Resonant interactions among two-dimensional nonlinear localized waves and lump molecules for the $(2+1)$-dimensional elliptic Toda equation

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The $(2+1)$-dimensional elliptic Toda equation is a high-dimensional generalization of the Toda lattice and a semi-discrete Kadomtsev–Petviashvili I equation. This paper focuses on investigating the resonant interactions between two breathers, a breather/lump and line solitons as well as lump molecules for the $(2+1)$-dimensional elliptic Toda equation. Based on the N-soliton solution, we obtain the hybrid solutions consisting of line solitons, breathers and lumps. Through the asymptotic analysis of these hybrid solutions, we derive the phase shifts of the breather, lump and line solitons before and after the interaction between a breather/lump and line solitons. By making the phase shifts infinite, we obtain the resonant solution of two breathers and the resonant solutions of a breather/lump and line solitons. Through the asymptotic analysis of these resonant solutions, we demonstrate that the resonant interactions exhibit the fusion, fission, time-localized breather and rogue lump phenomena. Utilizing the velocity resonance method, we obtain lump–soliton, lump–breather, lump–soliton–breather and lump–breather–breather molecules. The above works have not been reported in the $(2+1)$-dimensional discrete nonlinear wave equations.

Keywords: $(2+1)$-dimensional elliptic Toda equation, resonant interaction, lump molecules

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1. Introduction

The Toda lattice originally describes the motion of a chain of particles with exponential interactions between the nearest neighbors of particles, which has important applications in molecular dynamics,[1] statistical mechanics,[2,3] biology,[4,5] optics,[6] string theory,[7] two-dimensional gravity[8,9] etc. The $(1+1)$-dimensional Toda lattice equation has the following form:[10]

$$v_{tt}(n) = \mathrm{e}^{v(n-1)-v(n)} - \mathrm{e}^{v(n)-v(n+1)}, \quad (n \in \mathbf{Z}), \tag{1}$$

where we denote $v(n,t)$ as $v(n)$, and $v(n)$ represents the motion of the n-th particle in the chain. The $(2+1)$-dimensional elliptic Toda lattice is obtained by the natural dimensional generalization of Eq. (1) as follows:

$$\Delta v(n) = \mathrm{e}^{v(n-1)-v(n)} - \mathrm{e}^{v(n)-v(n+1)}, \quad (n \in \mathbf{Z}), \tag{2}$$

where $\Delta = \partial_{xx} + \partial_{yy}$ is a two-dimensional Laplacian operator. Here we denote $v(n,x,y)$ as $v(n)$ for simplicity. In 1983, the exact Bessel-type solutions of the $(2+1)$-dimensional elliptic Toda Eq. (2) were obtained by applying Hirota bilinear method.[11] In Ref. [12], line solitons and instantons of the $(2+1)$-dimensional elliptic Toda equation were obtained by the inverse scattering transform method. Reference [13] established a relation between the $(2+1)$-dimensional elliptic Toda lattice and the Ablowitz–Ladik hierarchy. In Ref. [14], an improved representation of N-soliton solution for the $(2+1)$-dimensional elliptic Toda equation was obtained. Furthermore, rational and N-breather solutions were derived. Utilizing the symbolic computation, three classes of lump solutions for the $(2+1)$-dimensional elliptic Toda equation were constructed in Ref. [15]. More recently, N-soliton, M-breather and lump solutions of the $(2+1)$-dimensional elliptic Toda equation have been investigated by the Bäcklund transformation and nonlinear superposition formula,[16] which provides the possibility to obtain more hybrid solutions.

The nonlinear localized waves such as line solitons,[17,18] breathers,[19] lumps[20,21] and rogue waves widely exist in various nonlinear physical systems such as nonlinear optics, oceanography, Bose–Einstein condensates, and plasma physics. In the system with two spatial dimensions, the peak of the line soliton maintains its peak transversely on a planar line, so the line soliton is also called long crested wave in physics. The lump has two-dimensional localized structures,[22] which decays algebraically in all directions. The breather is partially localized and periodic in space or time.[23] The interactions among the two-dimensional localized waves exhibit diverse and complicated nonlinear wave phenomena,[24] which have important applications in physics and engineering.[25,26] Resonant interactions between solitary waves contribute to the formation of various surface-wave patterns and instantons in the quantum field theory. Miles initially investigated the resonant interaction between line solitons in KP-II equation and showed the Y-shape waveforms are formed during the resonant interaction between two line solitons.[27,28] Subsequently, more general web-shaped waveforms were observed in a coupled KP (cKP) system[29,30] and KP-II equation.[31] Although

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the multi-line solitons of the KP-I equation can not realize the resonant interaction, the breathers of the KP-I equation can form the web-shaped waveforms.[32] Johnson and Thompson gave a resonant solution of a lump and a line soliton for KP-I equation.[33] During the partial resonant interaction between a lump and a line soliton, the lump exhibits two different dynamical behaviors.[34,35] One is the lump fuses into the line soliton, which is called fusion. The other is the lump separates from the line soliton and keeps its shape and velocity unchanged, which is called fission. Rao *et al.* investigated the complete resonant interactions among the lumps and line solitons/breathers in KP-I equation.[36,37] In this case, lumps separate from one line soliton/breather and fuse into the other line soliton/breather. The dynamical behaviors of these lumps are similar to the rogue waves in the ocean, so these lumps are called rogue lumps. The resonant interaction between the breathers and line solitons in Mel'nikov equation, KP-I equations and $(2+1)$-dimensional asymmetrical Nizhnik–Novikov–Veselov equation as well as the resonant interaction between breathers in two-dimensional multi-component long-wave–short-wave resonant interaction system were studied in Refs. [38–41]. In this paper, we investigate the resonant interaction among the line solitons, breathers and lumps for the $(2+1)$-dimensional elliptic Toda equation.

The concept of soliton molecules is proposed in optical system, which refers to the formation of bound states between solitons.[42–44] Soliton molecules can improve the carrying capacity of optical fiber. Experimentally, the internal dynamics of soliton molecules are intensely studied by means of real-time ultrafast measurements.[45–48] The soliton molecules, the breather molecules and breather–soliton molecules have been observed in mode-locked fiber lasers.[49–51] Soliton molecules are involved not only in optics, but also in physical fields such as Bose–Einstein condensates,[52] oceanography,[53] etc. In Ref. [54], Lou proposed a new mechanism to form soliton molecules by introducing velocity resonance, which provides an effective method for the study of soliton molecules. Besides the line soliton molecules, other types of soliton molecules such as dark soliton molecules, kink–kink molecules, dromion molecules, breather molecules, breather–soliton molecules, lump–breather–soliton molecules are also obtained by utilizing the velocity resonance method,[55–59] which enrich the bound state phenomena of nonlinear localized waves. In this paper, we investigate lump–soliton, lump–breather, lump–soliton–breather, and lump–breather–breather molecules for the $(2+1)$-dimensional elliptic Toda equation.

This paper is organized as follows. In Section 2, we investigate the resonant interaction between two y-breathers for the $(2+1)$-dimensional elliptic Toda equation, and show that the resonant interaction between two y-breathers exhibits the fusion and fission phenomena by analyzing the asymptotic forms of the resonant solution. In Section 3, we study the resonant interaction between a y-breather and line solitons arranged in the y direction for the $(2+1)$-dimensional elliptic Toda equation. We demonstrate the resonant interactions between a y-breather and a line soliton/two line solitons arranged in the y direction exhibit the fusion, fission and time-localized breather phenomena by analyzing the asymptotic forms of the resonant solutions. In Section 4, we investigate the resonant interaction between a lump and line solitons arranged in the y direction for the $(2+1)$-dimensional elliptic Toda equation. We demonstrate the resonant interactions between a lump and line solitons arranged in the y direction exhibit the fusion, fission and rogue lump phenomena by analyzing the asymptotic forms of the resonant solutions. In Section 5, we obtain lump–soliton molecule, lump–breather molecule, lump–soliton–breather molecule and lump–breather–breather molecule for the $(2+1)$-dimensional elliptic Toda equation by applying the velocity resonance method. A summary and discussion is given in Section 6.

2. Resonant interactions between two y-breathers

In this section, we construct the resonant solution consisting of two y-breathers. Through the asymptotic analysis of the resonant solution, we show the resonant interaction between two y-breathers exhibits the fusion and fission phenomena.

By introducing $u(n) = \mathrm{e}^{v(n-1)-v(n)}$, Eq. (2) can be rewritten as follows:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ln u(n) = u(n+1)-2u(n)+u(n-1), \quad (3)$$

where we denote $u(n,x,y)$ as $u(n)$. Through the transformation

$$u(n) = \frac{f(n+1)f(n-1)}{f(n)^2}, \quad (4)$$

Eq. (3) can be transformed into the bilinear form

$$(D_x^2+D_y^2)f(n)\cdot f(n) = (2\,\mathrm{e}^{Dn}-2)f(n)\cdot f(n),$$

where we denote $f(n,x,y)$ as $f(n)$.

The N-soliton solution of the $(2+1)$-dimensional elliptic Toda equation has been derived by the Bäcklund transformation and nonlinear superposition formula[16]

$$u(n) = \frac{f(n+1)f(n-1)}{f(n)^2}, \quad (5)$$

where

$$f(n) = \sum_{\mu=0,1}\exp\left(\sum_{i<j}^{N}\mu_i\mu_j A_{ij}+\sum_{i=1}^{N}\mu_i\eta_i\right),$$

$$\eta_j = 2\sinh(k_j)\cos(l_j)x+2\sinh(k_j)\sin(l_j)y+2k_jn+\beta_j^0,$$

$$(j=1,2,\ldots,N),\ \ \exp(A_{ij}) = \frac{\cos(l_i-l_j)-\cosh(k_i-k_j)}{\cos(l_i-l_j)-\cosh(k_i+k_j)},$$

where the notation $\sum_{\mu=0,1}$ implies the summation over all possible combinations of $\mu_1=0,1,\ldots,\mu_N=0,1$ and $\sum_{i<j}^{N}$ indicates the summation over all possible pairs chosen from N elements.

When we take $N=4$, $k_1=\alpha_1$, $k_2=\alpha_2$, $k_3=\alpha_1$, $k_4=\alpha_2$, $l_1=\delta_1\mathrm{i}$, $l_2=\delta_2\mathrm{i}$, $l_3=-\delta_1\mathrm{i}$, $l_4=-\delta_2\mathrm{i}$, $\beta_1^0=\beta_3^{0*}$, and $\beta_2^0=\beta_4^{0*}$ ($\alpha_i\neq 0$, $\delta_i\neq 0$ for $i=1,2$ and $*$ denotes the complex conjugate) in N-soliton solution (5), we obtain a hybrid solution composed of two y-breathers

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2}, \tag{6}$$

with

$$\begin{aligned} f(n) \;=\; & 1+\frac{K_1}{4\alpha_1^4}\exp(2\xi_1)+\frac{K_2}{4\alpha_2^4}\exp(2\xi_2) \\ & +\frac{K_1K_2L_1^2L_2^2}{16\alpha_1^4\alpha_2^4}\exp[2(\xi_1+\xi_2)] \\ & -\frac{1}{\alpha_1^2}\exp(\xi_1)\left[1+\frac{K_2L_1L_2}{4\alpha_2^4}\exp(2\xi_2)\right]\cos(\psi_1) \\ & -\frac{1}{\alpha_2^2}\exp(\xi_2)\left[1+\frac{K_1L_1L_2}{4\alpha_1^4}\exp(2\xi_1)\right]\cos(\psi_2) \\ & +\frac{1}{2\alpha_1^2\alpha_2^2}\exp(\xi_1+\xi_2)\big[L_1\cos(\psi_1+\psi_2) \\ & +L_2\cos(\psi_1-\psi_2)\big], \end{aligned}$$

in which

$$\begin{aligned} K_i &= \frac{\cosh(2\delta_i)-1}{\cosh(2\delta_i)-\cosh(2\alpha_i)}, \quad (i=1,2), \\ L_1 &= \frac{\cosh(\delta_1-\delta_2)-\cosh(\alpha_1-\alpha_2)}{\cosh(\delta_1-\delta_2)-\cosh(\alpha_1+\alpha_2)}, \\ L_2 &= \frac{\cosh(\delta_1+\delta_2)-\cosh(\alpha_1-\alpha_2)}{\cosh(\delta_1+\delta_2)-\cosh(\alpha_1+\alpha_2)}, \\ \xi_i &= 2\sinh(\alpha_i)\cosh(\delta_i)x+2\alpha_i n+\sigma_i, \quad (i=1,2), \\ \psi_i &= 2\sinh(\alpha_i)\sinh(\delta_i)y+\theta_i, \quad (i=1,2), \end{aligned}$$

where σ_1, σ_2, θ_1, θ_2 are real constants and satisfy

$$\exp(\mathrm{Re}(\beta_1^0)-\sigma_1)=\frac{1}{2\alpha_1^2},$$

$$\exp(\mathrm{Re}(\beta_2^0)-\sigma_2)=\frac{1}{2\alpha_2^2}, \tag{7}$$

$$\theta_1=\mathrm{Im}(\beta_1^0)-\pi, \quad \theta_2=\mathrm{Im}(\beta_2^0)-\pi. \tag{8}$$

From hybrid solution (6), we obtain the breather named as breather 1 moves along $\xi_1\approx 0$ and the breather named as breather 2 moves along $\xi_2\approx 0$. The breather 1 and the breather 2 can be expressed as follows:

$$u_i=\frac{K_i\cosh(2\xi_i+2\varepsilon_i)+K_i\cosh(4\alpha_i)-4\sqrt{K_i}\cosh(2\alpha_i)\cosh(\xi_i+\varepsilon_i)\cos(\psi_i)+2\cos^2(\psi_i)}{2(\sqrt{K_i}\cosh(\xi_i+\varepsilon_i)-\cos(\psi_i))^2},$$

with

$$\exp(\varepsilon_i)=\frac{\sqrt{K_i}}{2\alpha_i^2},$$

where subscript i denotes the breather i ($i=1,2$). The nonsingular conditions of the breather 1 and the breather 2 are $|\delta_1|>|\alpha_1|$ and $|\delta_2|>|\alpha_2|$. Reference [16] gives that the phase shifts of two breathers before and after the interaction between two breathers are both Γ, where $\Gamma=\ln|L_1L_2|$. When $\Gamma\to\infty$, the interaction between two breathers becomes resonant interaction. Because the resonant solutions are similar in form as $\Gamma\to\pm\infty$, we only consider $\Gamma\to-\infty$, which is equivalent to $L_1=0$. From $L_1=0$, we obtain a resonant condition

$$\alpha_1-\alpha_2=\delta_1-\delta_2. \tag{9}$$

By introducing the resonant condition (9) into the hybrid solution (6), we derive the resonant solution as follows:

