phase estimation operation commonly used in other quantum algorithms by taking full advantage of the special properties of Toeplitz matrices.

Currently, preparing an arbitrary initial quantum state is still challenging and deserves dedicated research. For simplicity, in the subsequent comparison, we follow most of the previous quantum algorithms and assume that the preparation of the initial quantum state $|x,0\rangle = |\chi\rangle = \frac{\sum_i \chi_i |i\rangle}{||\chi||}$ corresponding to $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is efficient; i.e., we assume that there is a unitary operation which can produce the quantum state $|\chi\rangle$ within time complexity $\mathcal{O}(\mathrm{polylog}n)$.[31]

The proposed quantum algorithm is formally introduced below.

(1) Generate the initial quantum state

$$\frac{\sum_i \chi_i |i\rangle}{||\chi||}. \tag{22}$$

(2) Apply an inverse quantum Fourier transform (QFT)[37] on $|\chi\rangle$, and yield

$$\sum_j \alpha_j |j, \tag{23}$$

where $\alpha_j = \sum_i \chi_i \mathrm{e}^{-\frac{2\pi \mathrm{i} ij}{2n}} / (\sqrt{2n}||\chi||)$. Here, we use the convention that the QFT has the same effect as the inverse DFT, and vice versa.

(3) Similar to Ref. [30], suppose there is an Oracle that accesses the values (eigenvalues of $C(c)$) of the generating function f:

$$\sum_j \alpha_j |j\rangle \left| f\left(\frac{2\pi j}{2n}\right) \right\rangle. \tag{24}$$

The cost of invoking the Oracle is $\mathcal{O}(1)$. It can be known from Eq. (19) that the eigenvalues of the circulant matrix $C(c)$ can be directly calculated by the generating function. From Eqs. (14) and (15), it can be seen that the element-by-element multiplication of the eigenvalues with the vector after the discrete Fourier transform will be used in the calculation of the product of the circulant matrix and the vector $\begin{pmatrix} x \\ 0 \end{pmatrix}$. In this quantum version of the operation, the implementation steps are as follows.

(4) Add an ancillary qubit and perform a controlled rotation on it to get the state

$$\sum_j \alpha_j |j\rangle |\lambda_j\rangle (\Upsilon \lambda_j |0\rangle + \sqrt{1 - \Upsilon^2 \lambda_j^2} |1\rangle), \tag{25}$$

where $\lambda_j = f\left(\frac{2\pi j}{2n}\right)$ and $\Upsilon = O\left(\frac{1}{|\lambda_j|}\right)$.

(5) Uncompute the second register, and then measure the last ancillary qubit to see $|0\rangle$, and get

$$|\tilde{y}\rangle = \frac{\sum_j \alpha_j \lambda_j |j\rangle}{\sqrt{\sum_j (\alpha_j \lambda_j)^2}}. \tag{26}$$

(6) Perform "incomplete" QFT on the quantum state in Eq. (26) to directly obtain the corresponding quantum state $|y\rangle$ of classical y (the first n components of $\begin{pmatrix} y \\ * \end{pmatrix}$). For example, given $|j\rangle = |j_1, j_2, \ldots, j_m, j_{m+1}\rangle$, only the first m qubits participate in the computation when performing the QFT operation, resulting in

$$\begin{aligned}|j\rangle \rightarrow\ & \frac{1}{2^{\frac{m}{2}}} (|0\rangle + \mathrm{e}^{2\pi \mathrm{i} 0.j_m} |1\rangle)(|0\rangle + \mathrm{e}^{2\pi \mathrm{i} 0.j_{m-1} j_m} |1\rangle) \cdots \\ & (|0\rangle + \mathrm{e}^{2\pi \mathrm{i} 0.j_1 \ldots j_m} |1\rangle) |j_{m+1}\rangle \\ =\ & \frac{1}{\sqrt{2^m}} \sum_{k=0}^{2m-1} \mathrm{e}^{\frac{2\pi \mathrm{i} ij}{2^m}} |k\rangle |j_{m+1}\rangle.\end{aligned} \tag{27}$$

(7) Discard the last qubit in the quantum register and yield $|y\rangle$ after performing the "incomplete" QFT.

4. Special case

In the previous section we gave a quantum algorithm for Toeplitz matrix-vector multiplication with a known generating function f. However, in some special cases only the Toeplitz matrices are given while the generating function is unknown. In view of these situations, in this section we will design another quantum algorithm.

Similarly, in the quantum random access memory (QRAM)[38,39] data structured model, one can also prepare the quantum state $O_c|0\rangle \to |c\rangle = \sum_i c_i|i\rangle/\|c\|$ with time complexity $\mathcal{O}(\text{polylog}n)$.

