Stability and optimal control for delayed rumor-spreading model with nonlinear incidence over heterogeneous networks

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On the multilingual online social networks of global information sharing, the wanton spread of rumors has an enormous negative impact on people's lives. Thus, it is essential to explore the rumor-spreading rules in multilingual environment and formulate corresponding control strategies to reduce the harm caused by rumor propagation. In this paper, considering the multilingual environment and intervention mechanism in the rumor-spreading process, an improved ignorants–spreaders-1–spreaders-2–removers (I2SR) rumor-spreading model with time delay and the nonlinear incidence is established in heterogeneous networks. Firstly, based on the mean-field equations corresponding to the model, the basic reproduction number is derived to ensure the existence of rumor-spreading equilibrium. Secondly, by applying Lyapunov stability theory and graph theory, the global stability of rumor-spreading equilibrium is analyzed in detail. In particular, aiming at the lowest control cost, the optimal control scheme is designed to optimize the intervention mechanism, and the optimal control conditions are derived using the Pontryagin's minimum principle. Finally, some illustrative examples are provided to verify the effectiveness of the theoretical results. The results show that optimizing the intervention mechanism can effectively reduce the densities of spreaders-1 and spreaders-2 within the expected time, which provides guiding insights for public opinion managers to control rumors.

Keywords: rumor propagation, heterogeneous network, nonlinear incidence, optimal control

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1. Introduction

As a typical social phenomenon, rumors are usually defined as disseminating of unofficially verified or false information.[1] With the rapid emergence of internet technology, rumor propagation is not confined to traditional small-scale oral communication, but depends on online social networks (OSNs) such as Facebook, Twitter, and WeChat. In recent years, many investigations have shown that rumors spread quickly and spread more widely on the OSNs.[2,3] Unfortunately, rumors cause people to panic and even bring massive damage to society in a short time.[4] For example, a rumor that drinking alcohol can eradicate COVID-19 has caused hundreds of deaths quickly.[5] Hence, formulating rumors control strategies to minimize the harm of rumors has become an urgent problem to solve.

The study of rumor propagation can be retrospected to the 1960s. A classic DK model in Refs. [6,7] was first proposed by Daley and Kendall, in which the population was segmented into three classes: ignorants, spreaders, and removers. Maki-Thomson further improved the DK model to a novel MT model.[8] Based on two traditional models, many scholars have studied the dynamical behaviors of rumor-spreading models.[9–14] However, these simple models cannot accurately describe the realistic characteristics of rumor propagation. As we all know, many social and communication systems can be correctly characterized by complex networks whose nodes represent individuals, and the links mimic their interactions.[15] According to the probability distribution of node degree in the topological structure of the network, complex networks can be divided into homogeneous and heterogeneous networks.[16] Homogeneous networks are mainly concentrated near the average degree and obey binomial or Poisson degree distribution, such as the small-world networks model.[17,18] The traditional oral communication population is usually considered a homogeneous network model with uniform mixing, that is, the contact probability between any nodes is equal. However, the topological structure of OSNs often shows the heterogeneity of information interaction between nodes. Heterogeneous networks mean that few nodes have a large number of links, while most nodes only have a small number of links, and the probability of any node connecting with other nodes obeys power law distribution, such as BA scale-free networks.[19] Hence, heterogeneous networks may better describe the spread of rumors in OSNs. In Ref. [20], Moreno developed the rumor propagation model by applying the mean-field theory in heterogeneous networks. Zhu *et al.*[21] considered the network topology in heterogeneous networks. Recently, scholars have discovered that heterogeneous networks can more accurately simulate actual data of rumor-spreading.[22,23] Therefore, it is essential to study the dynamics of rumor propagation in heterogeneous networks.

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Studying rumor propagation aims to control rumors. In recent years, the optimal control theory has played a significant role in epidemics[24,25] and computer viruses.[26] Considering the similarity between rumor propagation and propagation mentioned above, scholars gradually applied optimal control to the rumor-spreading process and achieved some excellent results. Huo *et al.*[12] used optimal control to explore the impact of media reports on rumor-spreading, and the results showed that the government's positive actions could slow down the spread of rumors. Cheng *et al.*[27] proposed an optimal control problem to minimize the scale of rumor-spreading, considering deleting rumor posts, popular science education, and immunotherapy. Zhong *et al.*[28] designed an aperiodically intermittent stochastic stabilization method to suppress the spread of rumors. Li *et al.*[10] studied various control problems from different angles and obtained different optimal control strategies, which provided a flexible control strategy for the public opinion management department.

With the globalization of information sharing on OSNs, a rumor may spread simultaneously in multiple languages, bringing significant challenges to preventing and controlling rumors. The languages of different nationalities and regions worldwide differ significantly, and people also show different communicative characteristics for the same kind of rumor information. The produced information versions in multilingual environment show diversity in the rumor-spreading process, which are far more complicated than in single-language environment, and reflect the heterogeneity of the rumor propagation. Therefore, studying the dynamic characteristics of rumor propagation in multilingual environment have far-reaching significance and application value. Up to now, some results have been achieved in the modeling rumor propagation under multilingual environment,[29–33] which motivates further exploration.

Noting that the above works only consider bilinear incidence in multilingual environment, which is somewhat inconsistent with the practical significance.[34] In reality, due to crowding or taking appropriate precautions to restrain rumor-spreading, the number of effective contacts between individuals or the ability to spread rumors through interpersonal networks is always limited, which may reach saturation at high transmission levels.[25] Moreover, rumor propagation is closely related to people's psychological quality. Therefore, it is more reasonable to introduce psychological factors and adopt nonlinear incidence,[35,36] which can more visually expound the changes in individual attitudes and behaviors towards rumors.[21,37] In addition, in the process of rumor-spreading, the phenomenon of time delay is widespread. Specifically, translation between different languages, the inability to update the internet in time, and the inability of users to check and respond to rumors in time will all cause the time delay of rumor-spreading. In the multilingual environment, the influence of time delay on the dynamic behavior of rumor-spreading is worth exploring. Based on the above statements, a new I2SR rumor propagation model is established in this paper. The main works of this paper are described as follows:

(i) Compared with existing articles,[29–33] a more general I2SR delayed rumor propagation model with nonlinear incidence is established in multilingual environment and heterogeneous networks. The nonlinear incidence can describe the individual behavior change and the group effect, which is more consistent with reality than the bilinear incidence and the standard incidence when describing the rule of rumor propagation.

(ii) By applying the Lyapunov stability theory and graph theory, the stability of rumor-free equilibrium (RFE) and rumor-spreading equilibrium (RSE) is analyzed in detail.

(iii) Based on the Pontryagin's minimum principle, an optimal control scheme is given to reduce the density of rumor-spreading individuals. Numerical simulations are carried out using the forward–backward sweep method. Furthermore, the results show that the optimal control scheme can effectively reduce the density of rumor spreaders within the expected time and minimize the cost.

The structure of this paper is arranged as follows. An I2SR rumor propagation model is established in Section 2. The basic reproduction number and existence of the equilibriums are discussed in Section 3. In Section 4, the stability of the RFE and RSE is proved. The optimal control strategy is shown in Section 5. The theoretical results are verified in Section 6. The conclusion is drawn in Section 7.

2. Model formulation

The OSNs are regarded as heterogeneous networks composed of nodes and the links among them. Each node represents an individual, and the links between nodes describe the communication among individuals. Assuming that rumors spread in bilingual environment, the total population $N(t)$ is divided into four categories: ignorants, spreaders-1, spreaders-2, and removers. Ignorants ($I(t)$) indicate people who do not know rumors but are easily infected and spread them. Spreaders-1 ($S^1(t)$) denote people who know rumors and can speak two languages but spread them in the first language. Spreaders-2 ($S^2(t)$) mean people who know the rumors but spread them in the second language. Removers ($R(t)$) show people who know rumors but no longer spread them. Due to individual differences, different individuals have different reactions and spreading abilities to rumors. Therefore, to more accurately describe the rumor propagation process in heterogeneous networks, it is assumed that individuals with the same degree of OSNs are divided into a group. Let $I_m(t)$, $S^1_m(t)$, $S^2_m(t)$, and $R_m(t)$ are the densities of four groups with

m degree at time t, respectively. According to the above analysis, the rumor-spreading diagram in heterogeneous networks is shown in Fig. 1.

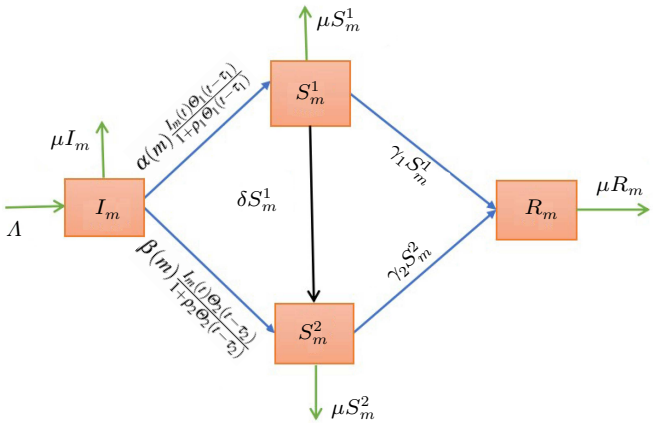


Fig. 1. I2SR rumor-spreading diagram in heterogeneous networks.

Based on the mean-field theory in heterogeneous networks, the rumor propagation model is established in multilingual environment as

$$\begin{cases} \frac{\mathrm{d}I_m(t)}{\mathrm{d}t} = \Lambda - \alpha(m)\frac{I_m(t)\Theta_1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)} \\ \qquad\qquad - \beta(m)\frac{I_m(t)\Theta_2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)} - \mu I_m(t), \\ \frac{\mathrm{d}S_m^1(t)}{\mathrm{d}t} = \alpha(m)\frac{I_m(t)\Theta_1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)} - (\delta+\gamma_1+\mu)S_m^1(t), \\ \frac{\mathrm{d}S_m^2(t)}{\mathrm{d}t} = \beta(m)\frac{I_m(t)\Theta_2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)} + \delta S_m^1(t) - (\gamma_2+\mu)S_m^2(t), \\ \frac{\mathrm{d}R_m(t)}{\mathrm{d}t} = \gamma_1 S_m^1(t) + \gamma_2 S_m^2(t) - \mu R_m(t), \end{cases} \tag{1}$$

where $m = 1,2,\ldots,n$, τ_1 and τ_2 denote the average time delay from spreaders-1 and spreaders-2 issuing or forwarding rumors to the ignorants receiving them, respectively. Λ represents the comimg rate of $I_m(t)$, and μ denotes the removal rate of $I_m(t)$, $S_m^1(t)$, $S_m^2(t)$, and $R_m(t)$. $\Theta_1(t-\tau_1)$ and $\Theta_2(t-\tau_2)$ represent the probability of any link pointing to spreaders-1 and spreaders-2, respectively. Considering the uncorrelated networks,[38] then $\Theta_z(t-\tau_z)$ $(0 \le \Theta_z(t-\tau_z) \le 1, z = 1,2)$ is defined as

$$\begin{aligned} \Theta_z(t-\tau_z) &= \sum_{i=1}^{n} \frac{\nu(i)}{i} p(i \mid m) S_i^z(t-\tau_z) \\ &= \frac{\sum_{i=1}^{n} \nu(i) p(i) S_i^z(t-\tau_z)}{\langle m \rangle}, \end{aligned}$$

where $p(i \mid m) = ip(i)/\langle m \rangle$ represents the probability that a node with degree i connects with degree m. $\langle m \rangle = \sum_{i=1}^{n} ip(i)$ is the average degree of the network and satisfies $\sum_{i=1}^{n} p(i) = 1$. $\nu(m)$ stands for the infectivity of an infected node with degree m. Assume that $\sum_{i=1}^{n} \nu(i)p(i) \le \langle m \rangle$ is due to the fact that it is impossible for each spreaders to contact all neighbors in one time step.[39] $\alpha(m)$ and $\beta(m)$ represent the propagation rates of spreaders-1 and spreaders-2 respectively, which change with the degree m of node. When the ignorants contact with the spreaders and are infected by rumor, the terms $\alpha(m)\frac{I_m(t)\Theta_1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)}$ and $\beta(m)\frac{I_m(t)\Theta_2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)}$ are the nonlinear incidences of spreaders-1 and spreaders-2, respectively. $\alpha(m)\Theta_1(t-\tau_1)$ and $\beta(m)\Theta_2(t-\tau_2)$ measure the infectivity of rumor-spreading. ρ_1 and ρ_2 represent psychological influence factors, which measure the psychological or suppressive effect of the public's attitude towards rumors. Since spreaders-1 ($S_m^1(t)$) can speak two languages, some of them may spread rumors through the second language, supposing $S_m^1(t)$ is transformed into $S_m^2(t)$ with probability δ. Under the constraint of the intervention mechanism, the probability that $S_m^1(t)$ (or $S_m^2(t)$) stops spreading rumors and becomes $R_m(t)$ is γ_1 (or γ_2). Assume that the above parameters are nonnegative.