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2}, \tag{10}$$

with

$$\begin{aligned} f(n) \;=\; & 1+\frac{K_1}{4\alpha_1^4}\exp(2\xi_1)+\frac{K_2}{4\alpha_2^4}\exp(2\xi_2) \\ & -\frac{1}{\alpha_1^2}\exp(\xi_1)\cos(\psi_1)-\frac{1}{\alpha_2^2}\exp(\xi_2)\cos(\psi_2) \\ & +\frac{L_2}{2\alpha_1^2\alpha_2^2}\exp(\xi_1+\xi_2)\cos(\psi_1-\psi_2), \end{aligned} \tag{11}$$

where

$$\begin{aligned} K_1 &= \frac{\cosh(2\delta_2+2\alpha_1-2\alpha_2)-1}{\cosh(2\delta_2+2\alpha_1-2\alpha_2)-\cosh(2\alpha_1)}, \\ K_2 &= \frac{\cosh(2\delta_2)-1}{\cosh(2\delta_2)-\cosh(2\alpha_2)}, \\ L_2 &= \frac{\cosh(\alpha_1-\alpha_2+2\delta_2)-\cosh(\alpha_1-\alpha_2)}{\cosh(\alpha_1-\alpha_2+2\delta_2)-\cosh(\alpha_1+\alpha_2)}, \\ \xi_1 &= 2\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)x+2\alpha_1 n+\sigma_1, \\ \xi_2 &= 2\sinh(\alpha_2)\cosh(\delta_2)x+2\alpha_2 n+\sigma_2, \\ \psi_1 &= 2\sinh(\alpha_1)\sinh(\alpha_1-\alpha_2+\delta_2)y+\theta_1, \\ \psi_2 &= 2\sinh(\alpha_2)\sinh(\delta_2)y+\theta_2. \end{aligned}$$

It should be noted that the resonant interaction between two breathers involves a resonant breather, so we analyze the asymptotic properties of three breathers. The nonsingular conditions of the breather 1 and the breather 2 are $|\delta_2+\alpha_1-\alpha_2|>|\alpha_1|$ and $|\delta_2|>|\alpha_2|$. From the resonant solution (10), we obtain the breather named as breather 3 moves along $\xi_1-\xi_2\approx 0$. By introducing

$$\varsigma=\begin{cases} 1, & L_2>0, \\ 0, & L_2<0, \end{cases}$$

the breather 3 can be expressed as follows:

$$u_3=\frac{K_3\cosh(2\xi_3+2\sigma'+2\varepsilon_3)+K_3\cosh(4\alpha_3)-4\sqrt{K_3}\cosh(2\alpha_3)\cosh(\xi_3+\sigma'+\varepsilon_3)\cos(\psi_3)+2\cos^2(\psi_3)}{2(\sqrt{K_3}\cosh(\xi_3+\sigma'+\varepsilon_3)-\cos(\psi_3))^2},$$

with

$$\alpha_3=\alpha_1-\alpha_2,\ \ \xi_3=\xi_1-\xi_2,\ \ \psi_3=\psi_1-\psi_2+\varsigma\pi,\ \ K_3=\frac{K_1K_2}{L_2^2},\ \ \sigma'=\ln(\frac{2|L_2|\alpha_2^2\alpha_3^2}{K_2\alpha_1^2}),\ \ \exp(\varepsilon_3)=\frac{\sqrt{K_3}}{2\alpha_3^2},$$

where subscript 3 denotes the breather 3. We derive $K_3>1$ under the resonant condition (9) and the nonsingular conditions of the breather 1 and the breather 2, so the breather 3 is nonsingular. We calculate the velocities of breather 1, breather 2 and breather 3 as follows:

$$V_1=\left(-\frac{\alpha_1}{\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)},0\right)^{\mathrm{T}},\ \ V_2=\left(-\frac{\alpha_2}{\sinh(\alpha_2)\cosh(\delta_2)},0\right)^{\mathrm{T}},$$
$$V_3=\left(-\frac{\alpha_1-\alpha_2}{\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)-\sinh(\alpha_2)\cosh(\delta_2)},0\right)^{\mathrm{T}},$$

where subscripts 1, 2 and 3 denote the breather 1, the breather 2 and the breather 3. In Table 1, we summarize that the three breathers admit six different types of the asymptotic behaviors in the resonant interaction under different parametric conditions. Here we analyze the case 1 with the parametric condition $\alpha_1>\alpha_2>0$, $|V_1|<|V_2|$ in detail, and the asymptotic processes of other cases are similar to the case 1.

Table 1. The asymptotic forms of three breathers under different parametric conditions.

Case	Condition	Before interaction ($n\to-\infty$)	After interaction ($n\to+\infty$)
1	$\alpha_1>\alpha_2>0,\ \|V_1\|<\|V_2\|$ or $\alpha_1<\alpha_2<0,\ \|V_1\|>\|V_2\|$	$u_{\mathrm{b}1}^{-}\simeq u_1,\ u_{\mathrm{b}2}^{-}\simeq 1,\ u_{\mathrm{b}3}^{-}\simeq 1$	$u_{\mathrm{b}1}^{+}\simeq 1,\ u_{\mathrm{b}2}^{+}\simeq u_2,\ u_{\mathrm{b}3}^{+}\simeq u_3$
2	$\alpha_1<\alpha_2<0,\ \|V_1\|<\|V_2\|$ or $\alpha_1>\alpha_2>0,\ \|V_1\|>\|V_2\|$	$u_{\mathrm{b}1}^{-}\simeq 1,\ u_{\mathrm{b}2}^{-}\simeq u_2,\ u_{\mathrm{b}3}^{-}\simeq u_3$	$u_{\mathrm{b}1}^{+}\simeq u_1,\ u_{\mathrm{b}2}^{+}\simeq 1,\ u_{\mathrm{b}3}^{+}\simeq 1$
3	$\alpha_2<\alpha_1<0,\ \|V_1\|>\|V_2\|$ or $\alpha_2>\alpha_1>0,\ \|V_1\|<\|V_2\|$	$u_{\mathrm{b}1}^{-}\simeq u_1,\ u_{\mathrm{b}2}^{-}\simeq 1,\ u_{\mathrm{b}3}^{-}\simeq u_3$	$u_{\mathrm{b}1}^{+}\simeq 1,\ u_{\mathrm{b}2}^{+}\simeq u_2,\ u_{\mathrm{b}3}^{+}\simeq 1$
4	$\alpha_2>\alpha_1>0,\ \|V_1\|>\|V_2\|$ or $\alpha_2<\alpha_1<0,\ \|V_1\|<\|V_2\|$	$u_{\mathrm{b}1}^{-}\simeq 1,\ u_{\mathrm{b}2}^{-}\simeq u_2,\ u_{\mathrm{b}3}^{-}\simeq 1$	$u_{\mathrm{b}1}^{+}\simeq u_1,\ u_{\mathrm{b}2}^{+}\simeq 1,\ u_{\mathrm{b}3}^{+}\simeq u_3$
5	$\alpha_1>0,\ \alpha_2<0,\ \|V_1\|>\|V_2\|$ or $\alpha_1<0,\ \alpha_2>0,\ \|V_1\|<\|V_2\|$	$u_{\mathrm{b}1}^{-}\simeq u_1,\ u_{\mathrm{b}2}^{-}\simeq u_2,\ u_{\mathrm{b}3}^{-}\simeq 1$	$u_{\mathrm{b}1}^{+}\simeq 1,\ u_{\mathrm{b}2}^{+}\simeq 1,\ u_{\mathrm{b}3}^{+}\simeq u_3$
6	$\alpha_1<0,\ \alpha_2>0,\ \|V_1\|>\|V_2\|$ or $\alpha_1>0,\ \alpha_2<0,\ \|V_1\|<\|V_2\|$	$u_{\mathrm{b}1}^{-}\simeq 1,\ u_{\mathrm{b}2}^{-}\simeq 1,\ u_{\mathrm{b}3}^{-}\simeq u_3$	$u_{\mathrm{b}1}^{+}\simeq u_1,\ u_{\mathrm{b}2}^{+}\simeq u_2,\ u_{\mathrm{b}3}^{+}\simeq 1$

For the asymptotic analysis of the breather 1, we have $\xi_1\approx 0$ and ξ_2 can be expressed as follows:

$$\xi_2=2\sinh(\alpha_2)\cosh(\delta_2)(|V_2|-|V_1|)n+\sigma_2-\frac{\sigma_1\sinh(\alpha_2)\cosh(\delta_2)}{\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)},\tag{12}$$

thus $\xi_2\to\mp\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the tau function of the breather 1 denoted as $f_{\mathrm{b}1}$ from Eq. (11) as follows:

$$f_{\mathrm{b}1}^{\mp}\simeq\begin{cases}1+\frac{K_1}{4\alpha_1^4}\exp(2\xi_1)-\frac{1}{\alpha_1^2}\exp(\xi_1)\cos(\psi_1), & n\to-\infty,\\ 1, & n\to+\infty,\end{cases}\tag{13}$$

where superscript $-$ and $+$ denote the asymptotic form as $n\to-\infty$ and $n\to+\infty$, respectively.

For the asymptotic analysis of the breather 2, we have $\xi_2\approx 0$ and ξ_1 can be expressed as follows:

$$\xi_1=2\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)(|V_1|-|V_2|)n+\sigma_1-\frac{\sigma_2\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)}{\sinh(\alpha_2)\cosh(\delta_2)},\tag{14}$$

thus $\xi_1\to\pm\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the tau function of the breather 2 denoted as $f_{\mathrm{b}2}$ from Eq. (11) as follows:

$$f_{\mathrm{b}2}^{\mp}\simeq\begin{cases}1, & n\to-\infty,\\ 1+\frac{K_2}{4\alpha_2^4}\exp(2\xi_2)-\frac{1}{\alpha_2^2}\exp(\xi_2)\cos(\psi_2), & n\to+\infty.\end{cases}\tag{15}$$

Under the parametric conditions $\alpha_1>\alpha_2>0$, $|V_1|<|V_2|$, we obtain the relation $|V_2|>|V_1|>|V_3|$. For the asymptotic analysis of the breather 3, we have $\xi_1-\xi_2\approx 0$, ξ_1 can be expressed as follows:

$$\xi_1=2\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)(|V_1|-|V_3|)n+\rho_1,$$

where

$$\rho_1=\sigma_1-\frac{\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)(\sigma_1-\sigma_2)}{\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)-\sinh(\alpha_2)\cosh(\delta_2)},$$

and ξ_2 can be expressed as follows:

$$\xi_2=2\sinh(\alpha_2)\cosh(\delta_2)(|V_2|-|V_3|)n+\rho_2,$$

in which

$$\rho_2=\sigma_2-\frac{\sinh(\alpha_2)\cosh(\delta_2)(\sigma_1-\sigma_2)}{\sinh(\alpha_1)\cosh(\alpha_1-\alpha_2+\delta_2)-\sinh(\alpha_2)\cosh(\delta_2)}.$$

Therefore, we obtain $\xi_1,\ \xi_2\to\mp\infty$ as $n\to\mp\infty$ and the asymptotic form of the tau function of the breather 3 denoted as $f_{\mathrm{b}3}$ from Eq. (11) as follows:

$$f_{\mathrm{b}3}^{\mp} \simeq \begin{cases} 1, & n \to -\infty, \\ \dfrac{K_2}{4\alpha_2^4}\exp(2\xi_2)\left(1+\dfrac{K_3}{4\alpha_3^4}\exp(2(\xi_3+\sigma'))-\dfrac{1}{\alpha_3^2}\exp(\xi_3+\sigma')\cos(\psi_3)\right), & n \to +\infty. \end{cases} \tag{16}$$

By introducing the asymptotic forms $f_{\mathrm{b}i}$ $(i=1,2,3)$ into the transformation Eq. (4), we derive the asymptotic forms for three breathers denoted as $u_{\mathrm{b}i}$ $(i=1,2,3)$ as follows:

(i) Before the interaction $(n \to -\infty)$

$$u_{\mathrm{b}1}^{-} \simeq u_1, \tag{17}$$

$$u_{\mathrm{b}2}^{-} \simeq 1, \tag{18}$$

$$u_{\mathrm{b}3}^{-} \simeq 1. \tag{19}$$

(ii) After the interaction $(n \to +\infty)$

$$u_{\mathrm{b}1}^{+} \simeq 1, \tag{20}$$

$$u_{\mathrm{b}2}^{+} \simeq u_2, \tag{21}$$

$$u_{\mathrm{b}3}^{+} \simeq u_3. \tag{22}$$

The specific process of the case 1 is shown in Fig. 1. Figure 1 illustrates fission of the breather 1 into the breather 2 and the breather 3. In case 2, the asymptotic expressions as $n \to -\infty$ in Eqs. (17)–(19) will interchange with the asymptotic expressions as $n \to +\infty$ in Eqs. (20)–(22). The case 2 describes the breather 2 merges with the breather 3, and they turn into the breather 1. We demonstrate the specific process in Fig. 2. In all cases, there are two different physical phenomena, which are fusion and fission. The case 4 and the case 6 describe the fission phenomena as well as the case 3 and the case 5 describe the fusion phenomena. Interestingly, the breathers that act as resonant breathers are different in different cases. The breather 1 acts as the resonant breather in case 1 and case 2, the breather 2 acts as the resonant breather in case 3 and case 4 as well as the breather 3 acts as the resonant breather in case 5 and case 6.

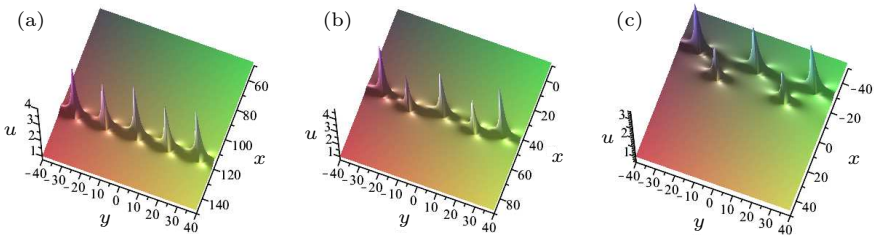


Fig. 1. Fission of two y-breathers with the parameter $\alpha_1=0.3$, $\alpha_2=0.2$, $\delta_1=0.6$, $\delta_2=0.5$, $\sigma_1=0$, $\sigma_2=0$, $\theta_1=0$, and $\theta_2=0$. (a) $n=-145$. (b) $n=-56$. (c) $n=25$.

