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Thermodynamic limit of the *XXZ* central spin model with an arbitrary central magnetic field

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The $U(1)$ symmetry of the XXZ central spin model with an arbitrary central magnetic field $\boldsymbol{B}$ is broken, since its total spin in the z-direction is not conserved. We obtain the exact solutions of the system by using the off-diagonal Bethe ansatz method. The thermodynamic limit is investigated based on the solutions. We find that the contribution of the inhomogeneous term in the associated T–Q relation to the ground state energy satisfies an N^{-1} scaling law, where N is the total number of spins. This result makes it possible to investigate the properties of the system in the thermodynamic limit. By assuming the structural form of the Bethe roots in the thermodynamic limit, we obtain the contribution of the direction of $\boldsymbol{B}$ to the ground state energy. It is shown that the contribution of the direction of the central magnetic field is a finite value in the thermodynamic limit. This is the phenomenon caused by the $U(1)$ symmetry breaking of the system.

Keywords: integrable spin chain, Bethe ansatz, T–Q relation, central spin model, thermodynamic limit

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1. Introduction

The central spin or Gaudin model[1] describes a single spin coupled to a bath of spins (see Fig. 1), and has played an essential role in condensed matter physics,[2,3] ultracold atoms,[4] quantum optics and quantum information.[5–9] The Hamiltonian of the XXZ central spin model with an arbitrary central magnetic field $\boldsymbol{B}$ reads

$$H = B^x\sigma_1^x + B^y\sigma_1^y + B^z\sigma_1^z + \sum_{k=2}^{N} A_k(\sigma_1^x\sigma_k^x + \sigma_1^y\sigma_k^y) + \Delta_k\sigma_1^z\sigma_k^z, \tag{1}$$

where $B^x = r\sin(\theta)\cos(\phi)$, $B^y = r\sin(\theta)\sin(\phi)$, $B^z = r\cos(\theta)$, $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$ and $r \geq 0$; $\boldsymbol{\sigma} = \{\sigma^x,\sigma^y,\sigma^z\}$ are the Pauli matrices, and $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm \mathrm{i}\sigma^y)$; the number of spins in the bath is $N-1$; A_k is the coupling amplitude, and Δ_k is the longitudinal interaction. The model is integrable for the coupling constant A_k and Δ_k with a constraint.[10–15]

For the $\theta = 0$ case, the eigenvalues and eigenstates of the Hamiltonian (1) can be exactly solved by the algebraic Bethe ansatz method.[16,17] Based on the exact solutions, on the one hand, an efficient technique has been devised for the numerical solution of the Bethe ansatz equations (BAEs);[18] on the other hand, the thermodynamic limit of the models is a subject of intense research.[19–23] For the $\theta \neq 0$ case, the integrability and solutions of the model have been studied when $\Delta_k^2 - A_k^2 = \text{Const.}$[24,25]

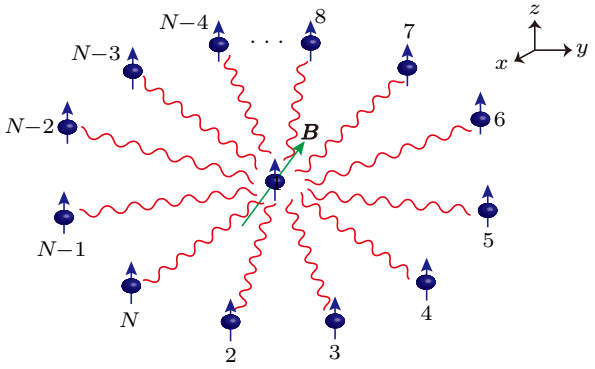


Fig. 1. The XXZ central spin model with an arbitrary central magnetic field $\boldsymbol{B}$.

In this paper, we consider the coupling constant $A_k = \frac{\varepsilon_1\varepsilon_k}{\varepsilon_1^2-\varepsilon_k^2}$, and $\Delta_k = \frac{\varepsilon_1^2}{\varepsilon_1^2-\varepsilon_k^2}$. In this case, the model is integrable. By using the off-diagonal Bethe ansatz (ODBA) method,[26,27] we derive the energies, eigenstates and Bethe ansatz equations of the model. In addition, we give numerical evidence that the whole spectrum and eigenstates are obtained by the ODBA method. Using the energies and Bethe ansatz equations, we study the thermodynamic limit of the XXZ central spin model with an arbitrary central magnetic field. It is found that the contribution of the inhomogeneous term in the associated T–Q relation to the ground state energy satisfies an N^{-1} scaling

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law, where N is the number of spin particles. This fact enables us to investigate the properties of the system when N tends to infinity. By assuming the structural form of the Bethe roots for $N\to\infty$, we calculate the contribution of the direction of $\boldsymbol{B}$ to the ground state energy, i.e., $E_\theta = r(1-|\cos(\theta)|)$. For the $r\neq 0$, $\theta\neq 0$, and $\theta\neq\pi$ case, the contribution of the direction of the central magnetic field does not reduce to zero when $N\to\infty$.