Remark 1 The intervention mechanism aims to suppress the spread of rumors by imposing certain measures on the rumor spreaders in reality, for example, legal punishment and public education,[10] debunking behavior,[11] media report,[12] identification and detection, etc. By implementing the intervention mechanism, OSNs users are encouraged to identify rumors actively, thereby effectively suppressing the spread of rumors on OSNs. Needless to say, more educated spreaders may not believe rumors. Therefore, it is necessary to strengthen publicity and education to guide OSNs users to change their attitudes towards rumors so that they no longer believe in rumors. Compared with the forgetting mechanism[29] and the hesitation mechanism[31] in existing multilingual environmental articles, considering the intervention mechanism has important practical significance in curbing the spread of rumors.

Remark 2 Compared with Refs. [29–33], the psychological factors are taken into account, and the nonlinear incidence in heterogeneous networks is introduced in this paper. In fact, the nonlinear incidence is more realistic and challenging than the bilinear incidence in heterogeneous networks. Specifically, to explore the effect of time delay on the real rumor propagation process, this paper adopts different time delays τ_1 and τ_2 for spreaders-1 and spreaders-2, respectively.

The initial conditions of system (1) are defined as follows:

$$\begin{aligned} &I_m(\vartheta) = \varphi_{m1}(\vartheta),\ S_m^1(\vartheta) = \varphi_{m2}(\vartheta),\ S_m^2(\vartheta) = \varphi_{m3}(\vartheta), \\ &R_m(\vartheta) = \varphi_{m4}(\vartheta),\ \varphi_{mi}(\vartheta) \ge 0,\ \varphi_{mi}(0) > 0, \\ &(\vartheta \in [\min\{-\tau_1,-\tau_2\},0],\ m = 1,2,\ldots,n,\ i = 1,2,3,4), \end{aligned} \tag{2}$$

where $\Upsilon = \{(\varphi_{m1}(\vartheta), \varphi_{m2}(\vartheta), \varphi_{m3}(\vartheta), \varphi_{m4}(\vartheta))\} \in \mathcal{C}$. $\mathcal{C}$ represents the Banach space $\mathcal{C}([\min\{-\tau_1,-\tau_2\},0], R_+^{4n})$ with norm

$$\begin{aligned} \|\Upsilon\| = \Big(&\sum_{m=1}^{n} (\mid \varphi_{m1}(\vartheta) \mid^2 + \mid \varphi_{m2}(\vartheta) \mid^2 \\ &+ \mid \varphi_{m3}(\vartheta) \mid^2 + \mid \varphi_{m4}(\vartheta) \mid^2) \Big)^{1/2}, \end{aligned}$$

where $\mid \varphi_{mi}(\vartheta) \mid = \sup_{\min\{-\tau_1,-\tau_2\} \le \vartheta \le 0} \mid \varphi_{mi}(\vartheta) \mid$.

Lemma 1 The solution $(I_m(t), S_m^1(t), S_m^2(t), R_m(t))$ of system (1) with initial condition (2) is positive for all $t > 0$.

Proof From initial condition (2), one has $\varphi_{mi}(\vartheta) \geq 0$, $\varphi_{mi}(0) > 0$, $(\vartheta \in [\min\{-\tau_1, -\tau_2\}, 0]$, $m = 1, 2, \ldots, n$, $i = 1, 2, 3, 4)$. Assume that there exists a constant $t_1^* > 0$, such that $I_m(t_1^*) = 0$, $\frac{\mathrm{d}I_m(t)}{\mathrm{d}t}|_{t=t_1^*} \leq 0$, and $I_m(t) > 0$ for $t \in (0, t_1^*)$. From system (1), it has $\frac{\mathrm{d}I_m(t)}{\mathrm{d}t}|_{t=t_1^*} = \Lambda > 0$, which contradicts with $\frac{\mathrm{d}I_m(t)}{\mathrm{d}t}|_{t=t_1^*} \leq 0$. Thus, $I_m(t) > 0$ for all $t > 0$.

Since $0 \leq \Theta_z(t - \tau_z) \leq 1$ $(z = 1, 2)$, from the second equation of system (1), one has

$$\begin{aligned}\frac{\mathrm{d}S_m^1(t)}{\mathrm{d}t} &= \alpha(m)\frac{I_m(t)\Theta_1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)} - (\delta+\gamma_1+\mu)S_m^1(t)\\ &\geq -(\delta+\gamma_1+\mu)S_m^1(t).\end{aligned}$$

By the comparison principle, there is $S_m^1(t) \geq \varphi_{m2}(0)\,\mathrm{e}^{-(\delta+\gamma_1+\mu)t} > 0$. Similarly, $S_m^2(t) > 0$ and $R_m(t) > 0$ can be verified by the above method.

Lemma 2 The feasible region Ω is a positive invariant set of system (1) with initial condition (2), which is defined as

$$\Omega = \left\{(I_m(t), S_m^1(t), S_m^2(t), R_m(t)) \in \mathcal{R}_+^{4n} \mid I_m(t) + S_m^1(t) + S_m^2(t) + R_m(t) \leq \frac{\Lambda}{\mu},\ m = 1, 2, \ldots, n\right\}.$$

Proof Let

$$N_m(t) = I_m(t) + S_m^1(t) + S_m^2(t) + R_m(t),\ \ m = 1, 2, \ldots, n$$

and $\sum_{m=1}^n N_m(t) = N(t)$, it has

$$\begin{aligned}\frac{\mathrm{d}N_m(t)}{\mathrm{d}t} &= \frac{\mathrm{d}I_m(t)}{\mathrm{d}t} + \frac{\mathrm{d}S_m^1(t)}{\mathrm{d}t} + \frac{\mathrm{d}S_m^2(t)}{\mathrm{d}t} + \frac{\mathrm{d}R_m(t)}{\mathrm{d}t}\\ &= \Lambda - \mu N_m(t).\end{aligned}$$

One has $\lim_{t\to\infty} N_m(t) = \Lambda/\mu$, which means that the feasible region Ω is a positively invariant with respect to system (1).

3. The existence of the equilibriums

Theorem 1 For system (1), the basic reproduction numbers are defined as

$$R_{01} = \frac{\Lambda\sum_{m=1}^n \nu(m)p(m)\alpha(m)}{\mu(\delta+\gamma_1+\mu)\langle m\rangle},$$
$$R_{02} = \frac{\Lambda\sum_{m=1}^n \nu(m)p(m)\beta(m)}{\mu(\gamma_2+\mu)\langle m\rangle},$$

the following statements hold.

Case 1 If $R_{01} < 1$ and $R_{02} < 1$, system (1) admits a unique RFE $E_0(I_m^\circ, S_m^{1\circ}, S_m^{2\circ}, R_m^\circ) = \{(\Lambda/u, 0, 0, 0), \ldots, (\Lambda/u, 0, 0, 0)\}$.

Case 2 If $R_{01} < 1$ and $R_{02} > 1$, system (1) admits a unique RSE $E_1^*(I_{1m}^*, S_{1m}^{1*}, S_{1m}^{2*}, R_{1m}^*) = \{(I_{11}^*, S_{11}^{1*}, S_{11}^{2*}, R_{11}^*), \ldots, (I_{1n}^*, S_{1n}^{1*}, S_{1n}^{2*}, R_{1n}^*)\}$, where $S_{1m}^{1*} = 0$,

$$I_{1m}^* = \frac{\Lambda(1+\rho_2\Theta_2^*)}{\beta(m)\Theta_2^* + \mu(1+\rho_2\Theta_2^*)},$$
$$S_{1m}^{2*} = \frac{\Lambda\beta(m)\Theta_2^*}{(\gamma_2+\mu)(\beta(m)\Theta_2^* + \mu(1+\rho_2\Theta_2^*))},$$
$$R_{1m}^* = \frac{\gamma_2\Lambda\beta(m)\Theta_2^*}{\mu(\gamma_2+\mu)(\beta(m)\Theta_2^* + \mu(1+\rho_2\Theta_2^*))},$$
$$\Theta_2^* = \frac{\sum_{m=1}^n \nu(m)p(m)S_{1m}^{2*}}{\langle m\rangle}.$$

Case 3 If $R_{01} > 1$ and $R_{02} < 1$ for $\delta = 0$, system (1) admits a unique RSE $E_2^*(I_{2m}^*, S_{2m}^{1*}, S_{2m}^{2*}, R_{2m}^*) = \{(I_{21}^*, S_{21}^{1*}, S_{21}^{2*}, R_{21}^*), \ldots, (I_{2n}^*, S_{2n}^{1*}, S_{2n}^{2*}, R_{2n}^*)\}$, where $S_{2m}^{2*} = 0$,

$$I_{2m}^* = \frac{\Lambda(1+\rho_1\Theta_1^*)}{\alpha(m)\Theta_1^* + \mu(1+\rho_1\Theta_1^*)},$$
$$S_{2m}^{1*} = \frac{\Lambda\alpha(m)\Theta_1^*}{(\gamma_1+\mu)(\alpha(m)\Theta_1^* + \mu(1+\rho_1\Theta_1^*))},$$
$$R_{2m}^* = \frac{\gamma_1\Lambda\alpha(m)\Theta_1^*}{\mu(\gamma_1+\mu)(\alpha(m)\Theta_1^* + \mu(1+\rho_1\Theta_1^*))},$$
$$\Theta_1^* = \frac{\sum_{m=1}^n \nu(m)p(m)S_{2m}^{1*}}{\langle m\rangle}.$$

Case 4 If $R_{01} > 1$ and $R_{02} > 1$, system (1) admits a unique positive RSE $E^*(I_m^*, S_m^{1*}, S_m^{2*}, R_m^*) = \{(I_1^*, S_1^{1*}, S_1^{2*}, R_1^*), \ldots, (I_n^*, S_n^{1*}, S_n^{2*}, R_n^*)\}$, where

$$\begin{cases} I_m^* = \frac{\Lambda(1+\rho_1\Theta_1^*)(1+\rho_2\Theta_2^*)}{\Delta},\ \ S_m^{1*} = \frac{1}{\delta+\gamma_1+\mu}\frac{\Lambda\alpha(m)\Theta_1^*(1+\rho_2\Theta_2^*)}{\Delta},\\ S_m^{2*} = \frac{\beta(m)\Theta_2^*(1+\rho_1\Theta_1^*)}{\Delta} + \frac{\delta\alpha(m)\Theta_1^*(1+\rho_2\Theta_2^*)}{(\delta+\gamma_1+\mu)\Delta},\\ R_m^* = \frac{1}{\mu}(\gamma_1 S_m^{1*} + \gamma_2 S_m^{2*}),\\ \Delta = \alpha(m)\Theta_1^*(1+\rho_2\Theta_2^*) + \beta(m)\Theta_2^*(1+\rho_1\Theta_1^*)\\ \qquad + \mu(1+\rho_1\Theta_1^*)(1+\rho_2\Theta_2^*),\\ \Theta_1^* = \frac{\sum_{m=1}^n \nu(m)p(m)S_m^{1*}}{\langle m\rangle},\ \ \Theta_2^* = \frac{\sum_{m=1}^n \nu(m)p(m)S_m^{2*}}{\langle m\rangle}. \end{cases}$$

Proof The steady state solution of system (1) can be expressed as

$$\frac{\mathrm{d}I_m(t)}{\mathrm{d}t} = 0,\ \ \frac{\mathrm{d}S_m^1(t)}{\mathrm{d}t} = 0,\ \ \frac{\mathrm{d}S_m^2(t)}{\mathrm{d}t} = 0,\ \ \frac{\mathrm{d}R_m(t)}{\mathrm{d}t} = 0.$$

For Case 1, it is easy to obtain the RFE E_0 for $R_{01} < 1$ and $R_{02} < 1$.

