Developing improved measures of non-Gaussianity and Gaussianity for quantum states based on normalized Hilbert–Schmidt distance

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Non-Gaussianity of quantum states is a very important source for quantum information technology and can be quantified by using the known squared Hilbert–Schmidt distance recently introduced by Genoni *et al.* (*Phys. Rev. A* **78** 042327 (2007)). It is, however, shown that such a measure has many imperfects such as the lack of the swapping symmetry and the ineffectiveness evaluation of even Schrödinger-cat-like states with small amplitudes. To deal with these difficulties, we propose an improved measure of non-Gaussianity for quantum states and discuss its properties in detail. We then exploit this improved measure to evaluate the non-Gaussianities of some relevant single-mode non-Gaussian states and multi-mode non-Gaussian entangled states. These results show that our measure is reliable. We also introduce a modified measure for Gaussianity following Mandilara and Cerf (*Phys. Rev. A* **86** 030102(R) (2012)) and establish a conservation relation of non-Gaussianity and Gaussianity of a quantum state.

Keywords: non-Gaussianity measure, non-Gaussian states, phase-space distribution function

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1. Introduction

In quantum optics communities, non-Gaussian states have been highlighted for decades, in large part because of their novel properties such as photon antibunching, squeezing effects, sub-Poissonian, and negative Wigner function. In particular, they have been long recognized as a precious resource for improving the performances of quantum information protocols including quantum teleportation,[1] quantum cloning,[2] entanglement distillation,[3] quantum secure communication,[4] and estimation problem.[5] However, compared to their Gaussian counterparts, non-Gaussian states are difficult to characterize mathematically since they possess not only the first- and second-order moments but also the higher-order moments. This nontrivial property that a quantum state deviates from Gaussian one is called the state's non-Gaussianity. Thus, a natural problem is how to quantify the degree of non-Gaussianity of a quantum state in a clear-cut universal approach.

Concerning this question, several non-Gaussianity measures have been introduced over the past two decades for different purposes. Genoni *et al.* first introduced the non-Gaussianity measures for a quantum state based on the Hilbert–Schmidt distance[6] and the quantum relative entropy.[7] They exploited these two measures to assess the non-Gaussianity of some relevant quantum states and found that such measures capture the same qualitative non-Gaussian behavior. However, these two measures are difficult to evaluate the non-Gaussianity of even Schrödinger-cat-like states (SCLSs) with the low values of the amplitude.[6–8] Soon afterwards, Marian *et al.* proposed another non-Gaussianity measure of one-mode states based on the quantum fidelity.[9] They found a remarkable consistency between this measure and the two previous measures for one-mode *m*-photon-added thermal states (mPATSs). In 2012, Ivan *et al.* suggested the Wehrl entropy as a measure of non-Gaussianity for quantum states,[10] by which they computed the non-Gaussianity of three classes of quantum states: photon number states, photon-added thermal states, and phase-averaged coherent states. Surprisedly, the non-Gaussianity of photon-added thermal states is demonstrated to be relatively insensitive to the temperature and is exactly equal to that of photon number states. But it is not case for other non-Gaussianity measures, which have shown that the non-Gaussianity of such states depends strongly on the thermal parameter.[11] Quite recently, Fu *et al.* presented a novel approach to quantify the non-Gaussianity of bosonic fields by using the uncertainty relation.[12] Their motivation comes from the fact that Gaussian states are the minimum uncertainty states of an informational uncertainty relation involving the Wigner–Yanase skew information.[13] However, in order to obtain the minimum uncertainty state, one will face the optimization of the conjugate rotated quadrature operator of single-mode bosonic field over the rotation angle, which is challenging for multi-mode case. In a word, at present, there is still much room and necessity for further improvement in quantifying non-Gaussianity of a quantum state.