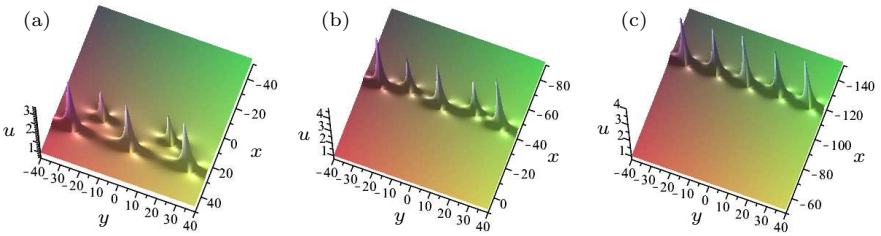


Fig. 2. Fusion of two y-breathers with the parameter $\alpha_1=-0.3$, $\alpha_2=-0.2$, $\delta_1=-0.6$, $\delta_2=-0.5$, $\sigma_1=0$, $\sigma_2=0$, $\theta_1=0$, and $\theta_2=0$. (a) $n=-25$. (b) $n=56$. (c) $n=145$.

3. Resonant interactions between a breather and line solitons

In this section, we construct the resonant solutions consisting of a breather and a line soliton/two line solitons. Through the asymptotic analysis of the resonant solutions, we show that the resonant interactions between a breather and a line soliton/two line solitons exhibit the fusion, fission and time-localized breather phenomena.

3.1. Resonant interactions between a breather and a line soliton

When we take $N=3$, $k_1=\alpha_1$, $k_2=\alpha_2$, $k_3=\alpha_1$, $l_1=\delta\mathrm{i}$, $l_2=0$, $l_3=-\delta\mathrm{i}$ and $\beta_1^0=\beta_3^{0*}$ ($\alpha_i\neq 0$ for $i=1,2$ and $\delta\neq 0$) in N-soliton solution (5), we obtain a hybrid solution consisting of a line soliton arranged in the y direction and a y-breather

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2}, \tag{23}$$

where

$$f(n) = 1+\frac{K}{4\alpha_1^4}\exp(2\xi_1)-\frac{1}{\alpha_1^2}\exp(\xi_1)\cos(\psi_1) + \exp(\xi_2)\left[1+\frac{KL^2}{4\alpha_1^4}\exp(2\xi_1)-\frac{L}{\alpha_1^2}\exp(\xi_1)\cos(\psi_1)\right],$$

in which

$$K=\frac{\cosh(2\delta)-1}{\cosh(2\delta)-\cosh(2\alpha_1)},\quad L=\frac{\cosh(\delta)-\cosh(\alpha_1-\alpha_2)}{\cosh(\delta)-\cosh(\alpha_1+\alpha_2)},$$

$$\xi_1=2\sinh(\alpha_1)\cosh(\delta)x+2\alpha_1 n+\sigma_1,$$

$$\psi_1 = 2\sinh(\alpha_1)\sinh(\delta)y+\theta,$$
$$\xi_2 = 2\sinh(\alpha_2)x+2\alpha_2 n+\beta_2^0,$$

where σ_1, θ, β_2^0 are real constants and satisfy

$$\exp(\mathrm{Re}(\beta_1^0)-\sigma_1)=\frac{1}{2\alpha_1^2},\quad \theta=\mathrm{Im}(\beta_1^0)-\pi. \tag{24}$$

$$\widehat{u}_1=\frac{K\cosh(2\xi_1+2\varepsilon_1)+K\cosh(4\alpha_1)-4\sqrt{K}\cosh(2\alpha_1)\cosh(\xi_1+\varepsilon_1)\cos(\psi_1)+2\cos^2(\psi_1)}{2(\sqrt{K}\cosh(\xi_1+\varepsilon_1)-\cos(\psi_1))^2}, \tag{25}$$

with

$$\exp(\varepsilon_1)=\frac{\sqrt{K}}{2\alpha_1^2},$$

where subscript 1 denotes the breather. The line soliton 1 can be expressed as follows:

$$\widehat{u}_2=\frac{\cosh(\xi_2)+\cosh(2\alpha_2)}{\cosh(\xi_2)+1}, \tag{26}$$

where subscript 2 denotes the line soliton 1. In order to derive the phase shifts of breather and line soliton 1 before and after the interaction, we analyze the asymptotic properties of the breather and the line soliton 1. We assume $\alpha_1>0$, $\alpha_2>0$, and $\alpha_2/\sinh(\alpha_2)>\alpha_1/\sinh(\alpha_1)\cosh(\delta)$. For the asymptotic analysis of the breather, we have $\xi_1\approx 0$ and ξ_2 can be expressed as follows:

$$\xi_2 = 2\sinh(\alpha_2)\left(\frac{\alpha_2}{\sinh(\alpha_2)}-\frac{\alpha_1}{\sinh(\alpha_1)\cosh(\delta)}\right)n +\beta_2^0-\frac{\sigma_1\sinh(\alpha_2)}{\sinh(\alpha_1)\cosh(\delta)}, \tag{27}$$

thus $\xi_2\to\mp\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the breather denoted as u_b from Eq. (23) as follows:

$$u_\mathrm{b}^{\mp}\simeq\begin{cases}\widehat{u}_1(\xi_1,\psi_1), & n\to-\infty,\\ \widehat{u}_1(\xi_1+\Gamma,\psi_1+\varepsilon\pi), & n\to+\infty,\end{cases} \tag{28}$$

where $\widehat{u}_1(\xi_1,\psi_1)$ represents $\widehat{u}_1$ in Eq. (25), $\Gamma=\ln|L|$ as well as $\varepsilon=0$ for $L>0$ and $\varepsilon=1$ for $L<0$.

For the asymptotic analysis of the line soliton 1, we have $\xi_2\approx 0$ and ξ_1 can be expressed as follows:

$$\xi_1 = 2\sinh(\alpha_1)\cosh(\delta)\left(\frac{\alpha_1}{\sinh(\alpha_1)\cosh(\delta)}-\frac{\alpha_2}{\sinh(\alpha_2)}\right)n +\sigma_1-\frac{\beta_2^0\sinh(\alpha_1)\cosh(\delta)}{\sinh(\alpha_2)}, \tag{29}$$

thus $\xi_1\to\pm\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the line soliton 1 denoted as u_{l1} from (23) as follows:

$$u_{l1}^{\mp}\simeq\begin{cases}\widehat{u}_2(\xi_2+2\Gamma), & n\to-\infty,\\ \widehat{u}_2(\xi_2), & n\to+\infty,\end{cases} \tag{30}$$

where $\widehat{u}_2(\xi_2)$ represents $\widehat{u}_2$ in Eq. (26).

From the asymptotic expressions (28) and (30), we obtain the phase shift of the breather is Γ and the phase shift of the line soliton 1 is 2Γ. When $\Gamma\to\infty$, the interaction between the breather and the line soliton 1 becomes resonant interaction. Because the resonant solutions are similar in form as $\Gamma\to\pm\infty$, we only consider $\Gamma\to-\infty$, which is equivalent to $L=0$. From $L=0$, we obtain a resonant condition

$$\delta=\alpha_2-\alpha_1. \tag{31}$$

By introducing the resonant condition (31) into the hybrid solution (23), we derive the resonant solution as follows:

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2}, \tag{32}$$

with

$$f(n)=1+\frac{K}{4\alpha_1^4}\exp(2\xi_1)-\frac{1}{\alpha_1^2}\exp(\xi_1)\cos(\psi_1)+\exp(\xi_2),$$

where

$$K=\frac{\cosh(2\alpha_2-2\alpha_1)-1}{\cosh(2\alpha_2-2\alpha_1)-\cosh(2\alpha_1)},$$
$$\xi_1=2\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)x+2\alpha_1 n+\sigma_1,$$
$$\psi_1=2\sinh(\alpha_1)\sinh(\alpha_2-\alpha_1)y+\theta,$$
$$\xi_2=2\sinh(\alpha_2)x+2\alpha_2 n+\beta_2^0.$$

The nonsingular condition of the resonant solution (32) is $|\alpha_2-\alpha_1|>|\alpha_1|$.

It should be noted that the resonant interaction between a breather and a line soliton involves a resonant line soliton which is different from the original line soliton, so we analyze the asymptotic properties of a breather and two line solitons. From the resonant solution (32), we obtain the line soliton named as line soliton 2 moves along $2\xi_1-\xi_2\approx 0$. The line soliton 2 can be expressed as follows:

$$\widehat{u}_3=\frac{\cosh(\xi_3+\zeta)+\cosh(2\alpha_3)}{\cosh(\xi_3+\zeta)+1},$$

with

$$\xi_3=2\xi_1-\xi_2,\quad \alpha_3=2\alpha_1-\alpha_2,\quad \exp(\zeta)=\frac{K}{4\alpha_1^4},$$

where subscript 3 denotes the line soliton 2. Furthermore we calculate the velocities of breather, line soliton 1 and line soliton 2 as follows:

$$V_\mathrm{b}=\left(-\frac{\alpha_1}{\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)},0\right)^\mathrm{T},$$

$$V_{l1}=\left(-\frac{\alpha_2}{\sinh(\alpha_2)},0\right)^{\mathrm{T}},$$

$$V_{l2}=\left(-\frac{2\alpha_1-\alpha_2}{2\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)-\sinh(\alpha_2)},0\right)^{\mathrm{T}},$$

where subscripts b, l1, and l2 denote the breather, the line soliton 1 and line soliton 2. In Table 2, we conclude that the breather and the line solitons admit four different types of the asymptotic behaviors in the resonant interaction under different parametric conditions. We obtain $|V_{l1}|>|V_{\mathrm{b}}|$ in all cases. Here we analyze the case 1 in detail, and the analyses of other cases are similar to the case 1.

For the asymptotic analysis of the breather, we have $\xi_1\approx 0$ and ξ_2 can be expressed as follows:

$$\begin{aligned}\xi_2 \;=\; & 2\sinh(\alpha_2)(|V_{l1}|-|V_{\mathrm{b}}|)n+\beta_2^0\\ & -\frac{\sigma_1\sinh(\alpha_2)}{\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)},\end{aligned}\tag{33}$$

thus $\xi_2\to\mp\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the breather from Eq. (32) as follows:

$$u_{\mathrm{b}}^{\mp}\simeq\begin{cases}\widehat{u}_1, & n\to-\infty,\\ 1, & n\to+\infty.\end{cases}\tag{34}$$

For the asymptotic analysis of the line soliton 1, we have $\xi_2\approx 0$ and ξ_1 can be expressed as follows:

$$\begin{aligned}\xi_1 \;=\; & 2\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)(|V_{\mathrm{b}}|-|V_{l1}|)n+\sigma_1\\ & -\frac{\beta_2^0\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)}{\sinh(\alpha_2)},\end{aligned}\tag{35}$$

thus $\xi_1\to\pm\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the line soliton 1 from Eq. (32) as follows:

$$u_{l1}^{\mp}\simeq\begin{cases}1, & n\to-\infty,\\ \widehat{u}_2, & n\to+\infty.\end{cases}\tag{36}$$

For the asymptotic analysis of the line soliton 2, we have $2\xi_1-\xi_2\approx 0$, ξ_1 can be expressed as follows:

$$\xi_1=2\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)(|V_{\mathrm{b}}|-|V_{l2}|)n+\gamma_1,$$

where

$$\gamma_1=\sigma_1-\frac{\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)(2\sigma_1-\beta_2^0)}{2\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)-\sinh(\alpha_2)},$$

and ξ_2 can be expressed as follows:

$$\xi_2=2\sinh(\alpha_2)(|V_{l1}|-|V_{l2}|)n+\gamma_2,$$

where

$$\gamma_2=\beta_2^0-\frac{\sinh(\alpha_2)(2\sigma_1-\beta_2^0)}{2\sinh(\alpha_1)\cosh(\alpha_2-\alpha_1)-\sinh(\alpha_2)}.$$

We can calculate $|V_{l2}|>|V_{l1}|>|V_{\mathrm{b}}|$ in case 1, thus $\xi_1,\xi_2\to\pm\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the line soliton 2 denoted as u_{l2} from (32) as follows:

$$u_{l2}^{\mp}\simeq\begin{cases}\widehat{u}_3, & n\to-\infty,\\ 1, & n\to+\infty.\end{cases}\tag{37}$$

We also calculate the peaks of the line soliton 1 and the line soliton 2 are $\cosh^2(\alpha_2)$ and $\cosh^2(2\alpha_1-\alpha_2)$, respectively. It should be noted that $\cosh^2(\alpha_2)>\cosh^2(2\alpha_1-\alpha_2)$ under the condition of case 1. Figure 3 shows the breather merges with the line soliton 2, and they turn into the line soliton 1. In case 2, the asymptotic expressions as $n\to-\infty$ in Eqs. (34), (36), and (37) will interchange with the asymptotic expressions as $n\to+\infty$ in Eqs. (34), (36), and (37). Therefore, the case 2 describes fission of the line soliton 1 into the line soliton 2 and breather. We demonstrate the specific process in Fig. 4. In all cases, there are two different physical phenomena, which are fusion and fission. The case 3 is fusion and the case 4 is fission. Interestingly, the line solitons that act as resonant solitons are different in different cases. The line soliton 1 acts as the resonant soliton in case 1 and case 2, and the line soliton 2 acts as the resonant soliton in case 3 and case 4.

Table 2. The asymptotic forms of a breather and two line solitons under different parametric conditions.