The paper is organized as follows. We derive the Bethe ansatz solutions of the XXZ central spin model with an arbitrary central magnetic field in Section 2. In Section 3, we focus on the contribution of the inhomogeneous terms to the ground state energy. In Section 4, based on the finite-size effects of the system, we study the thermodynamic limit of the model. We summarize and discuss our results in Section 5.

2. Bethe ansatz solutions

2.1. Integrability of the model

The integrability of the Hamiltonian (1) is associated with the rational R-matrix

$$R_{i,j}(u) = u + \eta P_{i,j}, \tag{2}$$

where u is the spectral parameter, η is the crossing (quasi-classical) parameter, and P is the permutation operator

$$P = \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix}. \tag{3}$$

Here and below, we adopt the standard notation: for any operator $O\in \mathrm{End}(\boldsymbol{V})$, O_j is an embedding operator in the tensor space $\boldsymbol{V}\otimes\boldsymbol{V}\otimes\cdots$, which acts as O on the j-th space and as an identity on the other factor spaces; $R_{i,j}(u)$ is an embedding operator of the R-matrix in the tensor space, which acts as an identity on the factor spaces except for the i-th and j-th ones.

The R-matrix satisfies the Yang–Baxter equation[28,29]

$$R_{1,2}(u-v)R_{1,3}(u)R_{2,3}(v) = R_{2,3}(v)R_{1,3}(u)R_{1,2}(u-v), \tag{4}$$

and possesses the following properties:

$$\text{Initial condition}: \ R_{1,2}(0) = \eta P_{1,2}, \tag{5}$$

$$\text{Unitarity relation}: \ R_{1,2}(u)R_{2,1}(-u) = -\rho(u)\mathrm{id}, \tag{6}$$

$$\text{Crossing relation}: \ R_{1,2}(u) = V_1 R^{t_2}_{1,2}(-u-\eta)V_1, \tag{7}$$

$$\text{Quasi-classical property}: \ R(u)|_{\eta\to 0} = u\times \mathrm{id}, \tag{8}$$

$$\text{Antisymmetry}: \ R_{1,2}(-\eta) = -\eta(1-P_{1,2}) = -2\eta P^{(-)}_{1,2}. \tag{9}$$

Here $\rho(u) = (u+\eta)(u-\eta)$, $V = -\mathrm{i}\sigma^y$, $R_{2,1}(u) = P_{1,2}R_{1,2}(u)P_{1,2}$. $P^{(\mp)}_{1,2} = (1/2)\{1\mp P_{1,2}\}$ is the antisymmetric (symmetric) projection operator in the tensor product space $\boldsymbol{V}\otimes\boldsymbol{V}$. The subscript t_2 denotes the transpose in the 2nd quantum space.

In this paper, we consider the reflection matrix $K^-(u)$,

$$K^-(u) = \begin{pmatrix} u & 0\\ 0 & -u \end{pmatrix}, \tag{10}$$

which satisfies the reflection equation (RE)

$$\begin{aligned} &R_{1,2}(u-v)K^-_1(u)R_{2,1}(u+v)K^-_2(v)\\ &= K^-_2(v)R_{1,2}(u+v)K^-_1(u)R_{2,1}(u-v), \end{aligned} \tag{11}$$

and the general non-diagonal matrix $K^+(u)$,

$$K^+(u) = \bar{\xi} + (u+\eta)\boldsymbol{h}_2\cdot\boldsymbol{\sigma}, \quad \boldsymbol{h}_2^2 = 1, \tag{12}$$

which satisfies the dual RE

$$\begin{aligned} &R_{1,2}(u-v)K^+_1(v)R_{2,1}(-u-v-2\eta)K^+_2(u)\\ &= K^+_2(u)R_{1,2}(-u-v-2\eta)K^+_1(v)R_{2,1}(u-v). \end{aligned} \tag{13}$$