From the viewpoint of measurement theory, a friend

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measure of non-Gaussianity should possess some beneficial properties: non-negativity, triangular inequality, symmetry, bounds, normalization, and physical implementation,[14,15] by which we return to the examination of the above-mentioned measures. Firstly, all of them are not a reasonably bounded and normalizable measure except the fidelity-based non-Gaussianity measure.[9] For example, for Fock states with infinite number of photons, one can see that the maximum value of non-Gaussianity based on the Hilbert–Schmidt distance is given by $1/2$,[6] while the non-Gaussianity based on quantum relative entropy,[7] Wehrl entropy,[10] and uncertainty relation[12] is boundless and tends to infinity, restricting us to gauge experimentally non-Gaussianity of quantum states. Secondly, we notice that the existing measures do not satisfy the exchange symmetry under the permutation of two quantum states.[8,10,12] Thirdly, although the fidelity-based non-Gaussianity is bounded and symmetric, the Uhlmann–Jozsa fidelity for multi-mode non-Gaussian states is very difficult to be calculated for a mixed quantum state. If quantum fidelity is replaced by the super-fidelity, it leads to have unclear physical meaning.[9,11,16] In view of these difficulties, in the present paper we shall modify the non-Gaussianity measure introduced by Genion *et al.* in Ref. [6] to become a welcome measure. In addition, for the integrity of the discussion we shall discuss the Gaussian properties of a quantum state and present an improved Gaussianity measure based on the Mandilara's Gaussianity measure.[17]

Our paper is structured as follows. In Section 2 we shall reexamine the problems involved in quantifying non-Gaussianity of quantum states based on the Hilbert–Schmidt distance and analyze its drawbacks. We put forward a modified measure of non-Gaussianity and study a set of its basic properties. Simultaneously, we develop a direct method for constructing the reference Gaussian state of a given quantum state by means of phase-space theory. In Section 3 we evaluate the non-Gaussianity of several families of non-Gaussian quantum states including single-mode Schrödinger-cat-like states, single-mode statistical mixture of vacuum and Fock states (SMVFSs), and multi-mode non-Gaussian entangled states by using our measure. Our examples show that this non-Gaussianity measure is exact and powerful. In Section 4 we introduce an improved Gaussianity measure of a quantum state and present a brief analysis on state's non-Gaussianity versus Gaussianity. In Section 5 we give a summary for our main results.

2. Non-Gaussianity measure: improvement and properties

In this section we begin to review the definition of non-Gaussianity measure based on the Hilbert–Schmidt distance, by which we discuss the problem in quantifying the non-Gaussianity of single-mode Schrödinger-cat-like states. Then, we propose a modified non-Gaussianity measure and a strategy for choosing the reference Gaussian state from a given quantum state. We also study the relevant properties of our non-Gaussianity measure. Finally, we briefly make a comparison between our non-Gaussianity measure and Genoni's non-Gaussianity one.

2.1. Review

In 2007, Genoni *et al.* first introduced a measure of quantifying non-Gaussian amount of a quantum state based on the Hilbert–Schmidt distance,[6] which is defined as

$$\delta_{\rm A}[\rho]=\frac{D^2_{\rm HS}[\rho,\rho_{\rm rG}]}{\mu[\rho]}, \tag{1}$$

where $D_{\rm HS}[\rho,\rho_{\rm rG}]$ is the Hilbert–Schmidt distance between a given quantum state ρ and its closest reference Gaussian state $\rho_{\rm rG}$, and the squared Hilbert–Schmidt distance is given by

$$\begin{aligned} D^2_{\rm HS}[\rho,\rho_{\rm rG}] &= \frac{1}{2}{\rm Tr}\left[(\rho-\rho_{\rm rG})^2\right] \\ &= \frac{\mu[\rho]+\mu[\rho_{\rm rG}]-2\kappa[\rho,\rho_{\rm rG}]}{2}, \end{aligned} \tag{2}$$

where the purity of a state ρ is given by $\mu[\rho]={\rm Tr}[\rho^2]$ and the trace overlap between ρ and $\rho_{\rm rG}$ is given by $\kappa[\rho,\rho_{\rm rG}]={\rm tr}[\rho\rho_{\rm rG}]$.

However, we note that such a measure fails to capture the non-Gaussianity of some quantum states. A typical example is single-mode Schrödinger-cat-like states, which are represented by a coherent superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, as follows:[18]

$$|\Psi\rangle_{\rm scs}=\frac{1}{N}\left[\cos(\phi)|\alpha\rangle+\sin(\phi)|-\alpha\rangle\right], \tag{3}$$