Case	Condition	Before interaction ($n\to-\infty$)	After interaction ($n\to+\infty$)
1	$\alpha_2>2\alpha_1>0$	$u_{\mathrm{b}}^{-}\simeq\widehat{u}_1$, $u_{l1}^{-}\simeq 1$, $u_{l2}^{-}\simeq\widehat{u}_3$	$u_{\mathrm{b}}^{+}\simeq 1$, $u_{l1}^{+}\simeq\widehat{u}_2$, $u_{l2}^{+}\simeq 1$
2	$\alpha_2<2\alpha_1<0$	$u_{\mathrm{b}}^{-}\simeq 1$, $u_{l1}^{-}\simeq\widehat{u}_2$, $u_{l2}^{-}\simeq 1$	$u_{\mathrm{b}}^{+}\simeq\widehat{u}_1$, $u_{l1}^{+}\simeq 1$, $u_{l2}^{+}\simeq\widehat{u}_3$
3	$\alpha_1<0$ $\alpha_2>0$	$u_{\mathrm{b}}^{-}\simeq\widehat{u}_1$, $u_{l1}^{-}\simeq\widehat{u}_2$, $u_{l2}^{-}\simeq 1$	$u_{\mathrm{b}}^{+}\simeq 1$, $u_{l1}^{+}\simeq 1$, $u_{l2}^{+}\simeq\widehat{u}_3$
4	$\alpha_1>0$ $\alpha_2<0$	$u_{\mathrm{b}}^{-}\simeq 1$, $u_{l1}^{-}\simeq 1$, $u_{l2}^{-}\simeq\widehat{u}_3$	$u_{\mathrm{b}}^{+}\simeq\widehat{u}_1$, $u_{l1}^{+}\simeq\widehat{u}_2$, $u_{l2}^{+}\simeq 1$

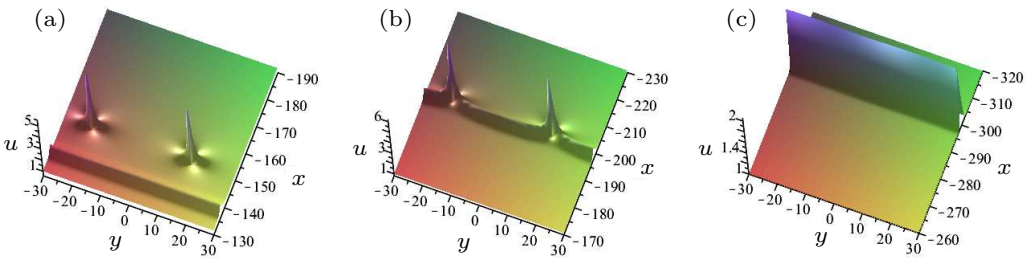


Fig. 3. Fusion of a y-breather and a line soliton arranged in the y direction with the parameter $\alpha_1=0.1$, $\alpha_2=0.9$, $\delta=0.8$, $\sigma_1=5$, $\theta=10$, and $\beta_2^0=5$. (a) $n=158$. (b) $n=225$. (c) $n=340$.

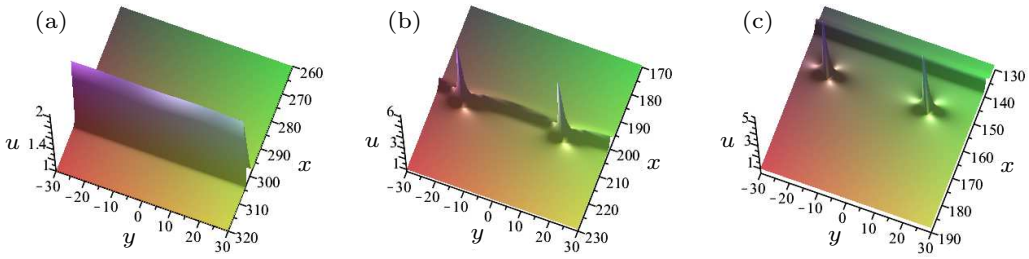


Fig. 4. Fission of a y-breather and a line soliton arranged in the y direction with the parameter $\alpha_1=-0.1$, $\alpha_2=-0.9$, $\delta=-0.8$, $\sigma_1=5$, $\theta=10$ and $\beta_2^0=5$. (a) $n=-340$. (b) $n=-225$. (c) $n=-158$.

3.2. Resonant interactions between a breather and two line solitons

When we take $N=4$, $k_1=\alpha_1$, $k_2=\alpha_2$, $k_3=\alpha_3$, $k_4=\alpha_1$, $l_1=\delta\mathrm{i}$, $l_2=0$, $l_3=0$, $l_4=-\delta\mathrm{i}$, and $\beta_1^0=\beta_4^{0*}$ ($\alpha_i\neq 0$ for $i=1,2,3$ and $\delta\neq 0$) in N-soliton solution (5), we obtain a hybrid solution consisting of a y-breather and two line solitons arranged in the y direction

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2}, \tag{38}$$

with

$$\begin{aligned} f(n) \;=\; & 1-\frac{1}{\alpha_1^2}\exp(\xi_1)\cos(\psi)+\frac{K_1}{4\alpha_1^4}\exp(2\xi_1) \\ & +\exp(\xi_2)\left[1-\frac{L_1}{\alpha_1^2}\exp(\xi_1)\cos(\psi)+\frac{K_1L_1^2}{4\alpha_1^4}\exp(2\xi_1)\right] \\ & +\exp(\xi_3)\left[1-\frac{L_2}{\alpha_1^2}\exp(\xi_1)\cos(\psi)+\frac{K_1L_2^2}{4\alpha_1^4}\exp(2\xi_1)\right] \\ & +K_2\exp(\xi_2+\xi_3)\left[1-\frac{L_1L_2}{\alpha_1^2}\exp(\xi_1)\cos(\psi)\right. \\ & \left.+\frac{K_1L_1^2L_2^2}{4\alpha_1^4}\exp(2\xi_1)\right], \end{aligned}$$

in which

$$\begin{aligned} K_1 &= \frac{\cosh(2\delta)-1}{\cosh(2\delta)-\cosh(2\alpha_1)},\quad K_2=\frac{1-\cosh(\alpha_2-\alpha_3)}{1-\cosh(\alpha_2+\alpha_3)}, \\ L_1 &= \frac{\cosh(\delta)-\cosh(\alpha_1-\alpha_2)}{\cosh(\delta)-\cosh(\alpha_1+\alpha_2)}, \\ L_2 &= \frac{\cosh(\delta)-\cosh(\alpha_1-\alpha_3)}{\cosh(\delta)-\cosh(\alpha_1+\alpha_3)}, \\ \xi_1 &= 2\sinh(\alpha_1)\cosh(\delta)x+2\alpha_1 n+\sigma_1, \\ \psi &= 2\sinh(\alpha_1)\sinh(\delta)y+\theta, \\ \xi_i &= 2\sinh(\alpha_i)x+2\alpha_i n+\beta_i^0,\quad (i=2,3), \end{aligned}$$

where σ_1, θ, β_2^0, β_3^0 are real constants and satisfy

$$\exp(\mathrm{Re}(\beta_1^0)-\sigma_1)=\frac{1}{2\alpha_1^2},\quad \theta=\mathrm{Im}(\beta_1^0)-\pi. \tag{39}$$

It should be noted that $K_2\geq 0$ when $\alpha_2+\alpha_3\neq 0$, so the nonsingular condition of the hybrid solution (38) is $|\delta|>|\alpha_1|$, $\alpha_2+\alpha_3\neq 0$, $|\delta|\neq|\alpha_1+\alpha_2|$, and $|\delta|\neq|\alpha_1+\alpha_3|$.

In Subsection 3.1, we have derived the phase shifts of the breather and the line soliton before and after the interaction between a y-breather and a line soliton arranged in the y direction. In the same way, we can obtain the phase shifts of the breather and two line solitons before and after the interaction among a y-breather and two line solitons arranged in the y direction. They are $\ln|L_1L_2|$, $\ln|K_2L_1^2|$, and $\ln|K_2L_2^2|$, respectively.

When phase shifts of the breather and two line solitons are both infinite, the interaction between the breather and two line solitons becomes a resonant interaction. Because the resonant solutions are similar as the phase shifts approach positive and negative infinity, we only consider the phase shifts approach negative infinity, in other words, $L_1=L_2=0$. From $L_1=L_2=0$, we obtain a resonant condition

$$\begin{cases} \alpha_1=\dfrac{\alpha_2+\alpha_3}{2}, \\ \delta=\dfrac{\alpha_2-\alpha_3}{2}. \end{cases} \tag{40}$$

By introducing the resonant condition (40) into the hybrid solution (38), we derive the resonant solution as follows:

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2}, \tag{41}$$

with

$$\begin{aligned} f(n) \;=\; & 1-\frac{4}{(\alpha_2+\alpha_3)^2}\exp(\xi_1)\cos(\psi)+\frac{4K_1}{(\alpha_2+\alpha_3)^4}\exp(2\xi_1) \\ & +\exp(\xi_2)+\exp(\xi_3)+K_2\exp(\xi_2+\xi_3), \end{aligned}$$

where

$$\begin{aligned} K_1 &= \frac{\cosh(\alpha_2-\alpha_3)-1}{\cosh(\alpha_2-\alpha_3)-\cosh(\alpha_2+\alpha_3)}, \\ K_2 &= \frac{1-\cosh(\alpha_2-\alpha_3)}{1-\cosh(\alpha_2+\alpha_3)}, \\ \xi_1 &= (\sinh(\alpha_2)+\sinh(\alpha_3))x+(\alpha_2+\alpha_3)n+\sigma_1, \\ \psi &= (\cosh(\alpha_2)-\cosh(\alpha_3))y+\theta, \\ \xi_2 &= 2\sinh(\alpha_2)x+2\alpha_2 n+\beta_2^0, \\ \xi_3 &= 2\sinh(\alpha_3)x+2\alpha_3 n+\beta_3^0. \end{aligned}$$

It should be noted that $K_2>0$, so the nonsingular condition of the resonant solution (41) is $|\alpha_2-\alpha_3|>|\alpha_2+\alpha_3|$.

From the resonant solution (41), we obtain the breather moves along $\xi_1\approx 0$, the line soliton named as line soliton 1 moves along $\xi_2\approx 0$, and the line soliton named as line soliton 2 moves along $\xi_3\approx 0$. The breather can be expressed as follows:

$$\widetilde{u}_1 = \frac{K_1\cosh(2\xi_1+2\varepsilon_1)+K_1\cosh(2(\alpha_2+\alpha_3))-4\sqrt{K_1}\cosh(\alpha_2+\alpha_3)\cosh(\xi_1+\varepsilon_1)\cos(\psi)+2\cos^2(\psi)}{2(\sqrt{K_1}\cosh(\xi_1+\varepsilon_1)-\cos(\psi))^2},$$

with

$$\exp(\varepsilon_1) = \frac{2\sqrt{K_1}}{(\alpha_2+\alpha_3)^2},$$

where subscript 1 denotes the breather. The line soliton 1 and the line soliton 2 can be expressed as follows:

$$\widetilde{u}_i = \frac{\cosh(\xi_i)+\cosh(2\alpha_i)}{\cosh(\xi_i)+1}, \quad (i=2,3) \tag{42}$$

where subscripts 2 and 3 denote the line soliton 1 and the line soliton 2, respectively. Furthermore we calculate the velocities of breathers, line soliton 1 and line soliton 2 as follows:

$$V_{\rm b} = \left(-\frac{\alpha_2+\alpha_3}{\sinh(\alpha_2)+\sinh(\alpha_3)},0\right)^{\rm T},$$

$$V_{\rm l1} = \left(-\frac{\alpha_2}{\sinh(\alpha_2)},0\right)^{\rm T}, \quad V_{\rm l2} = \left(-\frac{\alpha_3}{\sinh(\alpha_3)},0\right)^{\rm T},$$

where subscripts b, l1, and l2 denote the breather, the line soliton 1 and line soliton 2, respectively. We conclude that the breather and two line solitons admit two different types of the asymptotic behaviors in the resonant interaction under different parametric conditions.

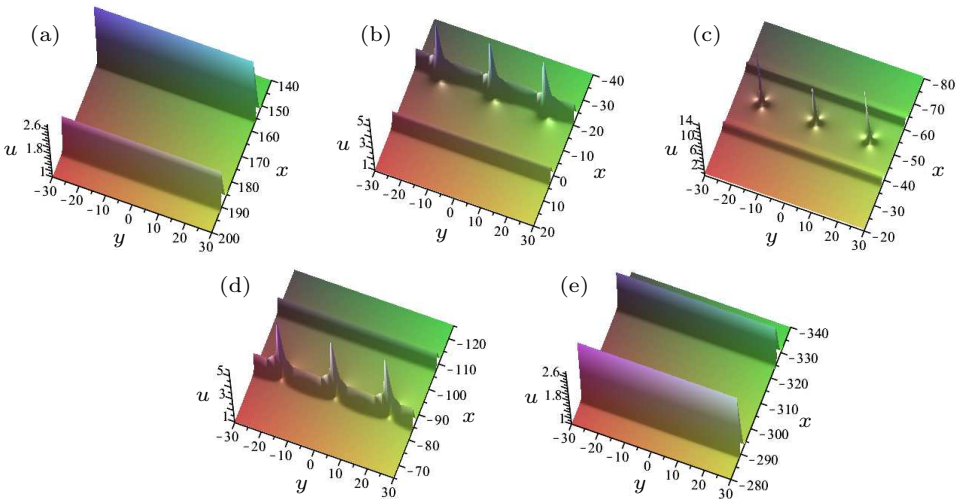


Fig. 5. Resonant interaction between a breather and two line solitons with the parameter $\alpha_1 = 0.1$, $\alpha_2 = 1.1$, $\alpha_3 = -0.9$, $\delta = 1.0$, $\sigma_1 = 0$, $\theta = 0$, $\beta_2^0 = -20$, and $\beta_3^0 = -20$. (a) $n = -199$, (b) $n = 14$, (c) $n = 60$, (d) $n = 114$, (e) $n = 363$.