The parameters $\bar{\xi}$ and $\boldsymbol{h}_2$ can depend on the crossing parameter η. Expanding $\bar{\xi}$ and $\boldsymbol{h}_2$ with respect to η, we have the following expressions:

$$\bar{\xi} = \xi^{(1)}\eta + \mathcal{O}(\eta^2), \tag{14}$$

$$h^x_2 = h^{(1)x}_2\eta + \frac{h^{(2)x}_2}{2!}\eta^2 + \mathcal{O}(\eta^3), \tag{15}$$

$$h^y_2 = h^{(1)y}_2\eta + \frac{h^{(2)y}_2}{2!}\eta^2 + \mathcal{O}(\eta^3), \tag{16}$$

$$h^z_2 = 1 + \frac{h^{(2)z}_2}{2!}\eta^2 + \mathcal{O}(\eta^3). \tag{17}$$

To describe the system, we introduce the one-row monodromy matrices

$$T_0(u) = R_{0,N}(u-\varepsilon_N)R_{0,N-1}(u-\varepsilon_{N-1})\cdots R_{0,1}(u-\varepsilon_1), \tag{18}$$

$$\hat{T}_0(u) = R_{1,0}(u+\varepsilon_1)\cdots R_{N-1,0}(u+\varepsilon_{N-1})R_{N,0}(u+\varepsilon_N), \tag{19}$$

and the double-row monodromy matrix

$$\mathcal{U}_0(u) = T_0(u)K^-_0(u)\hat{T}_0(u). \tag{20}$$

Here, $\{\varepsilon_j, j=1,\ldots,N\}$ are the inhomogeneous parameters. The transfer matrix is given by

$$t(u) = \mathrm{tr}_0\{K^+_0(u)\mathcal{U}_0(u)\}, \tag{21}$$

where tr_0 denotes the partial trace in auxiliary space. Following the same procedure as in Ref. [10], one can obtain the commutativity of the transfer matrices with different spectral parameters

$$[t(u), t(v)] = 0. \tag{22}$$

This ensures the integrability of the model described by the Hamiltonian (1). The Hamiltonian (1) can be obtained by

$$H = \frac{1}{\prod_{i=2}^N(\varepsilon_1-\varepsilon_i)\prod_{i=1}^N(\varepsilon_1+\varepsilon_i)}\frac{\partial(t(\varepsilon_1)/\eta)}{\varepsilon_1\partial\eta}\Big|_{\eta=0} - 1$$

$$-\frac{1}{2}\sum_{k=2}^{N}\left(\frac{\varepsilon_1}{\varepsilon_1-\varepsilon_k}+\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_k}\right). \tag{23}$$

The central magnetic field $\boldsymbol{B}$ in the Hamiltonian is described by the parameters in the K-matrices (10) and (12)

$$B^x=-\mathrm{i}\varepsilon_1 h_2^{(1)y},\quad B^y=\mathrm{i}\varepsilon_1 h_2^{(1)x},\quad B^z=\xi^{(1)}. \tag{24}$$

2.2. Energy spectrum and Bethe ansatz equations

In our current model, the K-matrices $K^-(u)$ (10) and $K^+(u)$ (12) do not commute. This leads to the fact that they cannot be diagonalized simultaneously (corresponding to the arbitrary central magnetic field). Therefore there is no obvious reference state of the system. The standard algebraic Bethe ansatz cannot be performed on this model. With the help of the ODBA method,[26] we construct the inhomogeneous T–Q relation

$$\Lambda(u)=\bar{a}(u)\frac{Q(u-\eta)}{Q(u)}+\bar{d}(u)\frac{Q(u+\eta)}{Q(u)}+cu(u+\eta)\frac{F(u)}{Q(u)}, \tag{25}$$

where the function $Q(u)$ is parameterized by N parameters $\{\lambda_j|j=1,\ldots,N\}$

$$Q(u)=\prod_{l=1}^{N}(u-\lambda_l)(u+\lambda_l+\eta)=Q(-u-\eta), \tag{26}$$

and the functions $\bar{a}(u)$, $\bar{d}(u)$, $F(u)$ and the constant c are given by

$$\bar{a}(u)=\frac{2u+2\eta}{2u+\eta}u(u+\bar{\xi})\prod_{l=1}^{N}(u+\varepsilon_l+\eta)(u-\varepsilon_l+\eta), \tag{27}$$

$$\begin{aligned}\bar{d}(u)&=\frac{2u}{2u+\eta}(u+\eta)((u+\eta)-\bar{\xi})\prod_{l=1}^{N}(u+\varepsilon_l)(u-\varepsilon_l)\\&=\bar{a}(-u-\eta),\end{aligned} \tag{28}$$

$$F(u)=\prod_{l=1}^{N}(u+\varepsilon_l)(u-\varepsilon_l)(u+\varepsilon_l+\eta)(u-\varepsilon_l+\eta), \tag{29}$$

$$c=2(h_2^z-1). \tag{30}$$

Based on the construction of the Hamiltonian equation (23), the energy spectrum can be derived from Eq. (25),

$$E=r\cos(\theta)+\sum_{i=2}^{N}\frac{\varepsilon_1^2}{\varepsilon_1^2-\varepsilon_i^2}-\sum_{k=1}^{N}\frac{2\varepsilon_1^2}{\varepsilon_1^2-\lambda_k^2}, \tag{31}$$