where $|\alpha\rangle$ denotes as a coherent state with amplitude $\alpha\in\mathbb{R}$, $N=\sqrt{1+{\rm e}^{-2|\alpha|^2}\sin(2\phi)}$ is a normalization factor, and ϕ is the relative phase. For convenience later, with the relative phase $\phi\in(m\pi,(m+\frac{1}{2})\pi)$, $m\in\mathbb{Z}$, Schrödinger-cat-like states are called even Schrödinger-cat-like states, and that with the relative phase $\phi\in((m+\frac{1}{2})\pi,(m+1)\pi)$, $m\in\mathbb{Z}$, they are called odd Schrödinger-cat-like states. We plot the non-Gaussianity of single-mode SCLSs as a function of phase ϕ and for different values of α, as shown in Fig. 1(a) (also see Fig. 1 in Ref. [6]). It is interesting to see from Fig. 1(a) that the non-Gaussianity measures $\delta_{\rm A}$ is almost zero for even Schrödinger-cat-like states with the low values of the amplitude α. But, we would verify that under the same conditions, such states are non-Gaussian by introducing the phase-averaged fourth-order cumulant, which is seen as a sufficient condition for non-Gaussianity, of the homodyne rotated quadrature operator $\hat{X}(\theta)$[19,20]

$$\kappa_4(\hat{X})=\frac{1}{2\pi}\int_0^{2\pi}\kappa_4(\theta)\,{\rm d}\theta, \tag{4}$$

where the fourth-order cumulant $\kappa_4(\theta)$ is given by

$$\kappa_4(\theta) = \langle \hat{X}_\theta^4 \rangle - 4\langle \hat{X}_\theta^3 \rangle \langle \hat{X}_\theta \rangle - 3\langle \hat{X}_\theta^2 \rangle^2 + 12\langle \hat{X}_\theta^2 \rangle \langle \hat{X}_\theta \rangle^2 - 6\langle \hat{X}_\theta \rangle^4, \tag{5}$$

where the homodyne rotated quadrature operator $\hat{X}_\theta = (\mathrm{e}^{-\mathrm{i}\theta}\hat{a} + \mathrm{e}^{\mathrm{i}\theta}\hat{a}^+)/\sqrt{2}$ with θ being the phase of the local oscillator in homodyne detection arrangement ($\theta \in [0, 2\pi]$) and $\hat{a}$ ($\hat{a}^+$) being the annihilation (creation) operator of the single-mode field. In Fig. 1(b), we plot $\kappa_4(\hat{X})$ as a function of the relative phase ϕ and for different values of α. It is obvious that the $\kappa_4(\hat{X})$ is not equal to zero in the mentioned-above parameters, and then such states are shown to be non-Gaussian, suggesting that we must modify Genoni's non-Gaussianity measure in order to faithfully quantify the tiny non-Gaussian features of a quantum state.

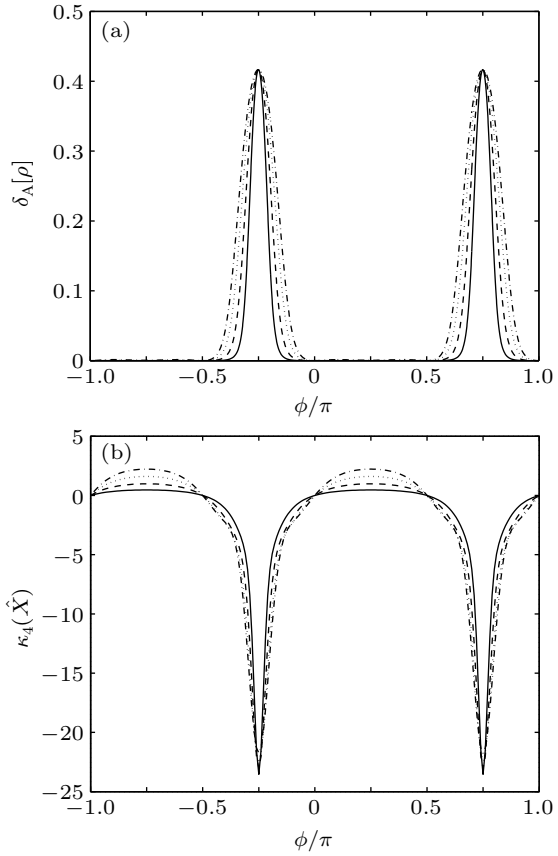


Fig. 1. (a) Non-Gaussianity $\delta_A[\rho]$ and (b) non-Gaussian criterion $\kappa_4(\hat{X})$ for single-mode SCLSs as a function of $\phi(\pi)$ and for different values of the amplitude α. Dot-dashed line, $\alpha = 0.5$; dotted line, $\alpha = 0.4$; dashed line, $\alpha = 0.3$; solid line, $\alpha = 0.2$.