Case 1 We consider the case when $\alpha_2 > 0$, $\alpha_3 < 0$, $|\alpha_2| > |\alpha_3|$ or $\alpha_2 < 0$, $\alpha_3 > 0$, $|\alpha_2| < |\alpha_3|$. Here we investigate the case with $\alpha_2 > 0$, $\alpha_3 < 0$, $|\alpha_2| > |\alpha_3|$. Under this parametric condition, we obtain $|V_{\rm l2}| > |V_{\rm l1}| > |V_{\rm b}|$. For the asymptotic analysis of the breather, we have $\xi_1 \approx 0$, ξ_2 can be expressed as follows:

$$\xi_2 = 2\sinh(\alpha_2)\,(|V_{\rm l1}|-|V_{\rm b}|)\,n+\beta_2^0-\frac{2\sigma_1\sinh(\alpha_2)}{\sinh(\alpha_2)+\sinh(\alpha_3)},$$

and ξ_3 can be expressed as follows:

$$\xi_3 = 2\sinh(\alpha_3)\,(|V_{\rm l2}|-|V_{\rm b}|)\,n+\beta_3^0-\frac{2\sigma_1\sinh(\alpha_3)}{\sinh(\alpha_2)+\sinh(\alpha_3)},$$

thus $\xi_2 \to \mp\infty$, $\xi_3 \to \pm\infty$ as $n \to \mp\infty$. We obtain the asymptotic form of the breather denoted as $u_{\rm b}$ from Eq. (41) as follows:

$$u_{\rm b}^{\mp} \simeq \begin{cases} 1, & n \to -\infty, \\ 1, & n \to +\infty. \end{cases} \tag{43}$$

For the asymptotic analysis of the line soliton 1, we have $\xi_2 \approx 0$, ξ_1 can be expressed as follows:

$$\xi_1 = (\sinh(\alpha_2)+\sinh(\alpha_3))\,(|V_{\rm b}|-|V_{\rm l1}|)\,n$$
$$+\sigma_1-\frac{\beta_2^0(\sinh(\alpha_2)+\sinh(\alpha_3))}{2\sinh(\alpha_2)},$$

and ξ_3 can be expressed as follows:

$$\xi_3 = 2\sinh(\alpha_3)\,(|V_{\rm l2}|-|V_{\rm l1}|)\,n+\beta_3^0-\frac{\beta_2^0\sinh(\alpha_3)}{\sinh(\alpha_2)},$$

thus $\xi_1 \to \pm\infty$, $\xi_3 \to \pm\infty$ as $n \to \mp\infty$. By utilizing the relation $2\xi_1-\xi_3 = \xi_2+2\sigma_1-\beta_3^0-\beta_2^0$, we obtain the asymptotic form of the line soliton 1 denoted as $u_{\rm l1}$ from Eq. (41) as follows:

$$u_{\rm l1}^{\mp} \simeq \begin{cases} \widetilde{u}_2(\xi_2+\zeta), & n \to -\infty, \\ \widetilde{u}_2(\xi_2), & n \to +\infty, \end{cases} \tag{44}$$

with

$$\exp(\zeta) = \frac{4K_1\exp(2\sigma_1-\beta_2^0-\beta_3^0)}{(\alpha_2+\alpha_3)^4}+K_2,$$

where $\widetilde{u}_2(\xi_2)$ represents $\widetilde{u}_2$ in Eq. (42).

For the asymptotic analysis of the line soliton 2, we have $\xi_3 \approx 0$, ξ_1 can be expressed as follows:

$$\xi_1 = (\sinh(\alpha_2)+\sinh(\alpha_3))\,(|V_{\rm b}|-|V_{\rm l2}|)\,n$$

$$+\sigma_1-\frac{\beta_3^0(\sinh(\alpha_2)+\sinh(\alpha_3))}{2\sinh(\alpha_3)},$$

and ξ_2 can be expressed as follows:

$$\xi_2=2\sinh(\alpha_2)\,(|V_{11}|-|V_{12}|)\,n+\beta_2^0-\frac{\beta_3^0\sinh(\alpha_2)}{\sinh(\alpha_3)},$$

thus $\xi_1\to\pm\infty$, $\xi_2\to\pm\infty$ as $n\to\mp\infty$. By utilizing the relation $2\xi_1-\xi_2=\xi_3+2\sigma_1-\beta_3^0-\beta_2^0$, we obtain the asymptotic form of the line soliton 2 denoted as u_{12} from (41) as follows:

$$u_{12}^{\mp}\simeq\begin{cases}\widetilde{u}_3(\xi_3+\zeta), & n\to-\infty,\\ \widetilde{u}_3(\xi_3), & n\to+\infty,\end{cases}\tag{45}$$

where $\widetilde{u}_3(\xi_3)$ represents $\widetilde{u}_3$ in Eq. (42).

In the same way, we obtain the asymptotic forms Eqs. (43)–(45) in the case with $\alpha_2<0$, $\alpha_3>0$, $|\alpha_2|<|\alpha_3|$. We calculate that the peaks of the line soliton 1 and the line soliton 2 are $\cosh^2(\alpha_2)$ and $\cosh^2(\alpha_3)$, respectively. We have $\cosh^2(\alpha_2)>\cosh^2(\alpha_3)$ under the condition $\alpha_2>0$, $\alpha_3<0$, $|\alpha_2|>|\alpha_3|$, and $\cosh^2(\alpha_3)>\cosh^2(\alpha_2)$ under the condition $\alpha_2<0$, $\alpha_3>0$, $|\alpha_2|<|\alpha_3|$.

The specific process of the resonant interaction between a breather and two line solitons is shown in Fig. 5. The breather detaches from the line soliton 1 with higher peak and fuses into the line soliton 2 with lower peak. In the whole process, the breather is localized in time, which exhibits time-localized breather phenomenon.

Case 2 We consider the case when $\alpha_2>0$, $\alpha_3<0$, $|\alpha_2|<|\alpha_3|$ or $\alpha_2<0$, $\alpha_3>0$, $|\alpha_2|>|\alpha_3|$. Similar to the Case 1, we derive the following asymptotic forms.

(i) Before the interaction ($n\to-\infty$)

Breather ($\xi_1\approx0$, $\xi_2\to-\infty$, $\xi_3\to+\infty$):

$$u_{\rm b}^{-}\simeq1.$$

Line soliton 1 ($\xi_2\approx0$, $2\xi_1-\xi_3\approx0$, $\xi_1\to-\infty$, $\xi_3\to-\infty$):

$$u_{11}^{-}\simeq\widetilde{u}_2(\xi_2).$$

Line soliton 2 ($\xi_3\approx0$, $2\xi_1-\xi_2\approx0$, $\xi_1\to-\infty$, $\xi_2\to-\infty$):

$$u_{12}^{-}\simeq\widetilde{u}_3(\xi_3).$$

After the interaction ($n\to+\infty$)

Breather ($\xi_1\approx0$, $\xi_2\to+\infty$, $\xi_3\to-\infty$):

$$u_{\rm b}^{+}\simeq1.$$

Line soliton 1 ($\xi_2\approx0$, $2\xi_1-\xi_3\approx0$, $\xi_1\to+\infty$, $\xi_3\to+\infty$):

$$u_{11}^{+}\simeq\widetilde{u}_2(\xi_2+\zeta).$$

Line soliton 2 ($\xi_3\approx0$, $2\xi_1-\xi_2\approx0$, $\xi_1\to+\infty$, $\xi_2\to+\infty$):

$$u_{12}^{+}\simeq\widetilde{u}_3(\xi_3+\zeta).$$

From above asymptotic analysis, we derive the case 2 with the parametric conditions $\alpha_2>0$, $\alpha_3<0$, $|\alpha_2|<|\alpha_3|$ describes the breather detaches from the line soliton 2 with higher peak and fuses into the line soliton 1 with lower peak, and the case 2 with the parametric conditions $\alpha_2<0$, $\alpha_3>0$, $|\alpha_2|>|\alpha_3|$ describes the breather detaches from the line soliton 1 with higher peak and fuses into the line soliton 2 with lower peak. After the resonant interaction, the original line soliton 2 becomes the line soliton 1 and the original line soliton 1 becomes the line soliton 2.

4. Resonant interactions between a lump and line solitons

In this section, we construct the resonant solutions consisting of a lump and a line soliton/two line solitons. Through the asymptotic analysis of the resonant solutions, we show the resonant interactions between a lump and line solitons arranged in the y direction exhibit the fusion, fission and rogue lump phenomena.

4.1. Resonant interactions between a lump and a line soliton

If the N-soliton solution (5) with $N=3$ satisfies the partial long wave limit constraints

$$k_1=K_1\varepsilon,\ \ k_2=K_2\varepsilon,\ \ \exp(\beta_1^0)=\exp(\beta_2^0)=-1,$$
$$K_1=K_2^*,\ \ l_1=l_2^*,\ \ \varepsilon\to0,$$

where $*$ denotes the complex conjugate, we obtain the hybrid solution consisting of a lump and a line soliton

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2},\tag{46}$$

in which

$$\begin{aligned}f(n)\ =\ &\theta_1\theta_2+a_{12}\\ &+\exp(\eta_3)(a_{13}a_{23}+a_{13}\theta_2+a_{23}\theta_1+\theta_1\theta_2+a_{12}),\end{aligned}$$

with

$$a_{12}=\frac{2K_1K_2}{\cos(l_1-l_2)-1},$$
$$a_{i3}=\frac{2K_i\sinh(k_3)}{\cos(l_i-l_3)-\cosh(k_3)},\ \ (i=1,2),$$
$$\theta_i=2K_i\cos(l_i)x+2K_i\sin(l_i)y+2K_in,\ \ (i=1,2),$$
$$\eta_3=2\sinh(k_3)\cos(l_3)x+2\sinh(k_3)\sin(l_3)y+2k_3n+\beta_3^0,$$

where β_3^0 is a arbitrary real constant.

By taking $K_1=a$, $K_2=a$, $k_3=\alpha$, $l_1=b\mathrm{i}$, $l_2=-b\mathrm{i}$, and $l_3=0$ ($a\neq0$, $\alpha\neq0$, and $b\neq0$) in the Eq. (46), we obtain the hybrid solution consisting of a lump and a line soliton arranged in the y-direction

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2},\tag{47}$$

with

$$f(n)=|\xi_1+\mathrm{i}\,\psi_1|^2+a_{12}+\exp(\eta_3)\left(|\xi_1+\mathrm{i}\,\psi_1+a_{13}|^2+a_{12}\right),$$

in which

$$a_{12}=\frac{2a^2}{\cosh(2b)-1},\ \ a_{13}=\frac{2a\sinh(\alpha)}{\cosh(b)-\cosh(\alpha)},$$
$$\xi_1=2a\cosh(b)x+2an,\ \ \psi_1=2a\sinh(b)y,$$
$$\eta_3=2\sinh(\alpha)x+2\alpha n+\beta_3^0,$$

where ξ_1, ψ_1 are real part and imaginary part of θ_1, respectively. It should be noted that $a_{12}>0$, so hybrid solution (47) is nonsingular.

From the hybrid solution (47), we obtain the lump moves along $\xi_1\approx 0$ and $\psi_1\approx 0$ as well as the line soliton moves along $\eta_3\approx 0$. The lump can be expressed as follows:

$$\overline{u}_1=\frac{(\xi_1^2-4a^2)^2+(\psi_1^2+a_{12})(\xi_1^2+4a^2)+2a_{12}\psi_1^2+\psi_1^4+a_{12}^2}{(\xi_1^2+\psi_1^2+a_{12})^2},\tag{48}$$

where subscript 1 denotes the lump. The line soliton can be expressed as follows:

$$\overline{u}_2=\frac{\cosh(\eta_3)+\cosh(2\alpha)}{\cosh(\eta_3)+1},\tag{49}$$

where subscript 2 denotes the line soliton. We assume $a>0$, $\alpha>0$ and $\alpha/\sinh(\alpha)>1/\cosh(b)$. For the asymptotic analysis of the lump, we have $\xi_1\approx 0$ and η_3 can be expressed as follows:

$$\eta_3=2\sinh(\alpha)\left(\frac{\alpha}{\sinh(\alpha)}-\frac{1}{\cosh(b)}\right)n+\beta_3^0,\tag{50}$$

thus $\eta_3\to\mp\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the lump denoted as $u_{\rm lump}$ from Eq. (47) as follows:

$$u_{\rm lump}^{\mp}\simeq\begin{cases}\overline{u}_1(\xi_1), & n\to-\infty,\\ \overline{u}_1(\xi_1+a_{13}), & n\to+\infty,\end{cases}\tag{51}$$

where $\overline{u}_1(\xi_1)$ represents $\overline{u}_1$ in Eq. (48).

For the asymptotic analysis of the line soliton, we have $\eta_3\approx 0$ and ξ_1 can be expressed as follows:

$$\xi_1=2a\cosh(b)\left(\frac{1}{\cosh(b)}-\frac{\alpha}{\sinh(\alpha)}\right)n-\frac{a\beta_3^0\cosh(b)}{\sinh(\alpha)},\tag{52}$$

thus $\xi_1\to\pm\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the line soliton denoted as $u_{\rm line}$ from (47) as follows:

$$u_{\rm line}^{\mp}\simeq\begin{cases}\overline{u}_2(\eta_3), & n\to-\infty,\\ \overline{u}_2(\eta_3), & n\to+\infty,\end{cases}\tag{53}$$

where $\overline{u}_2(\eta_3)$ represents $\overline{u}_2$ in Eq. (49).

From the asymptotic expressions (51) and (53), we obtain the line soliton remains unshifted during the process of the interaction and the phase shift of the lump before and after the interaction is a_{13}. In order to derive the resonant solution, we need to make the phase shift a_{13} infinite. To this end, we take the phase parameter β_3^0 as $\beta_3^0=\overline{\beta_3^0}-\ln(|a_{13}|^2)$, and then take the limit $\alpha\to b$. We obtain the resonant solution as follows:

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2},\tag{54}$$

in which

$$f(n)=|\xi_1+{\rm i}\,\psi_1|^2+a_{12}+\exp(\chi_2),$$

where

$$a_{12}=\frac{2a^2}{\cosh(2b)-1},$$
$$\xi_1=2a\cosh(b)x+2an,$$
$$\psi_1=2a\sinh(b)y,$$
$$\chi_2=2\sinh(b)x+2bn+\overline{\beta_3^0}.$$

It should be noted that $a_{12}>0$, so the resonant solution (54) is nonsingular.