and the Bethe roots $\{\lambda_k|k=1,\ldots,N\}$ need to satisfy a set of Bethe ansatz equations (BAEs)

$$\begin{aligned}&2\frac{1-r\cos(\theta)}{\lambda_m}-\sum_{k=1}^{N}\left(\frac{1}{\lambda_m+\varepsilon_k}+\frac{1}{\lambda_m-\varepsilon_k}\right)\\&+\sum_{k\neq m}\left(\frac{2}{\lambda_m-\lambda_k}+\frac{2}{\lambda_m+\lambda_k}\right)\\&+\frac{r^2\sin(\theta)^2\prod_{l=1}^{N}(\lambda_m+\varepsilon_l)(\lambda_m-\varepsilon_l)}{2\varepsilon_1^2\lambda_m\prod_{l\neq m}(\lambda_m-\lambda_l)(\lambda_m+\lambda_l)}=0.\end{aligned} \tag{32}$$

The numerical solutions of the BAEs (32) and the corresponding eigenvalues (energy spectrum) and eigenstates of the Hamiltonian (1) for $N=4$ are shown in Table 1. These numerical simulation results show that the BAEs (32) give the complete spectrum and eigenstates of the XXZ central spin model with an arbitrary central magnetic field $\boldsymbol{B}$.

Table 1. Solutions of BAEs (32) for the case of $N=4$, $\{\varepsilon_k=\exp(1-(k-1)/N), k=1,2,\ldots,N\}$, $r=\sqrt{2}$, $\theta=\pi/4$, $\phi=\pi/3$. The symbol n indicates the number of eigenvalues, and E_n is the corresponding energy. The energy E_n calculated from Eq. (31) is the same as that from the exact diagonalization of the Hamiltonian (1).

λ_1	λ_2	λ_3	λ_4	E_n	n
2.5340 − 0.0000i	5.2448 + 0.0000i	3.2728 − 4.3238i	3.2728 + 4.3238i	−8.5582	1
2.5378 − 0.0000i	3.3617 + 0.0000i	0.0000 + 7.1505i	0.0000 + 0.0391i	−7.6410	2
0.0357 − 0.0000i	2.4464 + 0.0000i	0.0000 + 7.0824i	0.0000 + 3.2666i	−7.1886	3
1.4347 − 0.0000i	2.3655 − 0.0000i	0.0000 + 7.1202i	0.0000 + 0.5340i	−6.7823	4
0.0000 + 7.1044i	0.0000 − 0.7655i	1.7426 − 0.1557i	1.7426 + 0.1557i	−2.3488	5
0.0459 + 0.0000i	1.8551 + 0.0000i	0.0000 + 3.9170i	0.0000 + 6.9857i	−0.2465	6
0.0779 + 0.0000i	1.4048 + 0.0000i	0.0000 + 4.3178i	0.0000 + 6.9018i	0.8440	7
1.8758 − 0.0000i	4.5410 − 0.0000i	0.0000 + 7.2122i	0.0000 + 0.0435i	1.4610	8
0.0000 + 1.5783i	0.0000 + 0.0261i	0.1361 − 6.0292i	0.1361 + 6.0292i	2.2401	9
1.4116 − 0.0000i	4.8134 − 0.0000i	0.0000 + 7.2495i	0.0000 + 0.0862i	2.3642	10
1.9223 + 0.0000i	5.1640 − 0.0000i	3.0613 − 3.7809i	3.0613 + 3.7809i	2.6470	11
0.0208 − 0.0000i	5.4871 + 0.0000i	0.0000 + 7.4128i	0.0000 + 1.9644i	3.5101	12
1.4507 + 0.0000i	4.9474 − 0.0000i	2.9169 − 3.2725i	2.9169 + 3.2725i	3.8581	13
0.0000 + 0.0164i	0.0000 + 4.3452i	4.8658 − 1.8970i	4.8658 + 1.8970i	4.5956	14
0.0159 − 0.0000i	5.3534 + 0.0000i	3.3592 − 3.4663i	3.3592 + 3.4663i	4.7109	15
7.5242 − 10.0807i	7.5242 + 10.0807i	11.2114 − 3.3813i	11.2114 + 3.3813i	6.5345	16

3. Finite-size effects

From now on, we consider the system with $A_k>0$ and $\Delta_k>0$. This corresponds to the parameter configuration $\{\varepsilon_1>\varepsilon_i>0, i=2,\ldots,N\}$ or $\{\varepsilon_1<\varepsilon_i<0, i=2,\ldots,N\}$. From Eqs. (31) and (32), one can conclude that $\{\varepsilon_1>\varepsilon_i>0, i=2,\ldots,N\}$ is equivalent to $\{\varepsilon_1<\varepsilon_i<0, i=2,\ldots,N\}$ for the

system. For the homogeneous cases, we have $\varepsilon_2 = \varepsilon_3 = \cdots = \varepsilon_N$. For the inhomogeneous cases, without loss of generality, we assume $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > \varepsilon_N > 0$.