2.2. Improved non-Gaussianity measure and its properties

To achieve our goal, we want to reexamine the Hilbert–Schmidt norm of two density operators ρ_1 and ρ_2, which is defined as[18]

$$d_{\mathrm{HS}}(\rho_1, \rho_2) \equiv \|\rho_1 - \rho_2\|_2 = \sqrt{\langle (\rho_1 - \rho_2)^2 \rangle} = \sqrt{\langle (\rho_1)^2 \rangle + \langle (\rho_2)^2 \rangle - 2\langle \rho_1 \rho_2 \rangle} = \sqrt{\mu[\rho_1] + \mu[\rho_2] - 2\kappa[\rho_1, \rho_2]}. \tag{6}$$

It is easy to verify that one has $d_{\mathrm{HS}}(\rho_1, \rho_2) = 0$ for two identical quantum states, while for two uncorrected quantum states, it is given by $d_{\mathrm{HS}}(\rho_1, \rho_2) = \sqrt{\mu[\rho_1] + \mu[\rho_2]}$ since $\kappa[\rho_1, \rho_2] = 0$. Therefore, we should renormalize the Hilbert–Schmidt norm (6) as

$$d_{\mathrm{NHS}}(\rho_1, \rho_2) = \sqrt{1 - \frac{2\kappa[\rho_1, \rho_2]}{\mu[\rho_1] + \mu[\rho_2]}}. \tag{7}$$

The Hilbert–Schmidt norm has played an important role in quantifying quantum correlations in a physical system. For example, it is used as a measure of non-classicality when ρ_2 being the nearest classical state[21,22] and as a entanglement measure when ρ_2 being the disentangled state.[23] Similarly, if ρ_2 is seen as the closest Gaussian state to ρ_1, then the normalized Hilbert–Schmidt norm will allow us to quantify the non-Gaussianity of quantum state ρ_1. For ease of calculation, we adopt the squared Hilbert–Schmidt norm as non-Gaussianity measure

$$\delta_{\mathrm{X}}[\rho] = d_{\mathrm{NHS}}^2(\rho, \rho_{\mathrm{rG}}) = 1 - \frac{2\kappa[\rho, \rho_{\mathrm{rG}}]}{\mu[\rho] + \mu[\rho_{\mathrm{rG}}]}, \tag{8}$$

where ρ_{rG} denotes a reference Gaussian state. Like other non-Gaussianity measures, our improved measure also encounters an intractable problem of how to construct a reference Gaussian state. As was stated in Ref. [6], the reference Gaussian state can be designed to be the most general Gaussian thermal one with the same first moments and the same covariance matrix as a given state ρ

$$\overline{X}[\rho_{\mathrm{rG}}] = \overline{X}[\rho], \sigma[\rho_{\mathrm{rG}}] = \sigma[\rho]. \tag{9}$$

Thus the purity of reference Gaussian state is given by $1/\prod_{j=1}^{n}(2\overline{n}_j + 1) < 1$ with the average number $\overline{n} = \sum_{j=1}^{n}\langle \hat{a}_j^+ \hat{a}_j \rangle_\rho$. For example, for single-mode continuous-variable system, a reference Gaussian state can be expressed in the form of $\rho_{\mathrm{rG}} = \hat{D}(\alpha)\hat{S}(\xi)\nu(\overline{n})\hat{S}(\xi)^+\hat{D}^+(\alpha)$ with $\hat{D}(\alpha) = \exp[\alpha\hat{a}^+ - \alpha^*\hat{a}]$ being the one-mode displacement operator, $\hat{S}(\xi) = \exp[\frac{1}{2}\xi\hat{a}^{+2} - \frac{1}{2}\xi^*\hat{a}^2]$ being the squeezing operator, $(\alpha, \xi) \in \mathbb{C}$, and $\nu(\overline{n}) = \sum_{l=0}^{\infty}(1+\overline{n})^{-1}[\overline{n}/(1+\overline{n})]^l |l\rangle\langle l|$ being a thermal state with an $\overline{n}$ average number of photon. Yet, such a treatment is not much easily applicable to a n-mode case since the corresponding reference Gaussian state is required to optimize the $n(2n+3)$ independent parameters,[7] which significantly increases the computation works. To overcome this obstacle, we provide a simple and direct strategy for constructing the reference Gaussian state from a given quantum state in the phase space.