A formal description of the quantum algorithm is given below.

(1) Prepare the initial quantum state

$$\frac{\sum_i \chi_i |i\rangle}{||\chi||}|0\rangle|0^{\log 2n}\rangle. \tag{28}$$

(2) Apply an inverse QFT on $|\chi\rangle$ and perform a Hadamard gate on the qubit $|0\rangle$ in the second register, resulting in

$$\sum_j \alpha_j |j\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0^{\log 2n}\rangle. \tag{29}$$

(3) Perform a controlled quantum gate to the third register of $\log 2n$ qubits, and obtain

$$\sum_j \alpha_j |j\rangle \frac{1}{\sqrt{2}}(|0\rangle|j\rangle + |1\rangle|c\rangle). \tag{30}$$

(4) Perform a controlled QFT operation on the qubits in the third register, and the quantum state becomes

$$\sum_j \alpha_j |j\rangle \frac{1}{\sqrt{2}}\left(|0\rangle \frac{\sum_{i'} e^{\frac{2\pi i i' j}{2n}}|i'\rangle}{\sqrt{2n}} + |1\rangle \frac{\sum_{i''} c_{i''}|i''\rangle}{\|c\|}\right). \tag{31}$$

(5) Repeat a Hadamard gate operation on the auxiliary qubit in the second register, get

$$\begin{aligned}
&\sum_j \alpha_j |j\rangle \frac{1}{\sqrt{2}}\left(|0\rangle\left(\frac{\sum_{i'} e^{\frac{2\pi i i' j}{2n}}|i'\rangle}{\sqrt{2n}} + \frac{\sum_{i''} c_{i''}|i''\rangle}{\|c\|}\right)\right.\\
&\left.+|1\rangle\left(\frac{\sum_{i'} e^{\frac{2\pi i i' j}{2n}}|i'\rangle}{\sqrt{2n}} - \frac{\sum_{i''} c_{i''}|i''\rangle}{\|c\|}\right)\right)\\
&= \sum_j \alpha_j |j\rangle (\sin\theta_j|0\rangle|\varphi_j^0\rangle + \cos\theta_j|1\rangle|\varphi_j^1\rangle)\\
&= \sum_j \alpha_j |j\rangle \frac{-i}{\sqrt{2}}(e^{i\theta_j}|\varphi_j^+\rangle - e^{-i\theta_j}|\varphi_j^-\rangle),
\end{aligned} \tag{32}$$

where

$$|\varphi_j^0\rangle = \frac{1}{2}\left(\frac{\sum_{i'} e^{\frac{2\pi i i' j}{2n}}|i'\rangle}{\sqrt{2n}} + \frac{\sum_{i''} c_{i''}|i''\rangle}{\|c\|}\right),$$

$$|\varphi_j^1\rangle = \frac{1}{2}\left(\frac{\sum_{i'} e^{\frac{2\pi i i' j}{2n}}|i'\rangle}{\sqrt{2n}} - \frac{\sum_{i''} c_{i''}|i''\rangle}{\|c\|}\right),$$

$$|\varphi_j^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\varphi_j^0\rangle \pm i|1\rangle|\varphi_j^1\rangle).$$

And

$$\begin{aligned}
\sin^2\theta_j &= \frac{1}{2}\left(1 + \frac{\sum_{i'} e^{\frac{2\pi i i' j}{2n}}\langle i'|\sum_{i''} c_{i''}|i''\rangle}{\sqrt{2n}\|c\|}\right)\\
&= \frac{1}{2}\left(1 + \frac{\sum_i c_i e^{\frac{2\pi i i j}{2n}}}{\sqrt{2n}\|c\|}\right).
\end{aligned} \tag{33}$$

(6) According to the quantum amplitude estimation (QAE)[37,40] algorithm, get

$$\sum_j \alpha_j |j\rangle \frac{-i}{\sqrt{2}}\left(e^{i\theta_j}|\varphi_j^+\rangle\left|\frac{\theta_j}{\pi}\right\rangle - e^{-i\theta_j}|\varphi_j^-\rangle\left|\frac{\pi-\theta_j}{\pi}\right\rangle\right). \tag{34}$$

(7) Append an extra register and perform a quantum arithmetic operation on it to obtain

$$\sum_j \alpha_j |j\rangle \frac{-i}{\sqrt{2}}\left(e^{i\theta_j}|\varphi_j^+\rangle\left|\frac{\theta_j}{\pi}\right\rangle - e^{-i\theta_j}|\varphi_j^-\rangle\left|\frac{\pi-\theta_j}{\pi}\right\rangle\right)|\lambda_j\rangle, \tag{35}$$

where $\lambda_j = 2\sin^2\theta_j - 1 = \sum_i c_i e^{\frac{2\pi i i j}{2n}}/(\sqrt{2n}\|c\|)$.