For Case 2, if $R_{01} < 1$ and $R_{02} > 1$, there exists T_1^* such that $S_{1m}^{1*} = \lim_{t\to T_1^*} S_{1m}^1(t) = 0$, then $\Theta_1^* = 0$. Through calculation, it is easy to obtain RSE E_1^*. Substituting S_{1m}^{2*} into Θ_2^*,

and defining Θ_2^* as $\widehat{\Theta}_2$, one has

$$\widehat{\Theta}_2 = \frac{1}{(\gamma_2+\mu)\langle m\rangle}\sum_{m=1}^{n}\nu(m)p(m)\frac{\Lambda\beta(m)\widehat{\Theta}_2}{(\beta(m)\widehat{\Theta}_2+\mu(1+\rho_2\widehat{\Theta}_2))}.$$

It is equivalent to $\widehat{\Theta}_2 F_1(\widehat{\Theta}_2)=0$. where

$$\begin{aligned}F_1(\widehat{\Theta}_2) &= 1-\frac{1}{(\gamma_2+\mu)\langle m\rangle}\\ &\quad\times\sum_{m=1}^{n}\nu(m)p(m)\frac{\Lambda\beta(m)}{(\beta(m)\widehat{\Theta}_2+\mu(1+\rho_2\widehat{\Theta}_2))}.\end{aligned}$$

Obviously, $\widehat{\Theta}_2=0$ is a trivial solution. Since

$$\begin{aligned}\frac{\mathrm{d}F_1(\widehat{\Theta}_2)}{\mathrm{d}\widehat{\Theta}_2} &= \frac{\Lambda}{(\gamma_2+\mu)\langle m\rangle}\frac{\sum_{m=1}^{n}\nu(m)p(m)\beta(m)(\beta(m)+\mu\rho_2)}{(\beta(m)\widehat{\Theta}_2+\mu(1+\rho_m\widehat{\Theta}_2))^2}\\ &> 0,\\ F_1\left(\frac{\Lambda}{\mu}\right) &= 1-\frac{\mu}{\gamma_2+\mu}\frac{\sum_{m=1}^{n}\nu(m)p(m)}{\langle m\rangle}\frac{\Lambda\beta(m)}{\Lambda\beta(m)+\mu^2+\mu\rho_2\Lambda}\\ &> 1-\frac{\mu}{\gamma_2+\mu}\frac{\sum_{m=1}^{n}\nu(m)p(m)}{\langle m\rangle}\geq 1-\frac{\mu}{\gamma_2+\mu}>0,\end{aligned}$$

then $F_1(\widehat{\Theta}_2^*)=0$ if and only if $\lim_{\widehat{\Theta}_2\to 0^+}F_1(\widehat{\Theta}_2)=1-R_{02}<0$, which means that system (1) has a unique positive RSE $E_1^*(I_{1m}^*,S_{1m}^{1*},S_{1m}^{2*},R_{1m}^*)$ for $R_{01}<1$ and $R_{02}>1$.

For Case 3, when $R_{01}>1$ and $R_{02}<1$, there exists T_2^* such that $S_{2m}^{2*}=\lim_{t\to T_2^*}S_{2m}^2(t)=0$ holds, then $\Theta_2^*=0$. Since $\frac{\mathrm{d}S_m^2(t)}{\mathrm{d}t}\,|_{t=T_2^*}=0$, it can be derived that $\delta=0$. Thus, the RSE E_2^* can be obtained. Substituting S_{2m}^{1*} into Θ_1^*, and defining Θ_1^* as $\overline{\Theta}_1$, one has

$$\overline{\Theta}_1 = \frac{1}{(\gamma_1+\mu)\langle m\rangle}\sum_{m=1}^{n}\nu(m)p(m)\frac{\Lambda\alpha(m)\overline{\Theta}_1}{\alpha(m)\overline{\Theta}_1+\mu(1+\rho_1\overline{\Theta}_1)}.$$

Let $\overline{\Theta}_1F_2(\overline{\Theta}_1)=0$, where

$$\begin{aligned}F_2(\overline{\Theta}_1) &= 1-\frac{1}{(\gamma_1+\mu)\langle m\rangle}\\ &\quad\times\sum_{m=1}^{n}\nu(m)p(m)\frac{\Lambda\alpha(m)\overline{\Theta}_1}{\alpha(m)\overline{\Theta}_1+\mu(1+\rho_1\overline{\Theta}_1)}.\end{aligned}$$

Similar to the analytical method in Case 2, it can be obtained that system (1) has a unique RSE E_2^* for $\delta=0$ and $R_{01}>1$, $R_{02}<1$.

For Case 4, substituting S_m^{1*} and S_m^{2*} into Θ_1^* and Θ_2^* respectively, and defining Θ_1^* and Θ_2^* as $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$ respectively, we have

$$\begin{aligned}\widetilde{\Theta}_1 &= \frac{\Lambda}{(\delta+\gamma_1+\mu)}\frac{\sum_{m=1}^{n}\nu(m)p(m)}{\langle m\rangle}\frac{\alpha(m)\widetilde{\Theta}_1(1+\rho_2\widetilde{\Theta}_2)}{\Delta},\\ \widetilde{\Theta}_2 &= \frac{\Lambda}{\gamma_2+\mu}\frac{\sum_{m=1}^{n}\nu(m)p(m)}{\langle m\rangle}\left(\frac{\beta(m)\widetilde{\Theta}_2(1+\rho_1\widetilde{\Theta}_1)}{\Delta}\right.\\ &\quad\left.+\frac{\delta\alpha(m)\widetilde{\Theta}_1(1+\rho_2\widetilde{\Theta}_2)}{(\delta+\gamma_1+\mu)\Delta}\right),\end{aligned}$$

where

$$\begin{aligned}\Delta &= \alpha(m)\widetilde{\Theta}_1(1+\rho_2\widetilde{\Theta}_2)+\beta(m)\widetilde{\Theta}_2(1+\rho_1\widetilde{\Theta}_1)\\ &\quad+\mu(1+\rho_1\widetilde{\Theta}_1)(1+\rho_2\widetilde{\Theta}_2).\end{aligned}$$

Let $\widetilde{\Theta}_1F_3(\widetilde{\Theta}_1,\widetilde{\Theta}_2)=0$, where

$$F_3(\widetilde{\Theta}_1,\widetilde{\Theta}_2)=1-\frac{\Lambda}{\delta+\gamma_1+\mu}\frac{\sum_{m=1}^{n}\nu(m)p(m)}{\langle m\rangle}\frac{\alpha(m)(1+\rho_2\widetilde{\Theta}_2)}{\Delta}$$

and let $\widetilde{\Theta}_2F_4(\widetilde{\Theta}_1,\widetilde{\Theta}_2)=0$, where

$$\begin{aligned}F_4(\widetilde{\Theta}_1,\widetilde{\Theta}_2) &= 1-\frac{\Lambda\sum_{m=1}^{n}\nu(m)p(m)}{(\gamma_2+\mu)\langle m\rangle}\\ &\quad\times\left(\frac{\beta(m)(1+\rho_1\widetilde{\Theta}_1)}{\Delta}+\frac{\delta\alpha(m)\widetilde{\Theta}_1(1+\rho_2\widetilde{\Theta}_2)}{(\delta+\gamma_1+\mu)\widetilde{\Theta}_2\Delta}\right).\end{aligned}$$

Since $\frac{\partial F_3(\widetilde{\Theta}_1,\widetilde{\Theta}_2)}{\partial\widetilde{\Theta}_1}>0$ and $F_3(\frac{\Lambda}{u},\widetilde{\Theta}_2)>0$, then $S_m^{1*}>0$ if $\lim_{\widetilde{\Theta}_1\to 0^+}F_3(\widetilde{\Theta}_1,0)<0$ and $F_3(0,0)=1-R_{01}$. Similarly, $S_m^{2*}>0$ if $\lim_{\widetilde{\Theta}_2\to 0^+}F_4(0,\widetilde{\Theta}_2)<0$. In other words, there is a unique RSE $E^*(I_m^*,S_m^{1*},S_m^{2*},R_m^*)$ if $R_{01}>1$ and $R_{02}>1$.

4. Stability analysis of equilibriums

Theorem 2 The RFE E_0 of system (1) is locally asymptotically stable if $R_{01}<1$ and $R_{02}<1$.

Proof The Jacobian matrix $\boldsymbol{J}(E_0)$ of system (1) is

$$\boldsymbol{J}(E_0)=\begin{pmatrix}\boldsymbol{D}_{11} & \boldsymbol{D}_{12} & \cdots & \boldsymbol{D}_{1n}\\ \boldsymbol{D}_{21} & \boldsymbol{D}_{22} & \cdots & \boldsymbol{D}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{D}_{n1} & \boldsymbol{D}_{n2} & \cdots & \boldsymbol{D}_{nn}\end{pmatrix},$$

where

$$\boldsymbol{D}_{ij}=\begin{pmatrix}0 & -\frac{\Lambda}{\mu}\frac{\alpha(i)\nu(j)p(j)}{\langle m\rangle}\mathrm{e}^{-\lambda\tau_1} & -\frac{\Lambda}{\mu}\frac{\beta(i)\nu(j)p(j)}{\langle m\rangle}\mathrm{e}^{-\lambda\tau_2} & 0\\ 0 & \frac{\Lambda}{\mu}\frac{\alpha(i)\nu(j)p(j)}{\langle m\rangle}\mathrm{e}^{-\lambda\tau_1} & 0 & 0\\ 0 & 0 & \frac{\Lambda}{\mu}\frac{\beta(i)\nu(j)p(j)}{\langle m\rangle}\mathrm{e}^{-\lambda\tau_2} & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\boldsymbol{D}_{ii}=\begin{pmatrix} -\mu & -\dfrac{\Lambda}{\mu}\dfrac{\alpha(i)\nu(i)p(i)}{\langle m\rangle}\mathrm{e}^{-\lambda\tau_1} & -\dfrac{\Lambda}{\mu}\dfrac{\beta(i)\nu(i)p(i)}{\langle m\rangle}\mathrm{e}^{-\lambda\tau_2} & 0\\ 0 & \dfrac{\Lambda}{\mu}\dfrac{\alpha(i)\nu(i)p(i)}{\langle m\rangle}\mathrm{e}^{-\lambda\tau_1}-(\delta+\gamma_1+\mu) & 0 & 0\\ 0 & \delta & \dfrac{\Lambda}{\mu}\dfrac{\beta(i)\nu(i)p(i)}{\langle m\rangle}\mathrm{e}^{-\lambda\tau_2}-(\gamma_2+\mu) & 0\\ 0 & \gamma_1 & \gamma_2 & -\mu \end{pmatrix}.$$

Then the characteristic equation of $J(E_0)$ is

$$(\lambda+\mu)^{2n}(\lambda+\delta+\gamma_1+\mu)^{n-1}(\lambda+\gamma_2+\mu)^{n-1}\times(\lambda-\lambda_1)(\lambda-\lambda_2)=0,$$

where

$$\begin{aligned}\lambda_1 &= \frac{\Lambda}{\mu}\frac{\sum_{m=1}^{n}\alpha(m)\nu(m)p(m)}{\langle m\rangle}\mathrm{e}^{-\lambda_1\tau_1}-(\delta+\gamma_1+\mu)\\ &= (\delta+\gamma_1+\mu)(R_{01}\mathrm{e}^{-\lambda_1\tau_1}-1), \qquad (3)\\ \lambda_2 &= \frac{\Lambda}{\mu}\frac{\sum_{m=1}^{n}\beta(m)\nu(m)p(m)}{\langle m\rangle}\mathrm{e}^{-\lambda_2\tau_2}-(\gamma_2+\mu)\\ &= (\gamma_2+\mu)(R_{02}\mathrm{e}^{-\lambda_2\tau_2}-1). \qquad (4)\end{aligned}$$

Obviously, the characteristic equation of $\boldsymbol{J}(E_0)$ has $4n$ eigenvalues, where $2n$ eigenvalues are $-\mu$, $n-1$ eigenvalues are $-(\delta+\gamma_1+\mu)$, $n-1$ eigenvalues are $-(\gamma_2+\mu)$, and the others two eigenvalues are λ_1 and λ_2.

(i) When $\tau_1=\tau_2=0$, from Eqs. (3) and (4), we can obtain

$$\lambda_1=(\delta+\gamma_1+\mu)(R_{01}-1),\ \ \lambda_2=(\gamma_2+\mu)(R_{02}-1),$$

which show that λ_1 and λ_2 have negative real parts only if $R_{01}<1$ and $R_{02}<1$.

(ii) When $\tau_1,\ \tau_2>0$, assume $\lambda_1=\mathrm{i}\omega_1(\omega_1>0)$, $\lambda_2=\mathrm{i}\omega_2(\omega_2>0)$. Then, separating the real and imaginary parts, one has

$$\begin{cases}\omega_1=-(\delta+\gamma_1+\mu)R_{01}\sin\omega_1\tau_1,\\ \delta+\gamma_1+\mu=(\delta+\gamma_1+\mu)R_{01}\cos\omega_1\tau_1,\end{cases} \qquad (5)$$

$$\begin{cases}\omega_2=-(\gamma_2+\mu)R_{02}\sin\omega_2\tau_2,\\ \gamma_2+\mu=(\gamma_2+\mu)R_{02}\cos\omega_2\tau_2.\end{cases} \qquad (6)$$

By squaring Eqs. (5) and (6), and adding them separately, one can derive

$$\omega_1^2=(\delta+\gamma_1+\mu)^2(R_{01}^2-1),\ \ \omega_2^2=(\gamma_2+\mu)^2(R_{02}^2-1).$$

When $R_{01}<1$ and $R_{02}<1$, there are no positive real roots ω_1 and ω_2 that satisfy Eqs. (5) and (6), respectively. Let $y(\lambda_1)=\lambda_1-(\delta+\gamma_1+\mu)(R_{01}\mathrm{e}^{-\lambda_1\tau_1}-1)$. Since $\dot{y}(\lambda_1)=1+\tau_1(\delta+\gamma_1+\mu)R_{01}\mathrm{e}^{-\lambda_1\tau_1}>0$, $y(\lambda_1)$ is strictly increasing. If $R_{01}<1$, $y(0)=-(\delta+\gamma_1+\mu)(R_{01}-1)>0$, then $y(\lambda_1)=0$ has a negative real root. Similarly, if $R_{02}<1$, $y(\lambda_2)=0$ has a negative real root. Thus, the RFE E_0 of system (1) is locally asymptotically stable for $R_{01}<1$ and $R_{02}<1$.