According to the theory of phase space, any n-mode quantum state ρ can be faithfully described by its s-ordered characteristic function

$$\chi_s(\eta, \rho) = \mathrm{Tr}[\hat{D}(\eta)\rho] \exp\left(\frac{s}{2}|\eta|^2\right), \tag{10}$$

where $\hat{D}(\boldsymbol{\eta}) = \bigotimes_{k=1}^{n} \hat{D}_k(\eta_k)$ is the n-mode displacement operator and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)^{\mathrm{T}}$ with $\eta_k \in \mathbb{C}$. The parameter s defines different orderings and then different quasi-probability functions, so we can obtain the corresponding quasi-probability distribution function from the Fourier transform of this characteristic function $\chi_s(\boldsymbol{\eta}, \rho)$ as

$$W_s(\boldsymbol{\xi}, \rho) = \int \chi_s(\boldsymbol{\eta}, \rho) \exp\left[\sum_{j=1}^{n} \left(\xi_j \eta_j^* - \xi_j^* \eta_j\right)\right] \frac{\mathrm{d}^2 \boldsymbol{\eta}}{\pi^{2n}}, \quad (11)$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}}$. We see from Eq. (11) that the values $s = 1$, 0, and -1 give the Glauber–Sudarshan P-function, the Wigner function, and the Husimi Q-function, respectively. These functions provide a way to characterize the non-classical nature of a quantum state since these functions can exhibit negative domains. But Gaussian states are an important exception due to the fact that their phase-space distribution functions are positive everywhere.[24] At the same time, they are fully characterized by the mean values d and covariance matrix $\boldsymbol{\sigma}$. Therefore, the construction problem of a reference Gaussian state reduces to finding the mean values and covariance matrix from the considered state ρ. Generally, they can be written as the following two forms in the phase-space.

Form I: The mean value is given by $d_{\mathrm{I}} \equiv \langle \hat{a}_1^+, \hat{a}_1, \ldots, \hat{a}_n^+, \hat{a}_n \rangle$ and covariance matrix is

$$\boldsymbol{V}_{\mathrm{I}} = \begin{pmatrix} \langle \hat{a}_1^+ \hat{a}_1 + \frac{1}{2} \rangle & -\langle \hat{a}_1^2 \rangle & \cdots & \langle \hat{a}_1 \hat{a}_n^+ \rangle & -\langle \hat{a}_1 \hat{a}_n \rangle \\ -\langle \hat{a}_1^{+2} \rangle & \langle \hat{a}_1^+ \hat{a}_1 + \frac{1}{2} \rangle & \cdots & -\langle \hat{a}_1^+ \hat{a}_n^+ \rangle & \langle \hat{a}_1^+ \hat{a}_n \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle \hat{a}_1^+ \hat{a}_n \rangle & -\langle \hat{a}_1 \hat{a}_n \rangle & \cdots & \langle \hat{a}_n^+ \hat{a}_n + \frac{1}{2} \rangle & -\langle \hat{a}_n^2 \rangle \\ -\langle \hat{a}_1^+ \hat{a}_n^+ \rangle & \langle \hat{a}_1 \hat{a}_n^+ \rangle & \cdots & \langle \hat{a}_n^{+2} \rangle & \langle \hat{a}_n^+ \hat{a}_n + \frac{1}{2} \rangle \end{pmatrix}. \quad (12)$$

Form II: The mean value is given by $d_{\mathrm{II}} \equiv \langle \hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n \rangle$ and covariance matrix is

$$\boldsymbol{V}_{\mathrm{II}} = \begin{pmatrix} V_{1,1} & V_{1,2} & \cdots & V_{1,n-1} & V_{1,n} \\ V_{2,1} & V_{2,2} & \cdots & V_{2,n-1} & V_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{n-1,1} & V_{n-1,2} & \cdots & V_{n-1,n-1} & V_{n-1,n} \\ V_{n,1} & V_{n,2} & \cdots & V_{n,n-1} & V_{n,n} \end{pmatrix}, \quad (13)$$