(8) Add an ancillary qubit and perform a controlled rotation on it, and obtain

$$\begin{aligned}
&\sum_j \alpha_j |j\rangle \frac{-i}{\sqrt{2}}\left(e^{i\theta_j}\left|\varphi_j^+\right\rangle\left|\frac{\theta_j}{\pi}\right\rangle - e^{-i\theta_j}\left|\varphi_j^-\right\rangle\left|\frac{\pi-\theta_j}{\pi}\right\rangle\right)|\lambda_j\rangle\\
&\times\left(\Upsilon\lambda_j|0\rangle + \sqrt{1-\Upsilon^2\lambda_j^2}|1\rangle\right).
\end{aligned} \tag{36}$$

(9) Uncompute quantum registers, except for the first and last registers, and then measure the last ancillary qubit to see $|0\rangle$, and transform the system to

$$|\tilde{y}\rangle = \frac{\sum_j \alpha_j \sum_i c_i e^{\frac{-2\pi i i j}{2n}}|j\rangle}{\sqrt{\sum_j\left(\alpha_j \sum_i c_i e^{\frac{-2\pi i i j}{2n}}\right)^2}}. \tag{37}$$

In this special case, when the generating function is unknown, to calculate the eigenvalues of the circulant matrix, it is necessary to calculate the discrete Fourier transform of a specific vector c. And this is exactly what the previous series of steps have done.

(10) Discard the last qubit in the quantum register and obtain $|y\rangle$ after performing the "incomplete" QFT on the quantum state in Eq. (37).

5. Complexity analysis

In this section, we will give the complexity analysis of the proposed quantum algorithms for Toeplitz matrix-vector multiplication. We first analyze the complexity of the first quantum algorithm with the generating function. The complexity of preparing the initial quantum state and that of performing the (inverse) QFT are both $\mathcal{O}(\text{polylog}n)$. In the measurement operation, the success probability of obtaining the measurement $|0\rangle$ is

$$P(0)=\sum_j \Upsilon^2\lambda_j^2=\Omega\left(\frac{1}{\kappa^2}\right), \tag{38}$$

where $\kappa=\max_j|\lambda_j|/\min_j|\lambda_j|$ is the condition number of the circulant matrix $C(c)$. Using quantum amplitude amplification (QAA),[37,40] $\mathcal{O}(\kappa)$ times of repetition is enough to get the state in Eq. (26). As a conclusion, the time complexity of the first proposed quantum algorithm is $\mathcal{O}(\kappa\text{polylog}n)$.

Next, we analyze the complexity of the second quantum algorithm with the unknown generating function f. The QAE[37,40] algorithm in step (6) can be implemented in $\mathcal{O}\left(\frac{\text{polylog}n}{\epsilon}\right)$ with error ϵ. Here, the estimate obtained by the algorithm is denoted as $\widetilde{\lambda}_j$

$$\begin{aligned}
\left|\widetilde{\lambda}_j-\lambda_j\right| &= \left|\frac{\widetilde{\sum_i c_i \mathrm{e}^{\frac{-2\pi \mathrm{i}ij}{2n}}}}{\sqrt{2n}\,\|c\|}-\frac{\sum_i c_i \mathrm{e}^{\frac{2\pi \mathrm{i}ij}{2n}}}{\sqrt{2n}\,\|c\|}\right| \\
&= \left|\sin^2\widetilde{\theta}_j-\sin^2\theta_j\right| \\
&= \left|\left(\sin\widetilde{\theta}_j-\sin\theta_j\right)\left(\sin\widetilde{\theta}_j+\sin\theta_j\right)\right| \\
&= \left|\left(2\sin\frac{\widetilde{\theta}_j-\theta_j}{2}\cos\frac{\widetilde{\theta}_j+\theta_j}{2}\right)\right. \\
&\quad \left.\times\left(2\sin\frac{\widetilde{\theta}_j+\theta_j}{2}\cos\frac{\widetilde{\theta}_j-\theta_j}{2}\right)\right| \\
&= \left|\left(2\sin\frac{\widetilde{\theta}_j-\theta_j}{2}\cos\frac{\widetilde{\theta}_j-\theta_j}{2}\right)\right. \\
&\quad \left.\times\left(2\sin\frac{\widetilde{\theta}_j+\theta_j}{2}\cos\frac{\widetilde{\theta}_j+\theta_j}{2}\right)\right| \\
&= \left|\sin\left(\widetilde{\theta}_j-\theta_j\right)\sin\left(\widetilde{\theta}_j+\theta_j\right)\right| \\
&\le \left|\sin\left(\widetilde{\theta}_j-\theta_j\right)\right|\le\left|\widetilde{\theta}_j-\theta_j\right|\le\epsilon.
\end{aligned} \tag{39}$$