Lemma 3[40] For any $r>0$ and $s>0$, if $\frac{\mathrm{d}x(t)}{\mathrm{d}t}\geq r-sx(t)$, we have $\liminf_{t\to\infty}x(t)\geq\frac{r}{s}$ for $t\geq0$ and $x(0)\geq0$; if $\frac{\mathrm{d}x(t)}{\mathrm{d}t}\leq r-sx(t)$, we have $\limsup_{t\to\infty}x(t)\leq\frac{r}{s}$ for $t\geq0$ and $x(0)\geq0$.

Theorem 3 The RFE E_0 of system (1) is globally asymptotically stable if $R_{01}<1$ and $R_{02}<1$.

Proof Firstly, we claim that $\lim_{t\to\infty}S_m^1(t)=0$. From the first equation of system (1), it follows that

$$\frac{\mathrm{d}I_m(t)}{\mathrm{d}t}\leq\Lambda-\mu I_m(t).$$

By using Lemma 3, we derive that

$$\limsup_{t\to\infty}I_m(t)\leq\frac{\Lambda}{\mu}=:I_m^\circ. \qquad (7)$$

Thus, for arbitrarily enough small ξ_1, there exists t_1 such that $I_m(t)\leq I_m^\circ+\xi_1$ for $t>t_1$. If $t>t_1$, we obtain

$$\frac{\mathrm{d}S_m^1(t)}{\mathrm{d}t}\leq\alpha(m)(I_m^\circ+\xi_1)\Theta_1(t-\tau_1)-(\delta+\gamma_1+\mu)S_m^1(t).$$

Consider the comparison system with the condition $A_m(0)=S_m^1(0)$ as follows:

$$\frac{\mathrm{d}A_m(t)}{\mathrm{d}t}=\alpha(m)(I_m^\circ+\xi_1)\widehat{\Theta}_1(t-\tau_1)-(\delta+\gamma_1+\mu)A_m(t),$$

where $\widehat{\Theta}_1(t-\tau_1)=\frac{1}{\langle m\rangle}\sum_{m=1}^{n}\nu(m)p(m)A_m(t-\tau_1)$. The following proves that $\lim_{t\to\infty}A_m(t)=0$. Constructing the Lyapunov functional

$$V_1(t)=\frac{\sum_{m=1}^{n}\nu(m)p(m)}{\langle m\rangle}\times\left(\alpha(m)(I_m^\circ+\xi_1)\int_{-\tau_1}^{0}\widehat{\Theta}_1(t+u)\mathrm{d}u+A_m(t)\right).$$

Calculating the derivative of $V_1(t)$ along the solution E_0, it shows that

$$\begin{aligned}\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} &= \frac{\sum_{m=1}^{n}\nu(m)p(m)}{\langle m\rangle}\times\left(\alpha(m)(I_m^\circ+\xi_1)\widehat{\Theta}_1(t)-(\delta+\gamma_1+\mu)A_m(t)\right)\\ &= (\delta+\gamma_1+\mu)\widehat{\Theta}_1(t)\times\left(R_{01}+\frac{\sum_{m=1}^{n}\nu(m)p(m)\alpha(m)\xi_1}{(\delta+\gamma_1+\mu)\langle m\rangle}-1\right).\end{aligned}$$

Since $R_{01}<1$, we choose $\xi_1>0$ sufficiently small such that $R_{01}+\frac{\sum_{m=1}^{n}\nu(m)p(m)\alpha(m)\xi_1}{(\delta+\gamma_1+\mu)\langle m\rangle}<1$. This guarantees that $R_{01}<1$, $\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}\leq0$ for all $A_m(t)\geq0$, and $\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}=0$ only if $A_m(t)=0$, then $\lim_{t\to\infty}A_m(t)=0$. By the comparison principle, one has $0<S_m^1(t)<A_m(t)$ for all $t_1>0$. Thus, $\lim_{t\to\infty}S_m^1(t)=0$.

Next, we will prove $\lim_{t\to\infty} S_m^2(t) = 0$. For arbitrary $\xi_2 > 0$ and $\widehat{\xi}_2 > 0$ small enough, there exists $t_2 > 0$ such that $I_m(t) \le I_m^\circ + \xi_2$ and $0 < S_m^1(t) < \widehat{\xi}_2$ for $t > t_2$. From system (1), one has

$$\frac{\mathrm{d}S_m^2(t)}{\mathrm{d}t} \le \beta(m)(I_m^\circ + \xi_2)\Theta_2(t-\tau_2) + \delta\widehat{\xi}_2 - (\gamma_1+\mu)S_m^1(t).$$

Denote $B_m(0) = S_m^2(0)$ and $\widehat{\Theta}_2(t) = \frac{1}{\langle m\rangle}\sum_{m=1}^n \nu(m)p(m)B_m(t-\tau_2)$. Based on the comparison principle, one has

$$\frac{\mathrm{d}B_m(t)}{\mathrm{d}t} = \beta(m)(I_m^\circ + \xi_2)\widehat{\Theta}_2(t-\tau_2) + \delta\widehat{\xi}_2 - (\gamma_2+\mu)B_m(t).$$

Let us consider the Lyapunov functional

$$V_2(t) = \frac{\sum_{m=1}^n \nu(m)p(m)}{\langle m\rangle} \times \left(\beta(m)(I_m^\circ+\xi_2)\int_{-\tau_2}^0 \widehat{\Theta}_2(t+u)\,\mathrm{d}u + B_m(t)\right).$$

Then, we have

$$\frac{\mathrm{d}V_2(t)}{\mathrm{d}t} = (\gamma_2+\mu)\widehat{\Theta}_2(t)\left(R_{02} + \frac{\sum_{m=1}^n \nu(m)p(m)\beta(m)\xi_2}{(\gamma_2+\mu)\langle m\rangle} + \frac{\delta\widehat{\xi}_2}{(\gamma_2+\mu)B_m(t)} - 1\right).$$

Similar to the previous analysis, we obtain $\lim_{t\to\infty} S_m^2(t) = 0$.

Due to $\lim_{t\to\infty} S_m^1(t) = 0$ and $\lim_{t\to\infty} S_m^2(t) = 0$, for $\xi_3 > 0$ arbitrarily small enough, there exists $t_3 > 0$ such that $0 < S_m^1(t),\ S_m^2(t) < \xi_3$ for $t > t_3$. Form the first equation of (1), it follows that

$$\frac{\mathrm{d}I_m(t)}{\mathrm{d}t} \ge \Lambda - (\alpha(m)+\beta(m))I_m(t)\widehat{\Theta}_{\xi_3} - \mu I_m(t),$$

where $\widehat{\Theta}_{\xi_3} = \frac{1}{\langle m\rangle}\sum_{m=1}^n \nu(m)p(m)\xi_3$. Then, one has

$$I_m(t) \ge \frac{\Lambda}{(\alpha(m)+\beta(m))\widehat{\Theta}_{\xi_3} + \mu}.$$

Let $\xi_3 \to 0$, there has

$$\liminf_{t\to\infty} I_m(t) \ge \frac{\Lambda}{\mu} =: I_m^\circ. \tag{8}$$

Combing Eqs. (7) and (8), it is clear that $\lim_{t\to\infty} I_m(t) = I_m^\circ = \Lambda/\mu$. This proves the RFE E_0 is globally attractive for $R_{01} < 1$ and $R_{02} < 1$ and it follows from Theorem 2 that E_0 is indeed globally asymptotically stable if $R_{01} < 1$ and $R_{02} < 1$.

Next, to conveniently prove the global asymptotic stability of the RSE, system (1) can be rewritten as

$$\begin{cases} \frac{\mathrm{d}I_m(t)}{\mathrm{d}t} = \Lambda - \sum_{i=1}^n q_{mi}\frac{I_m(t)S_i^1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)} - \sum_{i=1}^n h_{mi}\frac{I_m(t)S_i^2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)} - \mu I_m(t), \\ \frac{\mathrm{d}S_m^1(t)}{\mathrm{d}t} = \sum_{i=1}^n q_{mi}\frac{I_m(t)S_i^1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)} - (\delta+\gamma_1+\mu)S_m^1(t), \\ \frac{\mathrm{d}S_m^2(t)}{\mathrm{d}t} = \sum_{i=1}^n h_{mi}\frac{I_m(t)S_i^2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)} + \delta S_m^1(t) - (\gamma_2+\mu)S_m^2(t), \\ \frac{\mathrm{d}R_m(t)}{\mathrm{d}t} = \gamma_1 S_m^1(t) + \gamma_2 S_m^2(t) - \mu R_m(t), \end{cases} \tag{9}$$

where $q_{mi} = \alpha(m)\nu(i)p(i)/\langle m\rangle$, $h_{mi} = \beta(m)\nu(i)p(i)/\langle m\rangle$, $m,i = 1,\ldots,n$.

Theorem 4 If $R_{01} < 1$ and $R_{02} > 1$, the RSE E_1^* is globally asymptotically stable.

Proof If $R_{01} < 1$ and $R_{02} > 1$, from Case 2 in Theorem 1 and Eq. (9), we have

$$\begin{cases} \frac{\mathrm{d}I_m(t)}{\mathrm{d}t} = \Lambda - \sum_{i=1}^n h_{mi}\frac{I_m(t)S_i^2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)} - \mu I_m(t), \\ \frac{\mathrm{d}S_m^2(t)}{\mathrm{d}t} = \sum_{i=1}^n h_{mi}\frac{I_m(t)S_i^2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)} - (\gamma_2+\mu)S_m^2(t), \\ \frac{\mathrm{d}R_m(t)}{\mathrm{d}t} = \gamma_2 S_m^2(t) - \mu R_m(t). \end{cases} \tag{10}$$

Construct the Lyapunov functional

$$\begin{aligned} V_{1m}(t) = {} & \sum_{m=1}^n h_{mi}\int_{t-\tau_2}^t \left(\frac{I_{1m}^* S_i^2(s)}{1+\rho_2\Theta_2(s)} - \frac{I_{1m}^* S_{1i}^{2*}}{1+\rho_2\Theta_2^*} \right. \\ & \left. - \frac{I_{1m}^* S_{1i}^{2*}}{1+\rho_2\Theta_2^*}\ln\frac{(1+\rho_2\Theta_2^*)S_i^2(s)}{(1+\rho_2\Theta_2(s))S_{1i}^{2*}}\right)\mathrm{d}s \\ & + S_m^2(t) - S_{1m}^{2*} - S_{1m}^{2*}\ln\frac{S_m^2(t)}{S_{1m}^{2*}} + I_m(t) \\ & - I_{1m}^* - I_m^*\ln(I_m(t)/I_{1m}^*). \end{aligned} \tag{11}$$

Calculating the derivative of $V_{1m}(t)$ along solution E_1^*, we obtain

$$\begin{aligned} & \frac{\mathrm{d}V_{1m}(t)}{\mathrm{d}t} \\ = {} & -\mu\frac{(I_m(t)-I_{1m}^*)^2}{I_m(t)} + 2\sum_{i=1}^n h_{mi}\frac{I_{1m}^* S_{1i}^{2*}}{1+\rho_2\Theta_2^*} \\ & - \frac{I_{1m}^*}{I_m(t)}\sum_{i=1}^n h_{mi}\frac{I_{1m}^* S_{1i}^{2*}}{1+\rho_2\Theta_2^*} - \frac{S_m^2(t)}{S_{1m}^{2*}(t)}\sum_{i=1}^n h_{mi}\frac{I_{1m}^* S_{1i}^{2*}}{1+\rho_2\Theta_2^*} \\ & - \frac{S_{1m}^{2*}}{S_m^2(t)}\sum_{i=1}^n h_{mi}\frac{I_m(t)S_i^2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)} + \sum_{i=1}^n h_{mi}\frac{I_{1m}^* S_i^2(t)}{1+\rho_2\Theta_2(t)} \\ & - \sum_{i=1}^n h_{mi}\frac{I_m^* S_{1i}^{2*}}{1+\rho_2\Theta_2^*}\ln\frac{(1+\rho_2\Theta_2^*)S_i^2(t)}{(1+\rho_2\Theta_2(t))S_{1i}^{2*}} \\ & + \sum_{i=1}^n h_{mi}\frac{I_{1m}^* S_{1i}^{2*}}{1+\rho_2\Theta_2^*}\ln\frac{(1+\rho_2\Theta_2^*)S_i^2(t-\tau_2)}{(1+\rho_2\Theta_2(t-\tau_2))S_{1i}^{2*}}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} & -\ln\frac{(1+\rho_2\Theta_2^*)S_i^2(t)}{(1+\rho_2\Theta_2(t))S_{1i}^{2*}} = \ln\frac{1+\rho_2\Theta_2(t)}{1+\rho_2\Theta_2^*} - \ln\frac{S_i^2(t)}{S_{1i}^{2*}}, \\ & \ln\frac{(1+\rho_2\Theta_2^*)S_i^2(t-\tau_2)}{(1+\rho_2\Theta_2(t-\tau_2))S_{1i}^{2*}} - \ln\frac{I_{1m}^*}{I_m(t)} - \ln\frac{S_m^2(t)}{S_{1m}^{2*}} \\ = {} & \ln\frac{(1+\rho_2\Theta_2^*)S_i^2(t-\tau_2)I_m(t)S_{1m}^{2*}}{(1+\rho_2\Theta_2(t-\tau_2))S_i^{2*}I_{1m}^* S_m^2(t)}. \end{aligned} \tag{13}$$