From Eqs. (31) and (32), we know that the parameter ϕ makes no contribution to the energy expression. Therefore, let us set $\phi = 0$, and the Hamiltonian becomes

$$H = r\sin(\theta)\sigma_1^x + r\cos(\theta)\sigma_1^z + \sum_{k=2}^{N} A_k(\sigma_1^x\sigma_k^x + \sigma_1^y\sigma_k^y) + \Delta_k\sigma_1^z\sigma_k^z. \tag{33}$$

For $\theta \in [0, \pi/2]$, the Hamiltonian possesses the property

$$UHU^{-1} = r\sin(\pi-\theta)\sigma_1^x + r\cos(\pi-\theta)\sigma_1^z + \sum_{k=2}^{N} A_k(\sigma_1^x\sigma_k^x + \sigma_1^y\sigma_k^y) + \Delta_k\sigma_1^z\sigma_k^z, \tag{34}$$

where $U = \prod_{i=1}^{N}\sigma_i^x$. Thus, all we need to do is to research the parameter $\theta \in [0, \pi/2]$. Firstly, we study the case of $\theta = 0$, setting the stage for the $\theta \neq 0$ case.

3.1. The $\theta = 0$ case

For the $\theta = 0$ case, the total spin in the z-direction is conserved. The energy spectrum is

$$\bar{E} = r + \sum_{i=2}^{N}\frac{\varepsilon_1^2}{\varepsilon_1^2 - \varepsilon_i^2} - \sum_{k=1}^{M}\frac{2\varepsilon_1^2}{\varepsilon_1^2 - \lambda_k^2}, \tag{35}$$

and the associated BAEs are as follows:

$$2\frac{1-r}{\lambda_m} - \sum_{k=1}^{N}\left(\frac{1}{\lambda_m + \varepsilon_k} + \frac{1}{\lambda_m - \varepsilon_k}\right) + \sum_{k\neq m}\left(\frac{2}{\lambda_m - \lambda_k} + \frac{2}{\lambda_m + \lambda_k}\right) = 0. \tag{36}$$

The system is a single spin coupled to $N-1$ bath spins by inhomogeneous antiferromagnetic couplings ($A_k > 0$ and $\Delta_k > 0$). The numerical solutions of BAEs (36) and the corresponding eigenvalues (full energy spectrum) of the Hamiltonian (1) for $N = 3$ are shown in Table 2. We also give the numerical solutions of the ground state energy for larger system sizes ($N = 4$, 8, 12, 16, 20, 24), in Table 3. One can obtain the lowest energy state in the subspace $M = 1$ and a certain range of Bethe root.[30–32] For the ground state of the system, the BAE is

$$2\frac{1-r}{\lambda} = \sum_{k=1}^{N}\left(\frac{1}{\lambda + \varepsilon_k} + \frac{1}{\lambda - \varepsilon_k}\right). \tag{37}$$

The left hand side of BAE (37) has a finite value for a finite r and a nonzero λ. This indicates that the right hand side of BAE (37) also has a finite value when N is a finite value or $N \to \infty$. Taking $N \to \infty$ for Eq. (37), $\{\lambda \to \varepsilon_k, k = 1, 2, 3, \ldots, N-1\}$ should be the Bethe roots. Thus, in the thermodynamic limit, we can assume $\lambda = \varepsilon_k - \delta_k$ and $\delta_k > 0$. It is obvious that the system has the least energy when $\lambda \to \varepsilon_1$. Substituting $\lambda = \varepsilon_1 - \delta$ into Eq. (37), we obtain

$$2\frac{1-r}{\varepsilon_1 - \delta} - \left(\frac{1}{2\varepsilon_1 - \delta} - \frac{1}{\delta}\right) - \sum_{k=2}^{N}\left(\frac{1}{\varepsilon_1 + \varepsilon_k - \delta} + \frac{1}{\varepsilon_1 - \varepsilon_k - \delta}\right) = 0. \tag{38}$$

Noting that $\delta \to 0$ when $N \to \infty$, one can obtain

$$2\frac{1-r}{\varepsilon_1} - \left(\frac{1}{2\varepsilon_1} - \frac{1}{\delta}\right) - \sum_{k=2}^{N}\left(\frac{1}{\varepsilon_1 + \varepsilon_k} + \frac{1}{\varepsilon_1 - \varepsilon_k - \delta}\right) = 0. \tag{39}$$