We calculate the maximum amplitude point and two minima amplitude points of the lump as follows:

$$(x_1(n),y_1(n))=\left(-\frac{1}{\cosh(b)}n,0\right),$$
$$(x_2(n),y_2(n))=\left(-\frac{1}{\cosh(b)}n-\frac{\sqrt{4a^2+3a_{12}}}{2a\cosh(b)},0\right),$$
$$(x_3(n),y_3(n))=\left(-\frac{1}{\cosh(b)}n+\frac{\sqrt{4a^2+3a_{12}}}{2a\cosh(b)},0\right).$$

By introducing these points into the resonant solution (54), we obtain the maximum amplitude of the lump is

$$u_1=\frac{(4a^2+a_{12})^2+(4a^2+a_{12})\exp(\lambda_3t+\lambda_3+\overline{\beta_3^0})+(4a^2+a_{12})\exp(\lambda_3t-\lambda_3+\overline{\beta_3^0})+\exp(2\lambda_3t+2\overline{\beta_3^0})}{(a_{12}+\exp(\lambda_3t+\overline{\beta_3^0}))^2}$$

and two minima amplitudes of the lump are

$$u_2=\frac{\zeta_1+\zeta_2\exp(\lambda_3t+\lambda_3+\varphi)+\zeta_3\exp(\lambda_3t-\lambda_3+\varphi)+\exp(2\lambda_3t+2\varphi)}{(4a^2+4a_{12}+\exp(\lambda_3t+\varphi))^2},$$
$$u_3=\frac{\zeta_1+\zeta_3\exp(\lambda_3t+\lambda_3+\varphi')+\zeta_2\exp((\lambda_3t-\lambda_3+\varphi')+\exp(2\lambda_3t+2\varphi')}{(4a^2+4a_{12}+\exp(\lambda_3t+\varphi'))^2},$$

where

$$\zeta_1=16a_{12}^2+16a^2a_{12},\ \ \zeta_2=8a^2+3a_{12}+4a\sqrt{4a^2+3a_{12}},\ \ \zeta_3=8a^2+3a_{12}-4a\sqrt{4a^2+3a_{12}},$$
$$\lambda_3=2b-2\tanh(b),\ \ \varphi=-\frac{\sqrt{4a^2+3a_{12}}\tanh(b)}{a}+\overline{\beta_3^0},\ \ \varphi'=\frac{\sqrt{4a^2+3a_{12}}\tanh(b)}{a}+\overline{\beta_3^0}.$$

According to the sign of λ_3, the resonant interaction can be classified into two distinct cases.

(i) Fusion When $\lambda_3 > 0$, $(b > 0)$, we obtain

$$u_1 \to \begin{cases} \left(\dfrac{4a^2+a_{12}}{a_{12}}\right)^2, & n \to -\infty, \\ 1, & n \to +\infty, \end{cases} \tag{55}$$

$$u_2 \to \begin{cases} \dfrac{16a_{12}^2+16a^2a_{12}}{(4a^2+4a_{12})^2}, & n \to -\infty, \\ 1, & n \to +\infty, \end{cases} \tag{56}$$

$$u_3 \to \begin{cases} \dfrac{16a_{12}^2+16a^2a_{12}}{(4a^2+4a_{12})^2}, & n \to -\infty, \\ 1, & n \to +\infty. \end{cases} \tag{57}$$

From Eqs. (55)–(57), we derive that the lump only exists before the resonant interaction. Figure 6 shows that the lump fuses into the line soliton.

(ii) Fission When $\lambda_3 < 0$, $(b < 0)$, we obtain

$$u_1 \to \begin{cases} 1, & n \to -\infty, \\ \left(\dfrac{4a^2+a_{12}}{a_{12}}\right)^2, & n \to +\infty, \end{cases} \tag{58}$$

$$u_2 \to \begin{cases} 1, & n \to -\infty, \\ \dfrac{16a_{12}^2+16a^2a_{12}}{(4a^2+4a_{12})^2}, & n \to +\infty, \end{cases} \tag{59}$$

$$u_3 \to \begin{cases} 1, & n \to -\infty, \\ \dfrac{16a_{12}^2+16a^2a_{12}}{(4a^2+4a_{12})^2}, & n \to +\infty. \end{cases} \tag{60}$$

From Eqs. (58)–(60), we derive that the lump only exists after the resonant interaction. Figure 7 shows that the lump detaches from the line soliton.

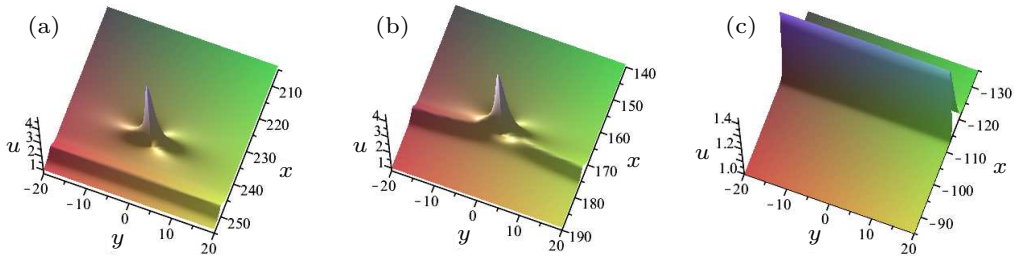


Fig. 6. Fusion of a lump and a line soliton with the parameter $a = 2.5$, $b = 0.6$, $\overline{\beta_3^0} = 28$. (a) $n = -280$, (b) $n = -200$, and (c) $n = 110$.

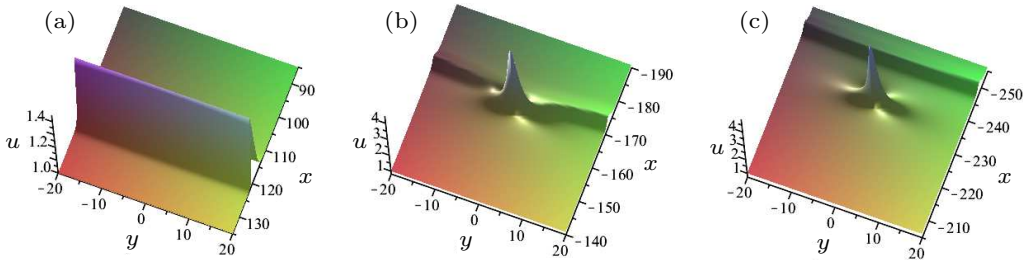


Fig. 7. Fission of a lump and a line soliton with the parameter $a = 2.5$, $b = -0.6$, $\overline{\beta_3^0} = 28$. (a) $n = -110$, (b) $n = 200$, and (c) $n = 280$.

4.2. Resonant interaction between a lump and two line solitons

If the N-soliton solution (5) with $N = 4$ satisfies the partial long wave limit constraints

$$k_1 = K_1\varepsilon, \quad k_2 = K_2\varepsilon, \quad \exp(\beta_1^0) = \exp(\beta_2^0) = -1,$$
$$K_1 = K_2^*, \quad l_1 = l_2^*, \quad \varepsilon \to 0,$$

we obtain the hybrid solution consisting of a lump and two line solitons

$$u(n) = \frac{f(n+1)f(n-1)}{f(n)^2}, \tag{61}$$

with

$$\begin{aligned} f(n) = {} & \exp(A_{34}+\eta_3+\eta_4)(a_{13}a_{23}+a_{13}a_{24}+a_{14}a_{23} \\ & +a_{14}a_{24}+a_{13}\theta_2+a_{14}\theta_2+a_{23}\theta_1+a_{24}\theta_1+\theta_1\theta_2 \\ & +a_{12})+(a_{13}a_{23}+a_{13}\theta_2+a_{23}\theta_1+\theta_1\theta_2+a_{12})\exp(\eta_3) \\ & +(a_{14}a_{24}+a_{14}\theta_2+a_{24}\theta_1+\theta_1\theta_2+a_{12})\exp(\eta_4) \\ & +\theta_1\theta_2+a_{12}, \end{aligned}$$

in which

$$a_{12} = \frac{2K_1K_2}{\cos(l_1-l_2)-1}, \quad a_{ij} = \frac{2K_i\sinh(k_j)}{\cos(l_i-l_j)-\cosh(k_j)},$$
$$(i = 1,2,\ j = 3,4), \quad \exp(A_{34}) = \frac{\cos(l_3-l_4)-\cosh(k_3-k_4)}{\cos(l_3-l_4)-\cosh(k_3+k_4)},$$
$$\theta_i = 2K_i\cos(l_i)x+2K_i\sin(l_i)y+2K_in, \quad (i = 1,2),$$
$$\eta_j = 2\sinh(k_j)\cos(l_j)x+2\sinh(k_j)\sin(l_j)y+2k_jn+\beta_j^0,$$
$$(j = 3,4),$$

where β_j^0 is an arbitrary real constant.

By taking $K_1 = 1$, $K_2 = 1$, $k_3 = \alpha_1$, $k_4 = \alpha_2$, $l_1 = b\mathrm{i}$, $l_2 = -b\mathrm{i}$, $l_3 = 0$, and $l_4 = 0$ ($\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $b \neq 0$) in the Eq. (61), we obtain a hybrid solution consisting of a lump and

two line solitons arranged in the y-direction

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2}, \tag{62}$$

with

$$\begin{aligned} f(n) \;=\; & |\xi_1+\mathrm{i}\psi_1|^2+a_{12}+\exp(\eta_3)\left(|\xi_1+\mathrm{i}\psi_1+a_{13}|^2+a_{12}\right)\\ & +\exp(\eta_4)\left(|\xi_1+\mathrm{i}\psi_1+a_{14}|^2+a_{12}\right)\\ & +\exp(\eta_3+\eta_4+A_{34})\left(|\xi_1+\mathrm{i}\psi_1+a_{13}+a_{14}|^2+a_{12}\right), \end{aligned}$$

in which

$$a_{12}=\frac{1}{\sinh^2(b)},\quad a_{13}=\frac{2\sinh(\alpha_1)}{\cosh(b)-\cosh(\alpha_1)},$$

$$a_{14}=\frac{2\sinh(\alpha_2)}{\cosh(b)-\cosh(\alpha_2)},\quad \exp(A_{34})=\frac{1-\cosh(\alpha_1-\alpha_2)}{1-\cosh(\alpha_1+\alpha_2)},$$

$$\xi_1=2\cosh(b)x+2n,\quad \psi_1=2\sinh(b)y,$$

$$\eta_3=2\sinh(\alpha_1)x+2\alpha_1 n+\beta_3^0,$$

$$\eta_4=2\sinh(\alpha_2)x+2\alpha_2 n+\beta_4^0,$$

where ξ_1 and ψ_1 are real part and imaginary part of θ_1, respectively. The nonsingular condition of the hybrid solution (62) is $\alpha_1+\alpha_2\neq 0$, $|b|\neq|\alpha_1|$, and $|b|\neq|\alpha_2|$.

The lump moves along $\xi_1\approx 0$ and $\psi_1\approx 0$. As $\eta_3,\eta_4\to-\infty$, We obtain the asymptotic form of the lump denoted as u_{lump} from Eq. (62) as follows:

$$u_{\mathrm{lump}}^{-}\simeq\frac{(\xi_1^2-4a^2)^2+(\psi_1^2+a_{12})(\xi_1^2+4a^2)+2a_{12}\psi_1^2+\psi_1^4+a_{12}^2}{(\xi_1^2+\psi_1^2+a_{12})^2}. \tag{63}$$

The lump is centered at $|\xi_1+\mathrm{i}\psi_1|=0$. As $\eta_3,\eta_4\to+\infty$, we obtain the asymptotic form of the lump as follows:

$$u_{\mathrm{lump}}^{+}\simeq\frac{((\xi_1+a_{13}+a_{14})^2-4a^2)^2+(\psi_1^2+a_{12})((\xi_1+a_{13}+a_{14})^2+4a^2)+2a_{12}\psi_1^2+\psi_1^4+a_{12}^2}{((\xi_1+a_{13}+a_{14})^2+\psi_1^2+a_{12})^2}. \tag{64}$$

The lump is centered at $|\xi_1+\mathrm{i}\psi_1+a_{13}+a_{14}|=0$. From the above analysis, we derive the phase shift of the lump before and after the interaction is $a_{13}+a_{14}$.

Now we investigate the resonant solution consisting of a lump and two line solitons. If a_{13}, $a_{14}\to\infty$, the phase shift of the lump before and after the interaction will become infinite. To this end, we take the phase parameter β_j^0 $(j=3,4)$ as $\beta_j^0=\overline{\beta_j^0}-\ln(|a_{1j}|^2)$, and take the limits $b\to p$, $\alpha_1\to p$, $\alpha_2\to -p$. We obtain the resonant solution as follows:

$$u(n)=\frac{f(n+1)f(n-1)}{f(n)^2}, \tag{65}$$

in which

$$\begin{aligned} f(n) \;=\; & |\xi_1+\mathrm{i}\psi_1|^2+a_{12}+\exp(\chi+\overline{\beta_3^0})+\exp(-\chi+\overline{\beta_4^0})\\ & +\sinh^2(p)\exp(\overline{\beta_3^0}+\overline{\beta_4^0}), \end{aligned} \tag{66}$$

where

$$a_{12}=\frac{1}{\sinh^2(p)},\quad \xi_1=2\cosh(p)x+2n,$$

$$\psi_1=2\sinh(p)y,\quad \chi=2\sinh(p)x+2pn.$$

It should be noted that $a_{12}>0$, so the resonant solution (65) is nonsingular.

The trajectories of two line solitons can be calculated from Eq. (65). Line soliton 1 moves along

$$L_1=0,$$

and the line soliton 2 moves along

$$L_2=0,$$

where

$$L_1=\chi-\overline{\beta_4^0}+L,\quad L_2=\chi+\overline{\beta_3^0}-L,$$

in which

$$L=\ln(|\xi_1+\mathrm{i}\psi_1|^2+a_{12}+\sinh^2(p)\exp(\overline{\beta_3^0}+\overline{\beta_4^0})).$$

The line soliton 1 and the line soliton 2 can be expressed as follows:

$$u_i(n)=\frac{f_i(n+1)f_i(n-1)}{f_i(n)^2}, \tag{67}$$

with

$$f_i(n)=1+\exp(L_i),\quad (i=1,2),$$

where subscript i $(i=1,2)$ denotes the line soliton i $(i=1,2)$.

For the asymptotic analysis of the lump, we have $\xi_1\approx 0$ and $\chi=(2p-2\sinh(p)/\cosh(p))n$, thus $|\chi|\to\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the lump from Eq. (65) as follows:

$$u_{\mathrm{lump}}^{\mp}\simeq\begin{cases}1, & n\to-\infty,\\ 1, & n\to+\infty.\end{cases} \tag{68}$$

For the asymptotic analysis of the line soliton 1, we have $L_1\approx 0$. We rewrite Eq. (66) in the following form:

$$f=\exp(-\chi+\beta_4^0)\left(1+\exp(L_1)+\frac{\exp(2L_1+\overline{\beta_3^0}+\overline{\beta_4^0})}{\exp(2L)}\right),$$

and $\exp(2L)\to+\infty$ as $n\to\mp\infty$. We obtain the asymptotic form of the line soliton 1 denoted as u_{line_1} as follows:

$$u_{\mathrm{line}_1}^{\mp}\simeq u_1(n),\quad n\to\mp\infty. \tag{69}$$

For the asymptotic analysis of the line soliton 2, we have $L_2\approx 0$. We rewrite Eq. (66) in the following form:

$$\begin{aligned} f \;=\; & \exp(-\chi+\beta_4^0)\Big(1+\exp(L_2-\overline{\beta_3^0}-\overline{\beta_4^0}+2L)\\ & +\exp(2L_2-\overline{\beta_3^0}-\overline{\beta_4^0}+2L)\Big), \end{aligned}$$

and $\exp(2L) \to +\infty$ as $n \to \mp\infty$, so we obtain the asymptotic form of the line soliton 2 denoted as u_{line_2} as follows:

$$u^{\mp}_{\text{line}_2} \simeq u_2(n), \quad n \to \mp\infty. \tag{70}$$

From above asymptotic analysis, we derive that the lump does not exist before and after the resonant interaction, and the line solitons do not change their velocities and shapes before and after the interaction. The process of the resonant interaction between a lump and two solitons is shown in Fig. 8. The lump separates from one line soliton and fuses into the other line soliton. In the whole process, the lump is localized in both time and space, which exhibits rogue lump phenomenon.