Let $M(x) = -x + \ln(x)$ and $N(x) = 1 - x + \ln(x)$. Obviously, $N(x) \le 0$ for $x > 0$. Substituting Eq. (13) into Eq. (12), it has

$$\frac{\mathrm{d}V_{1m}(t)}{\mathrm{d}t}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} h_{mi} \frac{I_{1m}^* S_{1i}^{2*}}{1+\rho_2\Theta_2^*} \left\{ N\left(\frac{I_{1m}^*}{I_m(t)}\right) \right. \\
&+ N\left(\frac{(1+\rho_2\Theta_2^*)S_i^2(t-\tau_2)I_m(t)S_{1m}^{2*}}{(1+\rho_2\Theta_2(t-\tau_2))S_{1i}^{2*}I_{1m}^*S_m^2(t)}\right) + N\left(\frac{1+\rho_2\Theta_2(t)}{1+\rho_2\Theta_2^*}\right) \Bigg\} \\
&+ \sum_{i=1}^{n} h_{mi} \frac{I_{1m}^* S_i^{2*}}{1+\rho_2\Theta_2^*} \left\{ M\left(\frac{S_m^2(t)}{S_{1m}^{2*}}\right) - M\left(\frac{S_i^2(t)}{S_{1i}^{2*}}\right) \right\} \\
&- \frac{\beta(m)\rho_2 I_{1m}^*(\Theta_2(t)-\Theta_2^*)^2}{(1+\rho_2\Theta_2^*)^2(1+\rho_2\Theta_2(t))} - \mu\frac{(I_m(t)-I_{1m}^*)^2}{I_m(t)}. \qquad (14)
\end{aligned}
$$

Note that the matrix

$$
(h_{mi}I_{1m}^*S_{1i}^{2*})_{n\times n} = \left(\frac{\beta(m)\nu(i)p(i)}{\langle m\rangle} I_{1m}^*S_{1i}^{2*}\right)_{n\times n}
$$

is irreducible. Then the matrix $\bar{\boldsymbol{H}}$ is also irreducible, where

$$
\bar{\boldsymbol{H}} = \begin{pmatrix}
\sum_{i\neq 1} h_{1i}I_{11}^*S_{1i}^{2*} & -h_{21}I_{12}^*S_{11}^{2*} & \cdots & -h_{n1}I_{1n}^*S_{11}^{2*} \\
-h_{12}I_{11}^*S_{12}^{2*} & \sum_{i\neq 2} h_{2i}I_{12}^*S_{1i}^{2*} & \cdots & -h_{n2}I_{1n}^*S_{12}^{2*} \\
\vdots & \vdots & \ddots & \vdots \\
-h_{1n}I_{11}^*S_{1n}^{2*} & -h_{2n}I_{12}^*S_{1n}^{2*} & \cdots & \sum_{i\neq n} h_{ni}I_{1n}^*S_{1i}^{2*}
\end{pmatrix}.
$$

Based on the Lemma 2.1 in Ref. [41], there exists a positive vector $\boldsymbol{\varpi} = (\varpi_1, \varpi_2, ..., \varpi_n)$ such that $\bar{\boldsymbol{H}}\boldsymbol{\varpi} = 0$. From $\bar{\boldsymbol{H}}\boldsymbol{\varpi} = 0$, one has

$$
\varpi_m \sum_{i=1}^{n} h_{mi}I_{1m}^*S_{1i}^{2*} = \sum_{i=1}^{n} \varpi_i h_{im} I_{1i}^* S_{1m}^{2*},
$$

then

$$
\begin{aligned}
&\sum_{m=1}^{n} \varpi_m \sum_{i=1}^{n} h_{mi}I_{1m}^*S_{1i}^{2*} M\left(\frac{S_m^2(t)}{S_{1m}^{2*}}\right) \\
&= \sum_{m=1}^{n}\sum_{i=1}^{n} \varpi_i h_{im} I_{1i}^* S_{1m}^{2*} M\left(\frac{S_m^2(t)}{S_{1m}^{2*}}\right) \\
&= \sum_{m=1}^{n}\sum_{i=1}^{n} \varpi_m h_{mi} I_{1m}^* S_{1i}^{2*} M\left(\frac{S_i^2(t)}{S_{1i}^{2*}}\right) \\
&= \sum_{m=1}^{n} \varpi_m \sum_{i=1}^{n} h_{mi}I_{1m}^*S_{1i}^{2*} M\left(\frac{S_i^2(t)}{S_{1i}^{2*}}\right), \qquad (15)
\end{aligned}
$$

which means that

$$
\sum_{m=1}^{n} \varpi_m \sum_{i=1}^{n} h_{mi}I_{1m}^*S_{1i}^{2*} \left\{ M\left(\frac{S_m^2(t)}{S_{1m}^{2*}}\right) - M\left(\frac{S_i^2(t)}{S_{1i}^{2*}}\right) \right\} = 0. \qquad (16)
$$

Combining Eqs. (14)–(16), define a Lyapunov functional as $V_m(t) = \sum_{m=1}^{n} \varpi_m V_{1m}(t)$, then

$$
\begin{aligned}
\frac{\mathrm{d}V_m(t)}{\mathrm{d}t} &= \sum_{m=1}^{n} \varpi_m \frac{\mathrm{d}V_{1m}(t)}{\mathrm{d}t} \\
&\leq \sum_{m=1}^{n} \varpi_m \sum_{i=1}^{n} h_{mi} \frac{I_{1m}^* S_i^{2*}}{1+\rho_2\Theta_2^*} \\
&\quad \times \left\{ M\left(\frac{S_m^2(t)}{S_{1m}^{2*}}\right) - M\left(\frac{S_i^2(t)}{S_{1i}^{2*}}\right) \right\} = 0,
\end{aligned}
$$

which shows that $\mathrm{d}V_m(t)/\mathrm{d}t \leq 0$ and $\mathrm{d}V_m(t)/\mathrm{d}t = 0$ if and only if $I_m(t) = I_{1m}^*$ and $S_m^2(t) = S_{1m}^{2*}$. By the LaSalle's invariance principle,[42] the RSE E_1^* is globally asymptotically stable for $R_{01} < 1$ and $R_{02} > 1$.

Theorem 5 If $R_{01} > 1$, $R_{02} < 1$ and $\delta = 0$, the RSE E_2^* is globally asymptotically stable.

Proof If $R_{01} > 1$, $R_{02} < 1$ and $\delta = 0$, based on Case 3 in Theorem 1 and Eq. (9), one has

$$
\begin{cases}
\frac{\mathrm{d}I_m(t)}{\mathrm{d}t} = \Lambda - \sum_{i=1}^{n} q_{mi} \frac{I_m(t)S_i^1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)} - \mu I_m(t), \\
\frac{\mathrm{d}S_m^1(t)}{\mathrm{d}t} = \sum_{i=1}^{n} q_{mi} \frac{I_m(t)S_i^1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)} - (\gamma_1+\mu)S_m^1(t), \\
\frac{\mathrm{d}R_m(t)}{\mathrm{d}t} = \gamma_1 S_m^1(t) - \mu R_m(t).
\end{cases}
$$

Consider the following Lyapunov functional:

$$
\begin{aligned}
V_{2m}(t) &= \sum_{m=1}^{n} q_{mi} \int_{t-\tau_1}^{t} \left(\frac{I_{2m}^* S_i^1(s)}{1+\rho_1\Theta_1(s)} - \frac{I_{2m}^* S_{2i}^{1*}}{1+\rho_1\Theta_1^*} \right. \\
&\quad \left. - \frac{I_{2m}^* S_{2i}^{1*}}{1+\rho_1\Theta_1^*} \ln \frac{(1+\rho_1\Theta_1^*)S_i^1(s)}{(1+\rho_1\Theta_1(s))S_{2i}^{1*}} \right) \mathrm{d}s \\
&\quad + S_m^1(t) - S_{2m}^{1*} - S_{2m}^{1*} \ln \frac{S_m^1(t)}{S_{2m}^{1*}} \\
&\quad + I_m(t) - I_{2m}^* - I_{2m}^* \ln \frac{I_m(t)}{I_{2m}^*}.
\end{aligned}
$$

Similar to the proof method of Theorem 4, it finally obtains that $\mathrm{d}V_{2m}(t)/\mathrm{d}t \leq 0$ and $\mathrm{d}V_{2m}(t)/\mathrm{d}t = 0$ if and only if $I_m(t) = I_{2m}^*$, $S_m^1(t) = S_{2m}^{1*}$. In the light of the LaSalle's invariance principle,[42] the RSE E_2^* is globally asymptotically stable.

Theorem 6 If $R_{01} > 1$, $R_{02} > 1$ and $\delta = 0$, the RSE E^* is globally asymptotically stable.

Proof Consider the Lyapunov functional

$$
\begin{aligned}
V_{3m}(t) &= I_m(t) - I_m^* - I_m^* \ln \frac{I_m(t)}{I_m^*} + S_m^1(t) - S_m^{1*} - S_m^{1*} \ln \frac{S_m^1(t)}{S_m^{1*}} \\
&\quad + \sum_{m=1}^{n} q_{mi} \int_{t-\tau_1}^{t} \left(\frac{I_m^* S_i^1(s)}{1+\rho_1\Theta_1(s)} - \frac{I_m^* S_i^{1*}}{1+\rho_1\Theta_1^*} - \frac{I_m^* S_i^{1*}}{1+\rho_1\Theta_1^*} \ln \frac{(1+\rho_1\Theta_1^*)S_i^1(s)}{(1+\rho_1\Theta_1(s))S_i^{1*}} \right) \mathrm{d}s \\
&\quad + \sum_{m=1}^{n} h_{mi} \int_{t-\tau_2}^{t} \left(\frac{I_m^* S_i^2(s)}{1+\rho_2\Theta_2(s)} - \frac{I_m^* S_i^{2*}}{1+\rho_2\Theta_2^*} - \frac{I_m^* S_i^{2*}}{1+\rho_2\Theta_2^*} \ln \frac{(1+\rho_2\Theta_2^*)S_i^2(s)}{(1+\rho_2\Theta_2(s))S_i^{2*}} \right) \mathrm{d}s + S_m^2(t) - S_m^{2*} - S_m^{2*} \ln \frac{S_m^2(t)}{S_m^{2*}}.
\end{aligned}
$$