To keep the equality, $\varepsilon_1 - \varepsilon_k$ must be greater than 2δ. Making the series expansion for $\frac{\delta}{\varepsilon_1 - \varepsilon_k - \delta}$, we get

$$\left(\frac{3}{2\varepsilon_1} - \frac{2r}{\varepsilon_1}\right)\delta + 1 - \sum_{k=2}^{N}\left(\frac{\delta}{\varepsilon_1 + \varepsilon_k} + \sum_{l=1}^{+\infty}\left(\frac{\delta}{\varepsilon_1 - \varepsilon_k}\right)^l\right) = 0. \tag{40}$$

For $l = 1$, one can obtain

$$\frac{1}{\delta} = -2\frac{1-r}{\varepsilon_1} + \frac{1}{2\varepsilon_1} + \sum_{k=2}^{N}\frac{2\varepsilon_1}{\varepsilon_1^2 - \varepsilon_k^2}. \tag{41}$$

Table 2. Solutions of BAEs (36) for the case of $N = 3$, $\{\varepsilon_k = \exp(1 - (k-1)/N), k = 1, 2, \ldots, N\}$, $r = \sqrt{2}$, $\theta = 0$. The symbol n indicates the number of eigenvalues, and E_n is the corresponding energy. The energy E_n calculated from Eq. (35) is the same as that from the exact diagonalization of the Hamiltonian (1).

λ_1	λ_2	λ_3	E_n	n
2.4618 + 0.0000i	–	–	−6.2955	1
2.3486 + 0.0000i	0.7367 + 0.0000i	–	−5.2205	2
1.5781 + 0.2387i	1.5781 − 0.2387i	–	−1.0020	3
0.5170 − 0.0000i	0.0000 + 1.8711i	–	1.3952	4
1.6850 − 0.0000i	–	–	1.5792	5
0.5513 − 0.0000i	0.0000 + 2.2371i	6.3440 − 0.0000i	1.9989	6
0.6204 − 0.0000i	–	–	2.7174	7
–	–	–	4.8273	8

Table 3. Solutions of BAEs (36) for the case of $\{\varepsilon_k = \exp(1 - (k-1)/N), k = 1, 2, \ldots, N\}$, $r = \sqrt{2}$, $\theta = 0$. The symbol λ indicates the Bethe root of the ground state, and E is the corresponding energy. The ground state energy E calculated from (35) is the same as that from the exact diagonalization of the Hamiltonian (1).

N	λ	E
4	2.5375	−8.7321
8	2.6398	−19.2911
12	2.6699	−30.7267
16	2.6838	−42.7544
20	2.6918	−55.2302
24	2.6969	−68.0668

3.2. The $\theta \neq 0$ case

For the $\theta \neq 0$ case, the $U(1)$ symmetry of the system is broken. The corresponding BAEs (32) obtained from the ODBA method have a much more complicated structure, which makes it difficult to approach the thermodynamic limit of the model. On this question, we have made some

progress.[30,31] To study the contribution of the inhomogeneous term (the last term of Eq. (25)) to the ground state energy, we first consider the T–Q relation without the inhomogeneous term, i.e.,

$$\tilde{\Lambda}(u)=\bar{a}(u)\frac{Q(u-\eta)}{Q(u)}+\bar{d}(u)\frac{Q(u+\eta)}{Q(u)}. \tag{42}$$

The function $Q(u)$ is parameterized by M parameters $\{\lambda_j|j=1,\ldots,N\}$

$$Q(u)=\prod_{l=1}^{M}(u-\lambda_l)(u+\lambda_l+\eta)=Q(-u-\eta). \tag{43}$$

Using the same procedure, we obtain

$$\tilde{E}=r\cos(\theta)+\sum_{i=2}^{N}\frac{\varepsilon_1^2}{\varepsilon_1^2-\varepsilon_i^2}-\sum_{k=1}^{M}\frac{2\varepsilon_1^2}{\varepsilon_1^2-\lambda_k^2}, \tag{44}$$

$$2\frac{1-r\cos(\theta)}{\lambda_m}-\sum_{k=1}^{N}\left(\frac{1}{\lambda_m+\varepsilon_k}+\frac{1}{\lambda_m-\varepsilon_k}\right)+\sum_{k\neq m}\left(\frac{2}{\lambda_m-\lambda_k}+\frac{2}{\lambda_m+\lambda_k}\right)=0. \tag{45}$$

We remark that, for a finite N and $\theta\neq 0$, $\tilde{E}$ is different from the exact eigenvalue E given by Eq. (31). To study the difference between $\tilde{E}$ and E, we define

$$\Delta E=\tilde{E}-E. \tag{46}$$

The ΔE is the contribution of the inhomogeneous term in the associated T–Q relation to the energy spectrum.