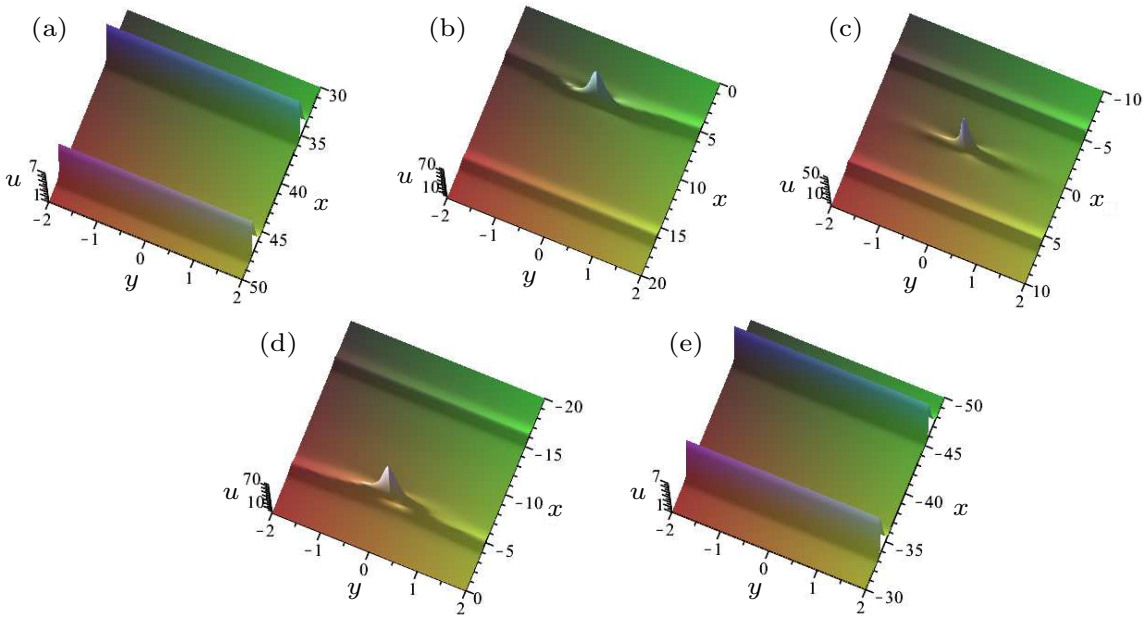


Fig. 8. Resonant interaction between a lump and two line solitons with the parameter $p = 1.6$, $\overline{\beta_3^0} = -20$, $\overline{\beta_4^0} = -20$. (a) $n = -60$, (b) $n = -15$, (c) $n = 0$, (d) $n = 15$, (e) $n = 60$.

5. Molecules between lump, breathers and line solitons

In this section, we investigate the lump–soliton molecule, the lump–breather molecule, the lump–breather–soliton molecule and the lump–breather–breather molecule for the $(2+1)$-dimensional elliptic Toda equation by utilizing the velocity resonance method.

Case 1 From the hybrid solution (46) composed of a lump and a line soliton, we obtain the velocities of the lump and the line soliton as follows:

$$V_{\text{lump}} = \left(\frac{\sin(l_1)-\sin(l_2)}{\sin(l_2-l_1)}, \frac{\cos(l_1)-\cos(l_2)}{\sin(l_1-l_2)}\right)^{\mathrm{T}},$$

$$V_{\text{line}} = \left(-\frac{k_3\cos(l_3)}{\sinh(k_3)}, -\frac{k_3\sin(l_3)}{\sinh(k_3)}\right)^{\mathrm{T}}.$$

Utilizing velocity resonance condition

$$V_{\text{lump}} = V_{\text{line}},$$

we derive a lump–line molecule which is shown in Fig. 9 by taking the parameters $K_1 = 1+0.6\mathrm{i}$, $K_2 = 1-0.6\mathrm{i}$, $k_3 = 0.5$, $l_1 = 0.8 + \operatorname{arccosh}(\frac{\sinh(0.5)}{0.5})\mathrm{i}$, $l_2 = 0.8 - \operatorname{arccosh}(\frac{\sinh(0.5)}{0.5})\mathrm{i}$, $l_3 = 0.8$, and $\beta_3^0 = 20$.

Case 2 From the hybrid solution (61) composed of a lump and a breather, we obtain the velocities of the lump and the breather as follows:

$$V_{\text{lump}} = \left(\frac{\sin(l_1)-\sin(l_2)}{\sin(l_2-l_1)}, \frac{\cos(l_1)-\cos(l_2)}{\sin(l_1-l_2)}\right)^{\mathrm{T}},$$

$$V_{\text{breather}} = \begin{pmatrix} -\frac{\mathrm{Re}(2k_3)\mathrm{Re}(2\sinh(k_3)\cos(l_3))}{\mathrm{Re}^2(2\sinh(k_3)\cos(l_3))+\mathrm{Re}^2(2\sinh(k_3)\sin(l_3))} \\ -\frac{\mathrm{Re}(2k_3)\mathrm{Re}(2\sinh(k_3)\sin(l_3))}{\mathrm{Re}^2(2\sinh(k_3)\cos(l_3))+\mathrm{Re}^2(2\sinh(k_3)\sin(l_3))} \end{pmatrix}.$$

Utilizing velocity resonance condition

$$V_{\text{lump}} = V_{\text{breather}},$$

we derive a lump–breather molecule which is shown in Fig. 10 by taking the parameter $K_1 = 1+0.6\mathrm{i}$, $K_2 = 1-0.6\mathrm{i}$, $k_3 = -0.3+\frac{\pi}{2}\mathrm{i}$, $k_4 = -0.3-\frac{\pi}{2}\mathrm{i}$, $l_1 = \pi + \operatorname{arccosh}(\frac{\cosh(0.3)\sinh(0.5)}{0.3})\mathrm{i}$, $l_2 = \pi - \operatorname{arccosh}(\frac{\cosh(0.3)\sinh(0.5)}{0.3})\mathrm{i}$, $l_3 = \frac{\pi}{2}+0.5\mathrm{i}$, $l_4 = \frac{\pi}{2}-0.5\mathrm{i}$, $\beta_3^0 = 15$, and $\beta_4^0 = 15$.

Case 3 If the N-soliton solution (5) with $N = 5$ satisfies the long wave limit constraints

$$k_1 = K_1\varepsilon, \quad k_2 = K_2\varepsilon, \quad \exp(\beta_1^0) = \exp(\beta_2^0) = -1,$$

$$K_1 = K_2^*, \quad l_1 = l_2^*, \quad k_3 = k_4^*, \quad l_3 = l_4^*, \quad \beta_3^0 = \beta_4^{0*}, \quad \varepsilon \to 0,$$

we obtain the hybrid solution consisting of a lump, a breather and a line soliton

$$u(n) = \frac{f(n+1)f(n-1)}{f(n)^2}, \tag{71}$$

with

$$
\begin{aligned}
f(n) = {} & \mathrm{e}^{A_{34}+A_{35}+A_{45}+\eta_3+\eta_4+\eta_5}(a_{13}a_{23}+a_{13}a_{24}+a_{13}a_{25}+a_{14}a_{23}+a_{14}a_{24}+a_{14}a_{25}+a_{15}a_{23}+a_{15}a_{24}+a_{15}a_{25}\\
&+a_{13}\theta_2+a_{14}\theta_2+a_{15}\theta_2+a_{23}\theta_1+a_{24}\theta_1+a_{25}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{34}+\eta_3+\eta_4}(a_{13}a_{23}+a_{13}a_{24}+a_{14}a_{23}\\
&+a_{14}a_{24}+a_{13}\theta_2+a_{14}\theta_2+a_{23}\theta_1+a_{24}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{35}+\eta_3+\eta_5}(a_{13}a_{23}+a_{13}a_{25}+a_{15}a_{23}\\
&+a_{15}a_{25}+a_{13}\theta_2+a_{15}\theta_2+a_{23}\theta_1+a_{25}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{45}+\eta_4+\eta_5}(a_{14}a_{24}+a_{14}a_{25}+a_{15}a_{24}\\
&+a_{15}a_{25}+a_{14}\theta_2+a_{15}\theta_2+a_{24}\theta_1+a_{25}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{\eta_3}(a_{13}a_{23}+a_{13}\theta_2+a_{23}\theta_1+\theta_1\theta_2+a_{12})\\
&+\mathrm{e}^{\eta_4}(a_{14}a_{24}+a_{14}\theta_2+a_{24}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{\eta_5}(a_{15}a_{25}+a_{15}\theta_2+a_{25}\theta_1+\theta_1\theta_2+a_{12})+\theta_1\theta_2+a_{12},
\end{aligned}
$$

in which

$$
a_{12}=\frac{2K_1K_2}{\cos(l_1-l_2)-1},\quad a_{ij}=\frac{2K_i\sinh(k_j)}{\cos(l_i-l_j)-\cosh(k_j)},
$$

$$
(i=1,2,j=3,4,5),\quad \exp(A_{ij})=\frac{\cos(l_i-l_j)-\cosh(k_i-k_j)}{\cos(l_i-l_j)-\cosh(k_i+k_j)},
$$

$$
(i=3,4,j=4,5,i<j),
$$

$$
\theta_i=2K_i\cos(l_i)x+2K_i\sin(l_i)y+2K_in,\quad (i=1,2),
$$

$$
\eta_j=2\sinh(k_j)\cos(l_j)x+2\sinh(k_j)\sin(l_j)y+2k_jn+\beta_j^0,
$$

$$
(j=3,4,5),
$$

where β_j^0 $(j=3,4,5)$ is an arbitrary real constant. From hybrid solution (71), we obtain the velocities of the lump, the breather and the line soliton as follows:

$$
V_{\mathrm{lump}}=\left(\frac{\sin(l_1)-\sin(l_2)}{\sin(l_2-l_1)},\frac{\cos(l_1)-\cos(l_2)}{\sin(l_1-l_2)}\right)^{\mathrm{T}},
$$

$$
V_{\mathrm{breather}}=\begin{pmatrix}-\frac{\mathrm{Re}(2k_3)\mathrm{Re}(2\sinh(k_3)\cos(l_3))}{\mathrm{Re}^2(2\sinh(k_3)\cos(l_3))+\mathrm{Re}^2(2\sinh(k_3)\sin(l_3))}\\-\frac{\mathrm{Re}(2k_3)\mathrm{Re}(2\sinh(k_3)\sin(l_3))}{\mathrm{Re}^2(2\sinh(k_3)\cos(l_3))+\mathrm{Re}^2(2\sinh(k_3)\sin(l_3))}\end{pmatrix},
$$

$$
V_{\mathrm{line}}=\left(-\frac{k_5\cos(l_5)}{\sinh(k_5)},-\frac{k_5\sin(l_5)}{\sinh(k_5)}\right)^{\mathrm{T}}.
$$

Utilizing velocity resonance condition

$$
V_{\mathrm{lump}}=V_{\mathrm{breather}}=V_{\mathrm{line}},
$$

we derive a lump–breather–line molecule which is shown in Fig. 11 by taking the parameter $K_1=1+0.6\mathrm{i}$, $K_2=1-0.6\mathrm{i}$, $k_3=-0.4789249532+\frac{\pi}{2}\mathrm{i}$, $k_4=-0.4789249532-\frac{\pi}{2}\mathrm{i}$, $k_5=1.6$, $l_1=\pi+\mathrm{arccosh}(\frac{\sinh(1.6)}{1.6})\mathrm{i}$, $l_2=\pi-\mathrm{arccosh}(\frac{\sinh(1.6)}{1.6})\mathrm{i}$, $l_3=\frac{\pi}{2}+0.6\mathrm{i}$, $l_4=\frac{\pi}{2}-0.6\mathrm{i}$, $l_5=\pi$, $\beta_3^0=-30$, $\beta_4^0=-30$, and $\beta_5^0=30$.