Based on Case 4 in Theorem 1, when $\delta = 0$, the derivative of $V_{3m}(t)$ is

$$\begin{aligned}\frac{\mathrm{d}V_{3m}(t)}{\mathrm{d}t} =& -\mu\frac{(I_m(t)-I_m^*)^2}{I_m(t)} + 2\sum_{i=1}^{n} q_{mi}\frac{I_m^* S_i^{1*}}{1+\rho_1\Theta_1^*} - \frac{I_m^*}{I_m(t)}\sum_{i=1}^{n} q_{mi}\frac{I_m^* S_i^{1*}}{1+\rho_1\Theta_1^*} + 2\sum_{i=1}^{n}\frac{h_{mi} I_m^* S_i^{2*}}{1+\rho_2\Theta_2^*} - \frac{I_m^*}{I_m(t)}\sum_{i=1}^{n}\frac{h_{mi} I_m^* S_i^{2*}}{1+\rho_2\Theta_2^*} \\ & - \frac{S_m^1(t)}{S_m^{1*}}\sum_{i=1}^{n}\frac{q_{mi} I_m^* S_i^{1*}}{1+\rho_1\Theta_1^*} - \frac{S_m^{1*}}{S_m^1(t)}\sum_{i=1}^{n}\frac{q_{mi} I_m(t) S_i^1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)} + \sum_{i=1}^{n}\frac{q_{mi} I_m^* S_i^{1*}}{1+\rho_1\Theta_1^*}\ln\frac{(1+\rho_1\Theta_1^*)S_i^1(t-\tau_1)}{(1+\rho_1\Theta_1(t-\tau_1))S_i^{1*}} \\ & - \sum_{i=1}^{n} q_{mi}\frac{I_m^* S_i^{1*}}{1+\rho_1\Theta_1^*}\ln\frac{(1+\rho_1\Theta_1^*)S_i^1(t)}{(1+\rho_1\Theta_1(t))S_i^{1*}} + \sum_{i=1}^{n} q_{mi}\frac{I_m^* S_i^1(t)}{1+\rho_1\Theta_1(t)} - \frac{S_m^2(t)}{S_m^{2*}(t)}\sum_{i=1}^{n} h_{mi}\frac{I_m^* S_i^{2*}}{1+\rho_2\Theta_2^*} \\ & - \frac{S_m^{2*}}{S_m^2(t)}\sum_{i=1}^{n} h_{mi}\frac{I_m(t) S_i^2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)} + \sum_{i=1}^{n} h_{mi}\frac{I_m^* S_i^2(t)}{1+\rho_2\Theta_2(t)} - \sum_{i=1}^{n} h_{mi}\frac{I_m^* S_i^{2*}}{1+\rho_2\Theta_2^*}\ln\frac{(1+\rho_2\Theta_2^*)S_i^2(t)}{(1+\rho_2\Theta_2(t))S_i^{2*}} \\ & + \sum_{i=1}^{n} h_{mi}\frac{I_m^* S_i^{2*}}{1+\rho_2\Theta_2^*}\ln\frac{(1+\rho_2\Theta_2^*)S_i^2(t-\tau_2)}{(1+\rho_2\Theta_2(t-\tau_2))S_i^{2*}}.\end{aligned}$$

Similar to the analysis method of Theorem 4, it can be derived as

$$\begin{aligned}\frac{\mathrm{d}V_{3m}(t)}{\mathrm{d}t} =& \sum_{i=1}^{n} q_{mi}\frac{I_m^* S_i^{1*}}{1+\rho_1\Theta_1^*}\left\{N\left(\frac{I_m^*}{I_m(t)}\right) + N\left(\frac{(1+\rho_1\Theta_1^*)S_i^1(t-\tau_1)I_m(t)S_m^{1*}}{(1+\rho_1\Theta_1(t-\tau_1))S_i^{1*}I_m^* S_m^1(t)}\right) + N\left(\frac{1+\rho_1\Theta_1(t)}{1+\rho_1\Theta_1^*}\right)\right\} \\ & + \sum_{i=1}^{n} q_{mi}\frac{I_m^* S_i^{1*}}{1+\rho_1\Theta_1^*}\left\{M\left(\frac{S_m^1(t)}{S_m^{1*}}\right) - M\left(\frac{S_i^1(t)}{S_i^{1*}}\right)\right\} \\ & + \sum_{i=1}^{n} h_{mi}\frac{I_m^* S_i^{2*}}{1+\rho_2\Theta_2^*}\left\{N\left(\frac{I_m^*}{I_m(t)}\right) + N\left(\frac{(1+\rho_2\Theta_2^*)S_i^2(t-\tau_2)I_m(t)S_m^{2*}}{(1+\rho_2\Theta_2(t-\tau_2))S_i^{2*}I_m^* S_m^2(t)}\right) + N\left(\frac{1+\rho_2\Theta_2(t)}{1+\rho_2\Theta_2^*}\right)\right\} \\ & + \sum_{i=1}^{n} h_{mi}\frac{I_m^* S_i^{2*}}{1+\rho_2\Theta_2^*}\left\{M\left(\frac{S_m^2(t)}{S_m^{2*}}\right) - M\left(\frac{S_i^2(t)}{S_i^{2*}}\right)\right\} \\ & - \frac{\alpha(m)\rho_1 I_m^*(\Theta_1(t)-\Theta_1^*)^2}{(1+\rho_1\Theta_1^*)^2(1+\rho_1\Theta_1(t))} - \frac{\beta(m)\rho_2 I_m^*(\Theta_2(t)-\Theta_2^*)^2}{(1+\rho_2\Theta_2^*)^2(1+\rho_2\Theta_2(t))} - \frac{\mu(I_m(t)-I_m^*)^2}{I_m(t)} \le 0.\end{aligned}$$

Note that $\mathrm{d}V_{2m}(t)/\mathrm{d}t = 0$ if and only if $I_m(t) = I_m^*$, $S_m^1(t) = S_m^{1*}$, and $S_m^2(t) = S_m^{2*}$. Clearly, the invariant set of $\mathrm{d}V_{3m}(t)/\mathrm{d}t = 0$ is a singleton E^*. According to the LaSalle's invariance principle,[42] the RSE E^* is globally asymptotically stable for $R_{01} > 1$ and $R_{02} > 1$.

Remark 3 The basic reproduction numbers R_{01} and R_{02} depend on some model parameters and the fluctuations of the degree distribution. We will discuss the effect of different parameters on rumor-spreading in model (1). The following results can be obtained by direct calculation:

$$\begin{aligned}&\frac{\partial R_{01}}{\partial\alpha(m)} = \frac{\Lambda\sum_{m=1}^{n}\nu(m)p(m)}{\mu(\delta+\gamma_1+\mu)\langle m\rangle} > 0,\\ &\frac{\partial R_{01}}{\partial\delta} = -\frac{\Lambda\sum_{m=1}^{n}\nu(m)p(m)\alpha(m)}{\mu(\delta+\gamma_1+\mu)^2\langle m\rangle} < 0,\\ &\frac{\partial R_{01}}{\partial\gamma_1} = -\frac{\Lambda\sum_{m=1}^{n}\nu(m)p(m)\alpha(m)}{\mu(\delta+\gamma_1+\mu)^2\langle m\rangle} < 0,\\ &\frac{\partial R_{02}}{\partial\beta(m)} = \frac{\Lambda\sum_{m=1}^{n}\nu(m)p(m)}{\mu(\gamma_2+\mu)\langle m\rangle} > 0,\\ &\frac{\partial R_{02}}{\partial\gamma_2} = -\frac{\Lambda\sum_{m=1}^{n}\nu(m)p(m)\beta(m)}{\mu(\gamma_2+\mu)^2\langle m\rangle} < 0.\end{aligned}$$

From the above analysis, we can obtain the following conclusions:

(i) R_{01} and R_{02} are proportional to the propagation rates $\alpha(m)$ and $\beta(m)$, respectively, which inspires us to reduce R_{01} and R_{02} by reducing the propagation rates $\alpha(m)$ and $\beta(m)$, thereby helping to suppress effectively the spread of rumors.

(ii) With the increase of δ, R_{01} gradually decreases, which means that increasing δ can effectively reduce the density of spreaders-1.

(iii) As γ_1 and γ_2 gradually increase, R_{01} and R_{02} finally decrease, indicating that the intervention mechanism significantly controls the rumor-spreading. The intensity of the intervention directly affects the rumor-spreading process. If the intensity of the intervention is too weak, it cannot restrain the rumor-spreading. If the intensity of the intervention is too great, there will have too high control costs. Therefore, designing an optimal control scheme to effectively restrain the spread of rumors without incurring high control costs will be our subsequent work.

5. Optimal control

In this section, an optimal control scheme is proposed to decrease the density of rumors spreading individuals at the lowest cost within the expected time.

In real life, the harm caused by rumors in multilingual en-

vironment may also be different due to differences in language and influence. When rumors spread, government departments will implement intervention mechanism, such as education, publishing information to refute rumors, and deleting posts, banning and punishing spreaders so that spreaders become removers. In fact, the practical meaning of optimal control is to minimize spreaders' density by optimizing the intervention mechanism. When the ability to spread rumors is strong, government departments should increase the intensity of intervention. When the ability to spread rumors is weak, government departments should reduce the intensity of intervention. Therefore, in controlling rumor-spreading, it is necessary to evaluate the spread of rumors in multilingual environment to dynamically optimize the corresponding intervention mechanism and reduce the density of spreaders with the minimum cost, thus minimizing the loss caused by rumor spread and the cost of intervention.

For system (1), there exists state variables $I_m(t)$, $S_m^z(t)$, $S_m^z(t-\tau_z)$, and $R_m(t)(z=1,2)$. Based on the Pontryagin's minimum principle,[43] we introduce the Lebesgue square-integrable control function pair $(\gamma_{1m}(t),\gamma_{2m}(t))$ $(0<t\le T)$ to optimize the intervention mechanism. Then, the admissible set of control function is given by

$$\Gamma=\{\gamma_{zm}(t)\in L^2(0,T],0\le\gamma_{zm}(t)\le 1,0<t\le T,z=1,2\}.$$

Therefore, system (1) can be converted into the following control system:

$$\begin{cases}\dfrac{\mathrm{d}I_m(t)}{\mathrm{d}t}=\Lambda-\alpha(m)\dfrac{I_m(t)\Theta_1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)}-\beta(m)\dfrac{I_m(t)\Theta_2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)}\\ \qquad\qquad -\mu I_m(t),\\ \dfrac{\mathrm{d}S_m^1(t)}{\mathrm{d}t}=\alpha(m)\dfrac{I_m(t)\Theta_1(t-\tau_1)}{1+\rho_1\Theta_1(t-\tau_1)}-(\delta+\gamma_{1m}(t)+\mu)S_m^1(t),\\ \dfrac{\mathrm{d}S_m^2(t)}{\mathrm{d}t}=\beta(m)\dfrac{I_m(t)\Theta_2(t-\tau_2)}{1+\rho_2\Theta_2(t-\tau_2)}+\delta S_m^1(t)-(\gamma_{2m}(t)+\mu)S_m^2(t),\\ \dfrac{\mathrm{d}R_m(t)}{\mathrm{d}t}=\gamma_{1m}(t)S_m^1(t)+\gamma_{2m}(t)S_m^2(t)-\mu R_m(t).\end{cases}\tag{17}$$

The control function pair $(\gamma_{1m}(t),\gamma_{2m}(t))$ is found to minimize the objective function

$$J\big((\gamma_{1m}(t),\gamma_{2m}(t))\big)$$

$$=\int_0^T\sum_{m=1}^n\left(\xi_m^1S_m^1(t)+\xi_m^2S_m^2(t)+\xi_m^3\frac{(\gamma_{1m}(t))^2}{2}+\xi_m^4\frac{(\gamma_{2m}(t))^2}{2}\right)\mathrm{d}t,\tag{18}$$

where ξ_m^1 and ξ_m^2 are the weight coefficients relate to spreaders-1 and spreaders-2. ξ_m^3 and ξ_m^4 represent the weight coefficients of the control variables $\gamma_{1m}(t)$ and $\gamma_{2m}(t)$. Next, we can take the Lagrangian as

$$L\big(S_m^z(t),(\gamma_{1m}(t),\gamma_{2m}(t))\big)=\sum_{m=1}^n\left(\xi_m^1S_m^1(t)+\xi_m^2S_m^2(t)+\xi_m^3\frac{(\gamma_{1m}(t))^2}{2}+\xi_m^4\frac{(\gamma_{2m}(t))^2}{2}\right),$$

and define the Hamiltonian function of the control system (17) as

$$\begin{aligned}&H\big(I_m(t),S_m^z(t),S_m^z(t-\tau_z),R_m(t),\gamma_{zm}(t),\lambda_{im}(t)\big)\\&=L\big(S_m^z(t),(\gamma_{1m}(t),\gamma_{2m}(t))\big)\\&\quad+\sum_{m=1}^n\left(\lambda_{1m}(t)\frac{\mathrm{d}I_m(t)}{\mathrm{d}t}+\lambda_{2m}(t)\frac{\mathrm{d}S_m^1(t)}{\mathrm{d}t}\right.\\&\quad\left.+\lambda_{3m}(t)\frac{\mathrm{d}S_m^2(t)}{\mathrm{d}t}+\lambda_{4m}(t)\frac{\mathrm{d}R_m^1(t)}{\mathrm{d}t}\right),\end{aligned}\tag{19}$$

where $z=1,2$, $\lambda_{im}(t)$ $(i=1,2,3,4)$ denotes the adjoint function, it can be appropriately determined.

Remark 4 Different from the literature,[30] this paper adopts different control functions for spreaders-1 and spreaders-2 respectively, which can flexibly implement the control strength while saving the control cost.

Lemma 4[26,43] There exists an optimal control pair $(\gamma_{1m}^*(t),\gamma_{2m}^*(t))=\big((\gamma_{11}^*(t),\gamma_{21}^*(t)),\ldots,(\gamma_{1n}^*(t),\gamma_{2n}^*(t))\big)$ such that $J(\gamma_{1m}^*(t),\gamma_{2m}^*(t))=\min J\big((\gamma_{1m}(t),\gamma_{2m}(t))\big)$ subjects to the control system (17) with initial condition (2).