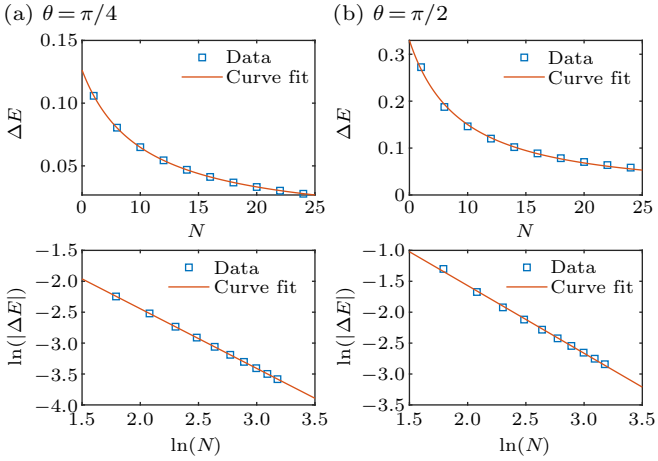


Fig. 2. The contribution of the inhomogeneous term to the ground state energy ΔE versus the spin particles N. The data can be fitted as $\Delta E=\gamma N^{\beta}$ [or $\ln(|\Delta E|)=p\ln(N)+q$]. Here $r=\sqrt{3}$, $\{\varepsilon_k=\exp(1-(k-1)/N), k=1,2,\ldots,N\}$, $\phi=0$, (a) $\theta=\pi/4$, $\gamma=0.5955$ and $\beta=-0.9633$; (b) $\theta=\pi/2$, $\gamma=2.0470$ and $\beta=-1.1350$. Due to the fact that $\beta<0$ [or $p<0$], when N tends to infinity, the contribution of the last term tends to zero.

Similar to the $\theta=0$ case, for the ground state of the system, we have $M=1$ and the BAE becomes

$$2\frac{1-r\cos(\theta)}{\lambda}-\sum_{k=1}^{N}\left(\frac{1}{\lambda+\varepsilon_k}+\frac{1}{\lambda-\varepsilon_k}\right)=0. \tag{47}$$

The values of ΔE versus the number of spins N are shown in Fig. 2. It indicates a power law (N^{-1} scaling law) (Based on the previous works,[30–32] we know that the contribution of the inhomogeneous term to the energy can have power decay or exponential decay, when N tends to infinity. In our current work, the contribution of the inhomogeneous term to the energy has power decay $\Delta E=\gamma N^{\beta}$, $(\beta\approx -1)$, $N\to\infty$. We call it the N^{-1} scaling law.) relation between ΔE and N, i.e., $\Delta E=\gamma N^{\beta}$. As $\beta\approx -1$, the value of ΔE tends to zero when $N\to\infty$. Thus, we can study the properties of the system in the thermodynamic limit via the BAE (47).

It is obvious that $\{\lambda\to\varepsilon_k|k=1,2,\ldots,N\}$ should be the Bethe roots for $N\to\infty$. From Eq. (44), the system has the least energy when $\lambda\to\varepsilon_1$. Substituting $\lambda=\varepsilon_1-\delta'$ into Eq. (47), we have

$$\left(\frac{3}{2\varepsilon_1}-\frac{2r\cos(\theta)}{\varepsilon_1}\right)\delta'+1-\sum_{k=2}^{N}\left(\frac{\delta'}{\varepsilon_1+\varepsilon_k}+\sum_{l=1}^{+\infty}\left(\frac{\delta'}{\varepsilon_1-\varepsilon_k}\right)^{l}\right)=0. \tag{48}$$

Considering the $l=1$ case, one can obtain

$$\frac{1}{\delta'}=\frac{1}{2\varepsilon_1}-2\frac{1-r\cos(\theta)}{\varepsilon_1}+\sum_{k=2}^{N}\left(\frac{1}{\varepsilon_1+\varepsilon_k}+\frac{1}{\varepsilon_1-\varepsilon_k}\right). \tag{49}$$

4. Contribution of θ to the ground state energy

In order to study the contribution of the direction of the central magnetic field to the ground state energy, we define

$$E_\theta\equiv\tilde{E}_{\rm g}-\bar{E}_{\rm g}\simeq r\cos(\theta)-r-\varepsilon_1\left(\frac{1}{\delta'}-\frac{1}{\delta}\right). \tag{50}$$