where the matrix elements are given by $V_{l,m} \equiv \frac{1}{2}\left\langle \hat{\boldsymbol{R}}_l \hat{\boldsymbol{R}}_m + \hat{\boldsymbol{R}}_m \hat{\boldsymbol{R}}_l \right\rangle - \left\langle \hat{\boldsymbol{R}}_l \right\rangle \left\langle \hat{\boldsymbol{R}}_m \right\rangle$ with the vector of field quadrature operators $\hat{\boldsymbol{R}} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)$ given by $\hat{x}_j = (\hat{a}_j + \hat{a}_j^+)/\sqrt{2}$ and $\hat{p}_j = (\hat{a}_j - \hat{a}_j^+)/(\mathrm{i}\sqrt{2})$. Hence, once the characteristic function for a quantum state is known, we can calculate the expectation values of the operators $\hat{a}_l^{+n} \hat{a}_j^m$ in Eq. (12) and $\hat{x}_l^n \hat{p}_j^m$ in Eq. (13) as

$$\langle \hat{a}_l^{+n} \hat{a}_j^m \rangle^{(s)} = \frac{(-)^m \partial^{n+m}}{\partial \eta_l^n \partial \eta_j^{*m}} \chi_s(\boldsymbol{\eta}, \rho)\Big|_{\eta_l = \eta_j^* = 0}, \quad (14a)$$

$$\langle \hat{x}_l^n \hat{p}_j^m \rangle^{(s)} = \frac{\partial^{n+m}}{\partial z_l^n \partial w_j^m} \chi_s(\boldsymbol{\eta}, \rho)\Big|_{z_l = w_j = 0}, \quad (14b)$$

where $\eta = w + \mathrm{i}z$.

In what follows, we study how to attain the closet reference Gaussian state, which equivalently reduces to finding the expectation values of the operators (14) of the reference Gaussian state ρ_{rG} from the state ρ. Let $\chi(\rho, \boldsymbol{\eta})$ and $\chi_{\mathrm{rG}}(\rho_{\mathrm{rG}}, \boldsymbol{\eta})$ act as the characteristic functions of the considered state ρ and its corresponding reference Gaussian state ρ_{rG}, respectively. Their difference is denoted by $f(\boldsymbol{\eta}) = \chi(\rho, \boldsymbol{\eta}) - \chi_{\mathrm{rG}}(\rho_{\mathrm{rG}}, \boldsymbol{\eta})$. In order to obtain the same first-order moments and covariance matrix as ρ, we set $\frac{\partial^m}{\partial \boldsymbol{\eta}^m} f(\boldsymbol{\eta})|_{\boldsymbol{\eta}=0} = 0$ with $m \leq 2$, yielding

$$d_{\rho_{\mathrm{rG}}} = d_\rho, \quad (15a)$$

$$\langle \hat{a}_l^{+n} \hat{a}_j^m \rangle_{\rho_{\mathrm{rG}}}^{(s)} = \langle \hat{a}_l^{+n} \hat{a}_j^m \rangle_{\rho}^{(s)}, \quad (15b)$$

$$\langle \hat{x}_l^n \hat{p}_j^m \rangle_{\rho_{\mathrm{rG}}}^{(s)} = \langle \hat{x}_l^n \hat{p}_j^m \rangle_{\rho}^{(s)}. \quad (15c)$$

Moreover, we can construct the characteristic function of the corresponding reference Gaussian state in the form of

$$\chi(\boldsymbol{\eta}, \rho_{\mathrm{rG}}) = \exp\left[-\frac{1}{2}\boldsymbol{\eta} \boldsymbol{V}_{\rho_{\mathrm{rG}}} \boldsymbol{\eta}^{\mathrm{T}} + \mathrm{i}\sqrt{2} d_{\rho_{\mathrm{rG}}} \boldsymbol{\Omega} \boldsymbol{\eta}^{\mathrm{T}}\right], \quad (16)$$

where the symplectic matrix is given by $\boldsymbol{\Omega} = \mathrm{i} \bigotimes_{l=1}^{n} \boldsymbol{\sigma}_2$ with $\boldsymbol{\sigma}_2$ being the y-Pauli matrix.