Theorem 7 Let $(\widetilde{I}_m^*(t),\widetilde{S}_m^{1*}(t),\widetilde{S}_m^{2*}(t),\widetilde{R}_m^*(t))$ is a unique optimal solutions associated with the optimal control variable $(\gamma_{1m}^*(t),\gamma_{2m}^*(t))$ for the optimal control problem Eqs. (17)–(19), where $m=1,\ldots,n$. Then, the corresponding adjoint function $\lambda_{im}(t)$ $(i=1,2,3,4)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}\lambda_{1m}(t)}{\mathrm{d}t}&=\lambda_{1m}(t)\left(\alpha(m)\frac{\widetilde{\Theta}_1^*(\widetilde{S}_m^{1*}(t))}{1+\rho_1\widetilde{\Theta}_1^*(\widetilde{S}_m^{1*}(t))}+\beta(m)\frac{\widetilde{\Theta}_2^*(\widetilde{S}_m^{2*}(t))}{1+\rho_2\widetilde{\Theta}_2^*(\widetilde{S}_m^{2*}(t))}+\mu\right)\\&\quad-\lambda_{2m}(t)\alpha(m)\frac{\widetilde{\Theta}_1^*(\widetilde{S}_m^{1*}(t))}{1+\rho_1\widetilde{\Theta}_1^*(\widetilde{S}_m^{1*}(t))}-\lambda_{3m}(t)\beta(m)\frac{\widetilde{\Theta}_2^*(\widetilde{S}_m^{2*}(t))}{1+\rho_2\widetilde{\Theta}_2^*(\widetilde{S}_m^{2*}(t))},\end{aligned}\tag{20}$$

$$\begin{aligned}\frac{\mathrm{d}\lambda_{2m}(t)}{\mathrm{d}t}&=-\xi_m^1+\lambda_{1m}(t)\times\lambda_{2m}(t+\tau_1)\frac{\nu(m)p(m)\alpha(m)}{\langle m\rangle}\frac{\widetilde{I}_m^*(t)}{\big(1+\rho_1\widetilde{\Theta}_1^*(\widetilde{S}_m^{1*}(t))\big)^2}\\&\quad-\lambda_{2m}(t)\left(\lambda_{2m}(t+\tau_1)\frac{\nu(m)p(m)\alpha(m)}{\langle m\rangle}\frac{\widetilde{I}_m^*(t)}{\big(1+\rho_1\widetilde{\Theta}_1^*(\widetilde{S}_m^{1*}(t))\big)^2}-(\delta+\gamma_{1m}^*(t)+\mu)\right)-\lambda_{3m}(t)\delta-\lambda_{4m}(t)\gamma_{1m}^*(t),\end{aligned}\tag{21}$$

$$\frac{\mathrm{d}\lambda_{3m}(t)}{\mathrm{d}t} = -\xi_m^2 + \lambda_{1m}(t)\times\lambda_{3m}(t+\tau_2)\frac{\nu(m)p(m)\beta(m)}{\langle m\rangle}\frac{\widetilde{I}_m^*(t)}{\left(1+\rho_2\widetilde{\Theta}_2^*(\widetilde{S}_m^{2*}(t))\right)^2}$$
$$-\lambda_{3m}(t)\left(\lambda_{3m}(t+\tau_2)\frac{\nu(m)p(m)\beta(m)}{\langle m\rangle}\frac{\widetilde{I}_m^*(t)}{\left(1+\rho_2\widetilde{\Theta}_2^*(\widetilde{S}_m^{2*}(t))\right)^2} - (\gamma_{2m}^*(t)+\mu)\right) - \lambda_{4m}(t)\gamma_{2m}^*(t), \tag{22}$$

$$\frac{\mathrm{d}\lambda_{4m}(t)}{\mathrm{d}t} = \lambda_{4m}(t)\mu. \tag{23}$$

The transversality conditions are

$$\lambda_{im}(T) = 0,\ \ i = 1,2,3,4.$$

The optimal control variables $\gamma_{1m}^*(t)$ and $\gamma_{2m}^*(t)$ are given by

$$\begin{cases} \gamma_{1m}^*(t) = \max\left\{\min\left\{\frac{(\lambda_{2m}-\lambda_{4m})\widetilde{S}_m^{1*}(t)}{\xi_m^3}, 1\right\}, 0\right\}, \\ \gamma_{2m}^*(t) = \max\left\{\min\left\{\frac{(\lambda_{3m}-\lambda_{4m})\widetilde{S}_m^{2*}(t)}{\xi_m^4}, 1\right\}, 0\right\}. \end{cases} \tag{24}$$

Proof By differentiating Hamiltonian (19) with respect to $I_m(t)$, $S_m^z(t)$, $S_m^z(t-\tau_z)$, $R_m(t)(z=1,2)$, and substituting $I_m(t) = \widetilde{I}_m^*(t)$, $S_m^z(t) = \widetilde{S}_m^{z*}(t)$, $S_m^z(t-\tau_z) = \widetilde{S}_m^{z*}(t)$, $R_m(t) = \widetilde{R}_m^*(t)$, $\gamma_{zm}(t) = \gamma_{zm}^*(t)$ $(z=1,2)$ into equations, Eqs. (20)–(23) can be obtained.

By using the optimal conditions, we find

$$\frac{\partial H}{\partial\gamma_{1m}(t)}\Big|_{I_m(t)=\widetilde{I}_m^*(t),S_m^z(t)=\widetilde{S}_m^{z*}(t),S_m^z(t-\tau_z)=\widetilde{S}_m^{z*}(t),R_m(t)=\widetilde{R}_m^*(t),\gamma_{1m}(t)=\gamma_{1m}^*(t)} = \sum_{m=1}^{n}\left(\xi_m^3\gamma_{1m}^*(t) - \lambda_{2m}\widetilde{S}_m^{1*}(t) + \lambda_{4m}\widetilde{S}_m^{1*}(t)\right) = 0,$$
$$\frac{\partial H}{\partial\gamma_{2m}(t)}\Big|_{I_m(t)=\widetilde{I}_m^*(t),S_m^z(t)=\widetilde{S}_m^{z*}(t),S_m^z(t-\tau_z)=\widetilde{S}_m^{z*}(t),R_m(t)=\widetilde{R}_m^*(t),\gamma_{2m}(t)=\gamma_{2m}^*(t)} = \sum_{m=1}^{n}\left(\xi_m^4\gamma_{2m}^*(t) - \lambda_{3m}\widetilde{S}_m^{2*}(t) + \lambda_{4m}\widetilde{S}_m^{2*}(t)\right) = 0,$$

which implies that

$$\gamma_{1m}^*(t) = \begin{cases} 0, & \frac{(\lambda_{2m}-\lambda_{4m})\widetilde{S}_m^{1*}(t)}{\xi_m^3} < 0, \\ \frac{(\lambda_{2m}-\lambda_{4m})\widetilde{S}_m^{1*}(t)}{\xi_m^3}, & 0 \le \frac{(\lambda_{2m}-\lambda_{4m})\widetilde{S}_m^{1*}(t)}{\xi_m^3} \le 1, \\ 1, & \frac{(\lambda_{2m}-\lambda_{4m})\widetilde{S}_m^{1*}(t)}{\xi_m^3} > 1, \end{cases}$$

$$\gamma_{2m}^*(t) = \begin{cases} 0, & \frac{(\lambda_{3m}-\lambda_{4m})\widetilde{S}_m^{2*}(t)}{\xi_m^4} < 0, \\ \frac{(\lambda_{3m}-\lambda_{4m})\widetilde{S}_m^{2*}(t)}{\xi_m^4}, & 0 \le \frac{(\lambda_{3m}-\lambda_{4m})\widetilde{S}_m^{2*}(t)}{\xi_m^4} \le 1, \\ 1, & \frac{(\lambda_{3m}-\lambda_{4m})\widetilde{S}_m^{2*}(t)}{\xi_m^4} > 1, \end{cases}$$

this is equivalent to Eq. (24).

6. Numerical simulations

In this section, some numerical simulations are carried out to examine the effectiveness of theoretical results. In the simulation, the degree m of the network nodes is selected as $1,2,\ldots,100$. Considering that the heterogeneous networks obey power law distribution, we choose $p(m) = \frac{m^{-2.1}}{1.5545}$ for $\sum_{m=1}^{n} p(m) = 1$. Through calculation, the average degree is $\langle m\rangle = 2.7520$. Although the infectivity of nodes with high degrees is very strong, the infectivity is always bounded. Therefore, a nonlinear infectivity function $\nu(m) = \frac{m^k}{1+m^k}$ [44] is selected, where $k = 0.9$. For convenience, assume that $\Lambda = \mu = 0.02$. The initial value of system (1) is selected as: $I_m(0) = 0.35 + \frac{x}{400}$, $S_m^1(0) = 0.35 - \frac{x}{800}$, $S_m^2(0) = 0.2 - \frac{x}{1600}$, $R_m(0) = 0.1 - \frac{x}{1600}$.

6.1. Stability of equilibriums

Case 5: Stability of RFE E_0 In system (1), select $\alpha(m) = 0.4 + 0.001m$, $\beta(m) = 0.35 + 0.001m$, $\delta = 0.05$, $\gamma_1 = 0.2$, $\gamma_2 = 0.31$, $\rho_1 = 3$, $\rho_2 = 2$, $\tau_1 = \tau_2 = 10$. $R_{01} = 0.3953 < 1$ and $R_{02} = 0.2911 < 1$ can be obtained by calculation. From Theorem 2, model (1) has a unique locally asymptotically stable RFE E_0, which is shown in Fig. 2(a). From Theorem 3, the RFE E_0 is globally asymptotically stable, which is verified in Fig. 2(b) with $m = 100$ and different initial values for $x = 10, 20, \ldots, 100$. It further shows that the rumors will eventually disappear when $R_{01} < 1$ and $R_{02} < 1$.

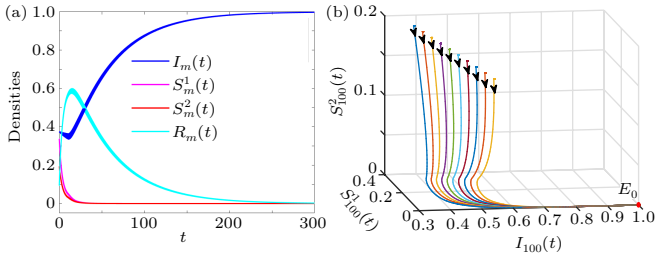


Fig. 2. The stability of RFE E_0 when $R_{01} < 1$ and $R_{02} < 1$.

Case 6: Stability of RSE E_1^* Choose $\alpha(m) = 0.2 + 0.001m$, $\beta(m) = 0.45 + 0.001m$, $\delta = 0.05$, $\gamma_1 = 0.013$, $\gamma_2 = 0.31$, $\rho_1 = 3$, $\rho_2 = 2$, $\tau_1 = \tau_2 = 10$. For $m = 100$, $R_{01} = 0.1685 < 1$ and $R_{02} = 3.5580 > 1$ can be obtained by calculation. Figure 3(a) describes the trajectories of system (1) with $m = 100$ and $x = 50$. From Theorem 4, the RSE E_1^* is globally asymptotically stable, which can be shown in Fig. 3(b) with ten different initial values.

Next, consider the influence of different transmission rates on rumor propagation. For $m \in [1, 100]$, $R_{01} < 1$ and $R_{02} > 1$ can also be obtained. As shown in the Figs. 3(c) and 3(d), the stability of RSE E_1^* is dependent on the degree of the network. In other words, the spreading rate not only affects the basic reproduction number, but also affects the value of the RSE E_1^*. To sum up, when $R_{01} < 1$ and $R_{02} > 1$, spreaders-2 will eventually continue to propagate.

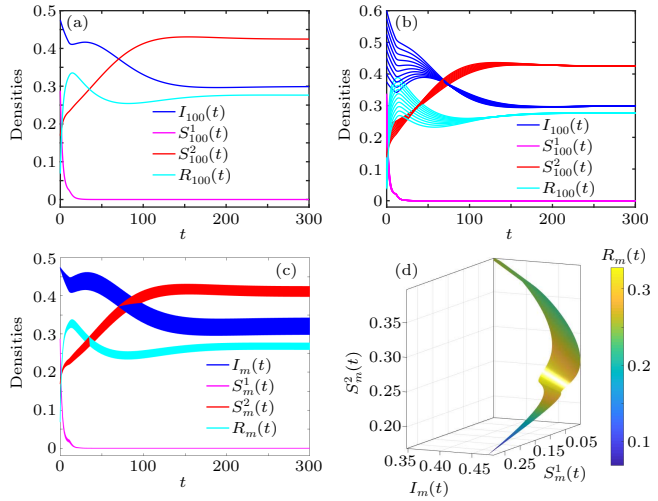


Fig. 3. The stability of RSE E_1^* when $R_{01} < 1$ and $R_{02} > 1$.