In step (9), the probability of getting the measurement outcome $|0\rangle$ is

$$P(0)=\sum_j \Upsilon^2\lambda_j^2=\Omega\left(\frac{1}{\kappa^2}\right). \tag{40}$$

Similarly, according to the error analysis in Ref. [4], introducing an error of $\mathcal{O}(\frac{\epsilon_0}{\kappa})$ when calculating λ_j will result in an error of ϵ_0 in the final result y. In conclusion, the time complexity of the second quantum algorithm with the unknown generating function f is $\mathcal{O}\left(\frac{\kappa^2\text{polylog}n}{\epsilon_0}\right)$.

6. Discussion

In this paper, we have proposed a quantum algorithm for Toeplitz matrix-vector multiplication by taking full advantage of the properties of Toeplitz matrices. The table below shows a comparison between our proposed quantum algorithms and other quantum algorithms.

Table 1. A comparison of existing quantum algorithms. κ' is the condition number of the Toeplitz matrix.

Algorithm	Time complexity	Remark
Our algorithms	$\mathcal{O}(\kappa\text{polylog}n)$	Nonasymptotic, f is provided
	$\mathcal{O}\left(\frac{\kappa^2\text{polylog}n}{\epsilon_0}\right)$	Nonasymptotic, no f
Zhou *et al.*[41]	$\mathcal{O}\left(\frac{\log n}{\||C\|x\rangle\|\|\|T\|x\rangle\|}\right)$	Nonasymptotic, no f
Wan *et al.*[30]	$\mathcal{O}(\kappa'\text{polylog}n)$	Asymptotic, f is provided
	$\mathcal{O}\left(\frac{\kappa'^2\text{polylog}n}{\delta\epsilon_0}\right)$	Asymptotic, no f
Wan *et al.*[31]	$\mathcal{O}\left(\kappa'\rho\text{polylog}\left(\frac{n}{\epsilon_0}\right)\right)$	Block-encoding based

In Ref. [41], Zhou *et al.* also presented a quantum algorithm for Toeplitz matrix-vector multiplication, for the case where the generating function is not known, and its time complexity is $\mathcal{O}\left(\frac{\log n}{\||C|x\rangle\|\|\|T|x\rangle\|}\right)$, which greatly depends on $\||C|x\rangle\|$ and $\||T|x\rangle\|$. In addition, in Ref. [30], Wan *et al.* gave two asymptotic quantum algorithm frameworks works that focused on resolving the Toeplitz system, corresponding to the cases with and without the generating function, respectively. But it can also be used to calculate the multiplication of the Toeplitz matrix and vector. Later, they proposed a block-encoding-based quantum algorithm,[31] whose time complexity has a linear dependence on ρ. As a conclusion, for the well-conditioned matrices, i.e., $\kappa=\kappa'=\mathcal{O}(\text{polylog}n)$, the proposed quantum algorithms are two exact algorithms with the time complexity independent of ρ (this number may be very large).

In addition, it is worth mentioning that the proposed quantum algorithms can be extended to the case with Hankel matrices. To be more specific, an $n\times n$ Hankel matrix is of the form

$$H=\begin{bmatrix}
h_{-(n-1)} & h_{-(n-2)} & \cdots & h_{-1} & h_0 \\
h_{-(n-2)} & h_{-(n-3)} & h_{-(n-4)} & \cdots & h_1 \\
\vdots & h_{-(n-4)} & h_{-(n-5)} & \ddots & \vdots \\
h_{-1} & \ddots & \ddots & \ddots & h_{n-2} \\
h_0 & h_1 & \cdots & h_{n-2} & h_{n-1}
\end{bmatrix}. \tag{41}$$

It can be observed that if the columns are permuted left-to-right, then H will become a Toeplitz matrix T. Therefore, the following relation holds:

$$Hx=Tx(n:-1:1). \tag{42}$$

Thus, it becomes a Toeplitz matrix-vector multiplication.

7. Conclusion

In this article, we present two accurate and efficient quantum algorithms for Toeplitz matrix-vector multiplication. Specifically, we first present a non-asymptotic quantum algorithm for Toeplitz matrix-vector multiplication with time complexity $\mathcal{O}(\kappa \text{polylog} n)$. For the case with an unknown generating function, we also give a corresponding non-asymptotic quantum version that eliminates the dependency on the L_1-norm ρ of the displacement of the structured matrices. Actually, the proposed quantum algorithms benefit from taking full advantage of the special properties of Toeplitz matrices. Here, we hope that the proposed quantum algorithms can provide some inspiration to researchers in related fields.

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