Then we have

$$E_\theta=r(1-\cos(\theta)),\quad \theta\in[0,\pi/2], \tag{51}$$

in both the homogeneous and inhomogeneous cases. Considering the Hamiltonian possesses the property (34), one can obtain

$$E_\theta=r(1-|\cos(\theta)|),\quad \theta\in[0,\pi]. \tag{52}$$

For the $r\neq 0$, $\theta\neq 0$ and $\theta\neq\pi$ case, the direction of the central magnetic field contributes nonzero values to the ground state energy in the thermodynamic limit. By using the relation (52), one can calculate the contribution of the direction of the central magnetic field to the ground state energy. The results are shown in Fig. 3, where the lines are the energy E_θ calculated by using the relation (52) and the green stars are data obtained by employing the BST algorithms[33] to solve the energy E_θ of the Hamiltonian (1) in the thermodynamic limit. Specifically, for one of the green stars, we first calculate the ground state energy $E_{\rm g}(N)$ with $N=4,\ldots,24$ by numerical diagonalization. Then, the large-N extrapolation of the

energy $E_g(N)$ was performed using BST algorithms from the sequence $\tilde{E}_g(4)-E_g(4)$, $\tilde{E}_g(6)-E_g(6)$, …, $\tilde{E}_g(24)-E_g(24)$. From Fig. 3, we can see that the analytical and numerical results agree with each other very well.

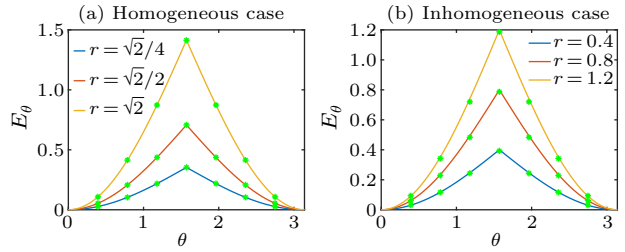


Fig. 3. The energy spectrum E_θ versus the parameter θ. The curves are calculated from Eq. (52), while the green stars are obtained from the Hamiltonian (1) with the BST algorithms. Here (a) $\varepsilon_1=1.8$, $\varepsilon_2=\varepsilon_N=0.7$; (b) $\varepsilon_k=\exp(1-(k-1)/N)$, $k=1,2,\ldots,N$.

The integrable models with $U(1)$ symmetry breaking caused by the direction of magnetic fields can be divided into two categories: One is where the model has a special direction, and the external magnetic field does not follow that direction (see Fig. 1); the other is the model with unparallel boundary magnetic fields (see Fig. 4). For the first category, the contribution of the direction of the magnetic field to the ground state energy is a finite value.[34–36] For the second category, the contribution of the relative angle between the boundary magnetic fields to the ground state energy reduces to zero in the thermodynamic limit.[30,31]

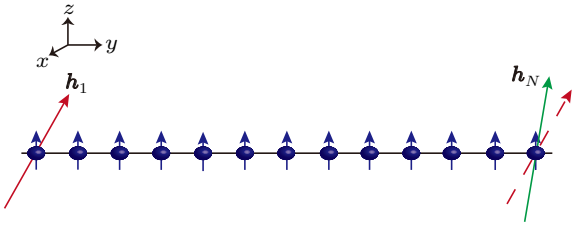


Fig. 4. The model with unparallel boundary fields.

From the point of view of the T–Q relation, when the parameters reflecting $U(1)$ symmetry breaking only appear in the inhomogeneous terms, the $U(1)$ symmetry breaking does not affect the properties of the system in the thermodynamic limit if the inhomogeneous terms can be ignored under the limit. However, when the parameters reflecting $U(1)$ symmetry breaking appear in both the homogeneous and inhomogeneous terms, the effect of $U(1)$ symmetry breaking still exists in the thermodynamic limit even if the inhomogeneous terms can be ignored in the limit.

5. Conclusion

In this paper, by using the off-diagonal Bethe ansatz method, we have presented the energy spectrum (31) and Bethe ansatz equations (32) of the model. In addition, we have given numerical evidence that the whole spectrum and eigenstates are obtained by the off-diagonal Bethe ansatz method.

Using the energies (31) and Bethe ansatz equations (32), we have studied the thermodynamic limit of the XXZ central spin model with an arbitrary central magnetic field. It is found that the contribution of the inhomogeneous term in the associated T–Q relation to the ground state energy satisfies an N^{-1} scaling law, where N is the number of spin particles. This fact enables us to explore the properties of the system in the thermodynamic limit. By assuming the Bethe roots $\lambda=\varepsilon_k-\delta_k$, the contribution of the direction of $\boldsymbol{B}$ to the ground state energy is given by $E_\theta=r(1-|\cos(\theta)|)$. For the $r\neq 0$, $\theta\neq 0$ and $\theta\neq\pi$ case, the contribution of the direction of the central magnetic field is a nonzero value. Our work provides a simple and powerful tool to study the thermodynamic limit of the integrable model without $U(1)$ symmetry.

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