Case 7: Stability of RSE E^* Choose $\alpha(m) = 0.405 + 0.001m$, $\beta(m) = 0.485 - 0.001m$, $\delta = 0$, $\gamma_1 = 0.0018$, $\gamma_2 = 0.0061$, $\rho_1 = 0.5$, $\rho_2 = 0.5$, $\tau_1 = \tau_2 = 10$. By calculation, $R_{01} = 4.9452 > 1$ and $R_{02} = 3.1490 > 1$ for $m = 100$. The trajectories of $I_{100}(t)$, $S_{100}^1(t)$, $S_{100}^2(t)$ and R_{100} are drawn in Fig. 4(a) with $x = 50$. According to Theorem 6, the RSE E^* is globally asymptotically stable, which is verified in Fig. 4(b) with different initial values $x = 10, 20, \ldots, 100$.

For $m \in [1, 100]$, $\alpha(m)$ and $\beta(m)$ have different propagation rates. Through calculation, $R_{01} > 1$ and $R_{02} > 1$ can be obtained. As shown in Fig. 4(c), the result shows that different transmission rates affect RSE E^*. The spatial state of the model (1) is given in Fig. 4(d). Obviously, it can be observed that the rumors continue to spread when $R_{01} > 1$ and $R_{02} > 1$.

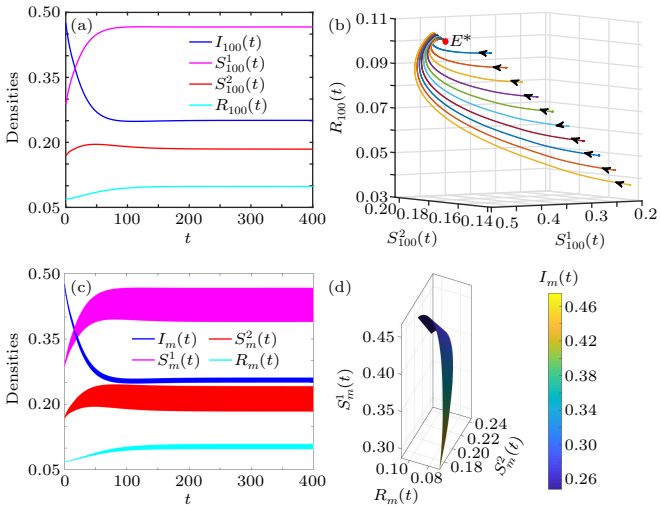


Fig. 4. The stability of RSE E^* when $R_{01} > 1$ and $R_{02} > 1$.

6.2. The influence of various parameters on rumor propagation

(i) The influence of time delay on rumor propagation Consider the time delay $\tau_2 = 0, 10, 20, 30, 40$, and the remaining parameters are shown in Case 7. Figure 5(a) shows that the speed at which ignorants turn into spreaders slows with the increase of time delay τ_2. It can be seen from Fig. 5(b) that as the time delay τ_2 increases, the peak densities of spreaders-2 decrease gradually at the beginning. When $\tau_2 \geq 40$, the density of spreaders-2 first reaches the peak and then fluctuates. The reason for the fluctuation is that the time delay is too significant to restrain the density increase of spreaders-2, which makes the ignorants instantly change to spreaders-1, but the density trajectory then returns to the final horizontal position. The results show that time delay does not change the final state of rumor-spreading but affects the outbreak time and peak value.

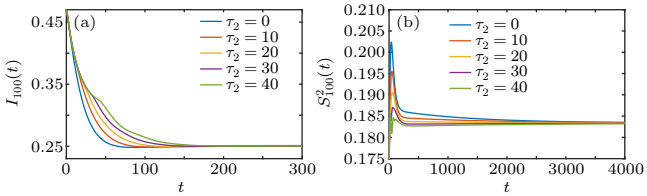


Fig. 5. The densities of $I_{100}(t)$ and $S_{100}^2(t)$ change with τ_2, respectively.

(ii) The influence of ρ_1 and ρ_2 on rumor propagation Choose ρ_1 and ρ_2 as 0.5, 1, 2, 5, 8, 10, 20, 50, 100, respectively, and the others parameters are selected as in Case 7. It can be seen from Fig. 6 that the density of spreaders-1 (or spreaders-2) decreases with the increase of ρ_1 (or ρ_2), which further proves that increasing psychological factors can inhibit the spread of rumors.

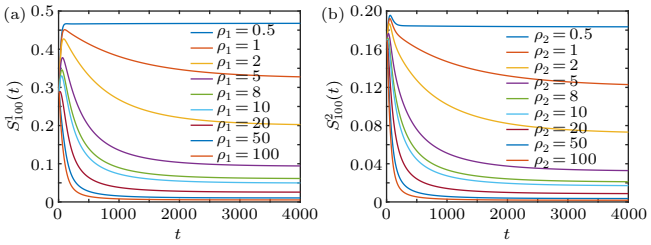


Fig. 6. The densities of $S^1_{100}(t)$ and $S^2_{100}(t)$ change with ρ_1 and ρ_2, respectively.

(iii) The influence of γ_1 and γ_2 on rumor propagation Consider γ_1 taking 0.0018, 0.003, 0.004, 0.005, 0.007, 0.008 and γ_2 taking 0.0061, 0.007, 0.008, 0.01, 0.015, 0.02, and fix the others parameters in Case 7. As shown in Fig. 7, the density of spreaders-1 (or spreaders-2) gradually decreases with the increase of γ_1 (or γ_2), and eventually tends to zero. The results show that increasing the intensity of intervention can inhibit rumor propagation.

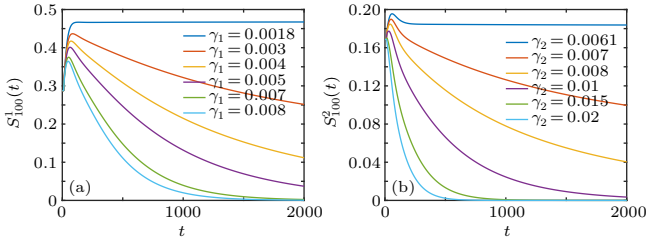


Fig. 7. The densities of $S^1_{100}(t)$ and $S^2_{100}(t)$ change with γ_1 and γ_2, respectively.

(iv) Relation between basic regeneration number and parameter value In Remark 3, we theoretically analyze the effect of the basic reproduction number on the parameters, and then these results are verified by numerical simulations. Consider the relationship between R_{01} and parameters $\alpha(m)$, γ_1, δ, R_{02} and $\beta(m)$, γ_2, respectively. Select variables $\alpha(m), \beta(m) \in [0,0.6]$, $\gamma_1, \gamma_2 \in [0,0.2]$, $\delta \in [0,0.1]$. It can be seen from Fig. 8(a) that R_{01} decreases as γ_1 and δ increase. As shown in Fig. 8(b), R_{02} decreases as γ_2 increases. This means that strengthening intervention can effectively reduce the threshold of rumor outbreak, thereby suppressing the spread of rumors.

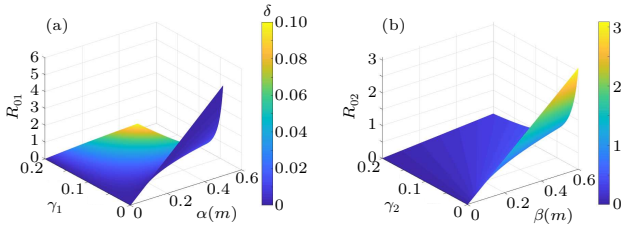


Fig. 8. (a) The relationship of R_{01} with parameters $\alpha(m), \gamma_1$ and δ. (b) The relationship of R_{02} with parameters $\beta(m)$ and γ_2.

6.3. Effectiveness of optimal control

In this subsection, the without control system (1) and the optimal control system (17) are simulated and compared to illustrate the effectiveness of the control scheme. Inspired by Ref. [25], the simulation adopts an efficient forward–backward sweep method to solve the control system (17). The weight coefficients of the objective function are as follows: $\xi^1_m = 1$, $\xi^2_m = 1$, $\xi^3_m = 0.5$, $\xi^4_m = 1.5$. It can be concluded from Case 3 that rumors continue to spread when $R_{01} > 1$ and $R_{02} > 1$. In order to verify the effectiveness of the optimal control strategy proposed by the control system (17), we choose $(\gamma_{1m}(t), \gamma_{2m}(t)) = (0,0)$ (i.e. without control), $(\gamma_{1m}(t), \gamma_{2m}(t))$ (i.e. with optimal control, $0 \le \gamma_{1m}(t), \gamma_{2m}(t) \le 1$), respectively.

Figures 9(a) and 9(b) show the densities changing trends of spreaders-1 and spreaders-2 under the conditions of without control, and optimal control, respectively. It is found that the optimal control strategy makes the densities of spreaders drop the fastest and tends to 0 within a short time. Figure 9(c) illustrates that the optimal control variables $\gamma_{1m}(t)$ and $\gamma_{2m}(t)$ with $m = 100$ gradually decrease and eventually tend to zero. It is further explained that the control intensity is the largest in the initial stage of adopting the control strategy, and the control intensity gradually weakens as the rumors are gradually eliminated. The corresponding control cost $J_{100}(t)$ is shown in Fig. 9(d). It can be seen that the total control cost accumulates as the control time increases, but the rate of change of $J_{100}(t)$ gradually slows down, and the control cost reaches its maximum when the control variables $\gamma_{1m}(t)$ and $\gamma_{2m}(t)$ decrease to zero. Therefore, the optimal control strategy proposed in this paper is effective.

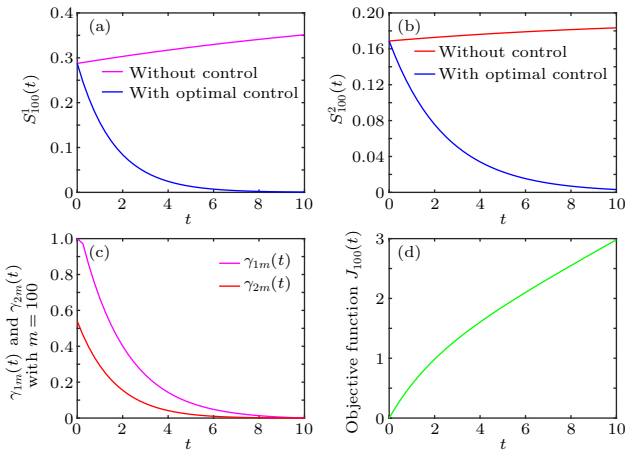


Fig. 9. Panels (a) and (b) are densities of $S^1_{100}(t)$ and $S^2_{100}(t)$ with and without optimal control. (c) The trend of the optimal control variables $\gamma_{1m}(t)$ and $\gamma_{2m}(t)$ with $m = 100$. (d) The trend of control cost $J_{100}(t)$.

7. Conclusion

The harm caused by rumors in multilingual OSNs is more severe than ever. In this paper, considering the effects of time delay, psychological factors, and intervention mechanisms on rumor propagation, an improved I2SR delayed rumor propagation model with nonlinear incidence is established in multilingual environment and heterogeneous networks. The nonlinear incidence can describe individual behavioral changes

and group effects. It can better reflect the characteristics of complex rumor propagation dynamics, which are more realistic than the bilinear and the standard incidence.

The positivity and boundedness of the model solution are studied, and the basic reproduction numbers R_{01} and R_{02} for rumor propagation are derived to ensure the existence and uniqueness of the equilibrium. The dynamic behavior of rumor propagation is studied in detail. We prove that when $R_{01} < 1$ and $R_{02} < 1$, the RFE E_0 is globally asymptotically stable, which means that rumors will eventually disappear; when $R_{01} > 1$ and $R_{02} > 1$, the RSE E^* is globally asymptotically stable, indicating that rumors will continue to spread. Meanwhile, we explore the influence of various parameters on rumor propagation through numerical simulation, and obtain the following results. (i) Time delay does not change the final state of rumor spreading, but it affects the outbreak time and peak value of rumor spreading. (ii) Increasing psychological factors can inhibit the spread of rumors. (iii) Increasing the intervention mechanism can effectively reduce the density of spreaders. (iv) The intervention mechanism has a significant effect on the basic reproduction numbers R_{01} and R_{02} of rumor-spreading.

To reduce the densities of spreaders-1 and spreaders-2, we propose an optimal control scheme to optimize the intervention mechanism. Based on the Pontryagin's minimum principle, we obtain the optimal control conditions. Numerical simulations are carried out using the forward–backward sweep method, and the results show that the optimized intervention mechanism can effectively reduce the density of rumor spreaders within the expected time and minimize the cost.

Acknowledgments

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