Case 4 If the N-soliton solution (5) with $N=6$ satisfies the long wave limit constraints

$$
\begin{aligned}
&k_1=K_1\varepsilon,\ \ k_2=K_2\varepsilon,\ \ \exp(\beta_1^0)=\exp(\beta_2^0)=-1,\\
&K_1=K_2^*,\ \ l_1=l_2^*,\ \ k_3=k_4^*,\ \ l_3=l_4^*,\\
&k_5=k_6^*,\ \ l_5=l_6^*,\ \ \beta_3^0=\beta_4^{0*},\ \ \beta_5^0=\beta_6^{0*},\ \ \varepsilon\to 0,
\end{aligned}
$$

we obtain the hybrid solution consisting of a lump and two breathers

$$
u(n)=\frac{f(n+1)f(n-1)}{f(n)^2},\tag{72}
$$

with

$$
\begin{aligned}
f(n) = {} & \mathrm{e}^{A_{34}+A_{35}+A_{36}+A_{45}+A_{46}+A_{56}+\eta_3+\eta_4+\eta_5+\eta_6}(a_{13}a_{23}+a_{13}a_{24}+a_{13}a_{25}+a_{13}a_{26}+a_{14}a_{23}+a_{14}a_{24}\\
&+a_{14}a_{25}+a_{14}a_{26}+a_{15}a_{23}+a_{15}a_{24}+a_{15}a_{25}+a_{15}a_{26}+a_{16}a_{23}+a_{16}a_{24}+a_{16}a_{25}+a_{16}a_{26}+a_{13}\theta_2\\
&+a_{14}\theta_2+a_{15}\theta_2+a_{16}\theta_2+a_{23}\theta_1+a_{24}\theta_1+a_{25}\theta_1+a_{26}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{34}+A_{35}+A_{45}+\eta_3+\eta_4+\eta_5}\times\\
&(a_{13}a_{23}+a_{13}a_{24}+a_{13}a_{25}+a_{14}a_{23}+a_{14}a_{24}+a_{14}a_{25}+a_{15}a_{23}+a_{15}a_{24}+a_{15}a_{25}+a_{13}\theta_2+a_{14}\theta_2\\
&+a_{15}\theta_2+a_{23}\theta_1+a_{24}\theta_1+a_{25}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{34}+A_{36}+A_{46}+\eta_3+\eta_4+\eta_6}(a_{13}a_{23}+a_{13}a_{24}+a_{13}a_{26}\\
&+a_{14}a_{23}+a_{14}a_{24}+a_{14}a_{26}+a_{16}a_{23}+a_{16}a_{24}+a_{16}a_{26}+a_{13}\theta_2+a_{14}\theta_2+a_{16}\theta_2+a_{23}\theta_1+a_{24}\theta_1\\
&+a_{26}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{35}+A_{36}+A_{56}+\eta_3+\eta_5+\eta_6}(a_{13}a_{23}+a_{13}a_{25}+a_{13}a_{26}+a_{15}a_{23}+a_{15}a_{25}\\
&+a_{15}a_{26}+a_{16}a_{23}+a_{16}a_{25}+a_{16}a_{26}+a_{13}\theta_2+a_{15}\theta_2+a_{16}\theta_2+a_{23}\theta_1+a_{25}\theta_1+a_{26}\theta_1+\theta_1\theta_2+a_{12})\\
&+\mathrm{e}^{A_{45}+A_{46}+A_{56}+\eta_4+\eta_5+\eta_6}(a_{14}a_{24}+a_{14}a_{25}+a_{14}a_{26}+a_{15}a_{24}+a_{15}a_{25}+a_{15}a_{26}+a_{16}a_{24}+a_{16}a_{25}\\
&+a_{16}a_{26}+a_{14}\theta_2+a_{15}\theta_2+a_{16}\theta_2+a_{24}\theta_1+a_{25}\theta_1+a_{26}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{34}+\eta_3+\eta_4}(a_{13}a_{23}\\
&+a_{13}a_{24}+a_{14}a_{23}+a_{14}a_{24}+a_{13}\theta_2+a_{14}\theta_2+a_{23}\theta_1+a_{24}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{35}+\eta_3+\eta_5}(a_{13}a_{23}\\
&+a_{13}a_{25}+a_{15}a_{23}+a_{15}a_{25}+a_{13}\theta_2+a_{15}\theta_2+a_{23}\theta_1+a_{25}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{36}+\eta_3+\eta_6}(a_{13}a_{23}\\
&+a_{13}a_{26}+a_{16}a_{23}+a_{16}a_{26}+a_{13}\theta_2+a_{16}\theta_2+a_{23}\theta_1+a_{26}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{45}+\eta_4+\eta_5}(a_{14}a_{24}\\
&+a_{14}a_{25}+a_{15}a_{24}+a_{15}a_{25}+a_{14}\theta_2+a_{15}\theta_2+a_{24}\theta_1+a_{25}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{46}+\eta_4+\eta_6}(a_{14}a_{24}\\
&+a_{14}a_{26}+a_{16}a_{24}+a_{16}a_{26}+a_{14}\theta_2+a_{16}\theta_2+a_{24}\theta_1+a_{26}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{A_{56}+\eta_5+\eta_6}(a_{15}a_{25}
\end{aligned}
$$

$$+a_{15}a_{26}+a_{16}a_{25}+a_{16}a_{26}+a_{15}\theta_2+a_{16}\theta_2+a_{25}\theta_1+a_{26}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{\eta_3}(a_{13}a_{23}+a_{13}\theta_2$$
$$+a_{23}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{\eta_4}(a_{14}a_{24}+a_{14}\theta_2+a_{24}\theta_1+\theta_1\theta_2+a_{12})+\mathrm{e}^{\eta_5}(a_{15}a_{25}+a_{15}\theta_2+a_{25}\theta_1$$
$$+\theta_1\theta_2+a_{12})+\mathrm{e}^{\eta_6}(a_{16}a_{26}+a_{16}\theta_2+a_{26}\theta_1+\theta_1\theta_2+a_{12})+\theta_1\theta_2+a_{12},$$

in which

$$a_{12}=\frac{2K_1K_2}{\cos(l_1-l_2)-1},\quad a_{ij}=\frac{2K_i\sinh(k_j)}{\cos(l_i-l_j)-\cosh(k_j)},$$
$$(i=1,2,j=3,4,5,6),$$
$$\exp(A_{ij})=\frac{\cos(l_i-l_j)-\cosh(k_i-k_j)}{\cos(l_i-l_j)-\cosh(k_i+k_j)},$$
$$(i=3,4,5,j=4,5,6,i<j),$$
$$\theta_i=2K_i\cos(l_i)x+2K_i\sin(l_i)y+2K_in,\quad(i=1,2),$$
$$\eta_j=2\sinh(k_j)\cos(l_j)x+2\sinh(k_j)\sin(l_j)y+2k_jn+\beta_j^0,$$
$$(j=3,4,5,6),$$

where β_j^0 (j = 3,4,5,6) is an arbitrary real constant. From hybrid solution (72), we obtain the velocities of the lump and two breathers as follows:

$$V_{\text{lump}}=\left(\frac{\sin(l_1)-\sin(l_2)}{\sin(l_2-l_1)},\frac{\cos(l_1)-\cos(l_2)}{\sin(l_1-l_2)}\right)^{\mathrm{T}},$$

$$V_{\text{breather1}}=\begin{pmatrix}-\frac{\mathrm{Re}(2k_3)\mathrm{Re}(2\sinh(k_3)\cos(l_3))}{\mathrm{Re}^2(2\sinh(k_3)\cos(l_3))+\mathrm{Re}^2(2\sinh(k_3)\sin(l_3))}\\-\frac{\mathrm{Re}(2k_3)\mathrm{Re}(2\sinh(k_3)\sin(l_3))}{\mathrm{Re}^2(2\sinh(k_3)\cos(l_3))+\mathrm{Re}^2(2\sinh(k_3)\sin(l_3))}\end{pmatrix},$$

$$V_{\text{breather2}}=\begin{pmatrix}-\frac{\mathrm{Re}(2k_5)\mathrm{Re}(2\sinh(k_5)\cos(l_5))}{\mathrm{Re}^2(2\sinh(k_5)\cos(l_5))+\mathrm{Re}^2(2\sinh(k_5)\sin(l_5))}\\-\frac{\mathrm{Re}(2k_5)\mathrm{Re}(2\sinh(k_5)\sin(l_5))}{\mathrm{Re}^2(2\sinh(k_5)\cos(l_5))+\mathrm{Re}^2(2\sinh(k_5)\sin(l_5))}\end{pmatrix}.$$

Utilizing velocity resonance condition

$$V_{\text{lump}}=V_{\text{breather1}}=V_{\text{breather2}},$$

we derive a lump–breather–breather molecule which is shown in Fig. 12 by taking the parameter $K_1=1+0.8\mathrm{i}$, $K_2=1-0.8\mathrm{i}$, $k_3=-0.5128078901+\frac{\pi}{2}\mathrm{i}$, $k_4=-0.5128078901-\frac{\pi}{2}\mathrm{i}$, $k_5=-0.4+\frac{3\pi}{2}\mathrm{i}$, $k_6=-0.4-\frac{3\pi}{2}\mathrm{i}$, $l_1=\pi+\operatorname{arccosh}(\frac{\cosh(0.5128078901)\sinh(0.6)}{0.5128078901})\mathrm{i}$, $l_2=\pi-\operatorname{arccosh}(\frac{\cosh(0.5128078901)\sinh(0.6)}{0.5128078901})\mathrm{i}$, $l_3=\frac{\pi}{2}+0.6\mathrm{i}$, $l_4=\frac{\pi}{2}-0.6\mathrm{i}$, $l_5=\frac{3\pi}{2}+0.5\mathrm{i}$, $l_6=\frac{3\pi}{2}-0.5\mathrm{i}$, $\beta_3^0=-20$, $\beta_4^0=-20$, $\beta_5^0=20$, and $\beta_6^0=20$.

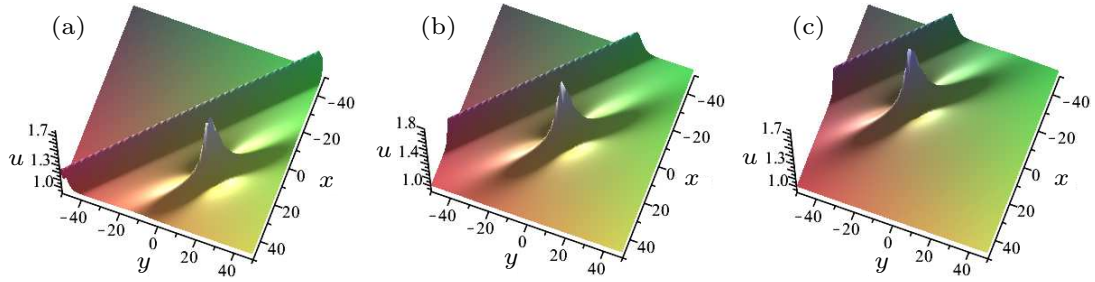


Fig. 9. Lump–line molecule. (a) $n=-20$, (b) $n=0$, and (c) $n=20$.

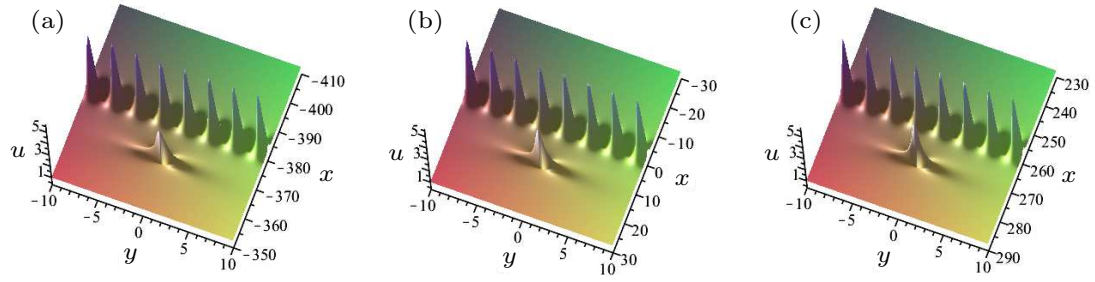


Fig. 10. Lump–breather molecule. (a) $n=-666$, (b) $n=24$, and (c) $n=496$.

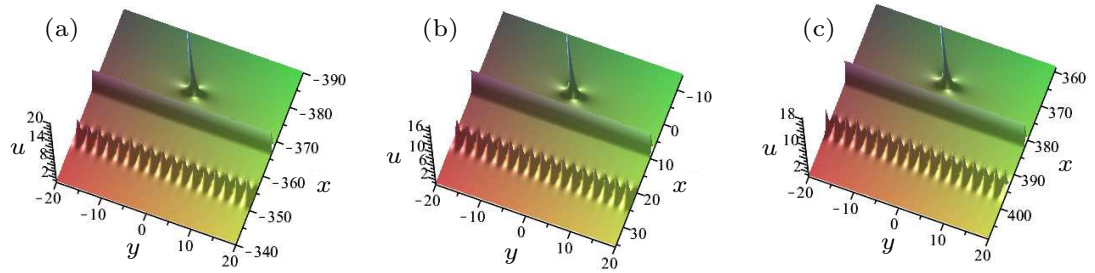


Fig. 11. Lump–breather–line molecule. (a) $n=-555$, (b) $n=3$, and (c) $n=556$.

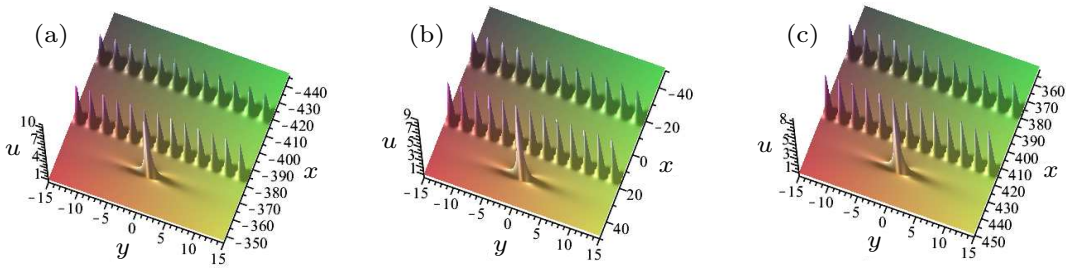


Fig. 12. Lump–breather–breather molecule. (a) $n = -560$, (b) $n = 0$, and (c) $n = 560$.

6. Conclusion

By introducing different parameter constraints and utilizing the partial long wave limits on the N-soliton solution, we obtain various hybrid solutions for the $(2+1)$-dimensional elliptic Toda equation. Based on these hybrid solutions, we demonstrate different interactions that exhibit interesting physical phenomena. When the phase shifts of solitary waves before and after the interaction are infinite, the elastic interaction becomes a resonant interaction. The resonant interaction between two breathers and the resonant interactions between a breather/lump and line solitons exhibit the fusion, fission, time-localized breather and rogue lump phenomena. When the interactions between the solitary waves make each other moves at the same speed, the molecular structure is formed between the solitary waves. We derive the lump–soliton molecule, the lump–breather molecule, the lump–soliton–breather molecule and the lump–breather–breather molecule for the $(2+1)$-dimensional elliptic Toda equation through the velocity resonance method. Our work on the diverse interactions among nonlinear localized waves for the $(2+1)$-dimensional elliptic Toda equation contributes to the study of complicated nonlinear wave phenomena for the $(2+1)$-dimensional discrete nonlinear wave equations.

A natural extension of the present work may be to investigate the resonant interaction between the lumps and breathers as well as the resonant interactions among the line solitons, breathers, lumps for the $(2+1)$-dimensional elliptic Toda equation, which may exhibit various interesting physical phenomena. Recently, the resonant interactions between the lumps and homoclinic orbits for some continuous $(2+1)$-dimensional integrable equation have been investigated.[60–62] It is interesting for us to study the resonant interaction between the lumps and homoclinic orbits for the $(2+1)$-dimensional elliptic Toda equation. Furthermore, pattern formation in higher-order lumps of the Kadomtsev–Petviashvili I equation at large time is investigated in Ref. [63]. We are also interested in studying the patterns of higher-order lumps for the $(2+1)$-dimensional elliptic Toda equation in future work.

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