Thus, we can easily obtain the purity of the reference Gaussian state as

$$\mu[\rho_{\mathrm{rG}}] = \frac{1}{2^n \sqrt{\det \boldsymbol{V}_{\rho_{\mathrm{rG}}}}}, \quad (17)$$

and the trace overlap between ρ and ρ_{rG} is

$$\begin{aligned} \kappa[\rho, \rho_{\mathrm{rG}}] &= \pi^n \int W(\boldsymbol{\xi}, \rho) W(\boldsymbol{\xi}, \rho_{\mathrm{rG}}) \mathrm{d}^{2n} \boldsymbol{\xi} \\ &= \frac{1}{\pi^n} \int \chi(\boldsymbol{\eta}, \rho) \chi(\boldsymbol{\eta}, \rho_{\mathrm{rG}}) \mathrm{d}^{2n} \boldsymbol{\eta}. \end{aligned} \quad (18)$$

Finally, we study some properties of our improved formula (8) as a measure of non-Gaussianity. Obviously, our improved measure naturally satisfies the symmetry under swapping the two states. Apart from this property, the $\delta_{\mathrm{X}}[\rho]$ also possesses the relevant ones summarized by the following lemmas.[11,16,25]

Lemma 1 Boundedness: $0 \leq \delta_{\mathrm{X}}[\rho] \leq 1$ and $\delta_{\mathrm{X}}[\rho] = 0$ if and only if ρ is a Gaussian state.

Proof According to the Cauchy–Schwarz inequality, we easily obtain

$$\kappa[\rho, \rho_{\mathrm{rG}}] = \mathrm{Tr}[\rho_{\mathrm{rG}} \rho] \leq \sqrt{\mu[\rho] \mu[\rho_{\mathrm{rG}}]}, \quad (19)$$

while

$$\mu[\rho] + \mu[\rho_{\mathrm{rG}}] \geq 2\sqrt{\mu[\rho] \mu[\rho_{\mathrm{rG}}]}. \quad (20)$$

So we have

$$0 \leq \frac{2\kappa[\rho, \rho_{\mathrm{rG}}]}{\mu[\rho] + \mu[\rho_{\mathrm{rG}}]} \leq 1. \quad (21)$$

It shows that $\delta_{\rm X}[\rho]$ varies between 0 and 1. If ρ is a Gaussian state, then $\mathrm{Tr}[\rho^2]=\mu[\rho_{\rm rG}]$ and $\kappa[\rho,\rho_{\rm rG}]=\mu[\rho_{\rm rG}]$, which leads to $\delta_{\rm X}[\rho]=0$.

Lemma 2 Invariance under a unitary map $\hat{U}$: $\delta_{\rm X}[\hat{U}\rho\hat{U}^{+}]=\delta_{\rm X}[\rho]$.

Proof Let us consider $\rho'=\hat{U}\rho\hat{U}^{+}$ where $\hat{U}_{\rm G}$ is a unitary map corresponding to a symplectic transformation in phase space and generally is described by $\hat{U}=\exp\left(-\mathrm{i}\hat{H}\right)$ with Hermitian $\hat{H}$ at most bilinear in field operators. After such a map, we have the purity of quantum state ρ' as

$$\mu[\rho']=\mathrm{Tr}[\rho'^2]=\mathrm{Tr}[\hat{U}\rho^2\hat{U}^{+}]=\mu[\rho], \tag{22}$$

where we have used the cyclic property of the trace. It shows that the purity of a quantum state is invariant under unitary map $\hat{U}$.

Similarity, we have

$$\mu[\rho'_{\rm rG}]=\mathrm{Tr}[\rho'^2_{\rm rG}]=\mathrm{Tr}[\hat{U}\rho^2_{\rm rG}\hat{U}^{+}]=\mu[\rho_{\rm rG}], \tag{23a}$$

$$\kappa[\rho',\rho'_{\rm rG}]=\mathrm{Tr}[\rho'\rho'_{\rm rG}]=\kappa[\rho,\rho_{\rm rG}]. \tag{23b}$$

Thus we have

$$\delta_{\rm X}[\rho']=\delta_{\rm X}[\rho]. \tag{24}$$

As is well known, a Gaussian map belongs to one of unitary maps. Therefore, this property ensures that the displacement and squeezing operators do not change the non-Gaussian nature of a quantum state.

Lemma 3 Consider a bipartite factorized state $\rho_{AB}=\rho_A\otimes\rho_B$. If ρ_B is a Gaussian state, then $\delta_{\rm X}[\rho_{AB}]=\delta_{\rm X}[\rho_A]$.

Proof For a bipartite factorized state $\rho_{AB}=\rho_A\otimes\rho_B$, its reference Gaussian state can be written as $\rho_{AB,\rm rG}=\rho_{A,\rm rG}\otimes\rho_{B,\rm rG}$. According to the definition of the purity of a quantum state, we have

$$\mu[\rho_{AB}]=\mu[\rho_A]\mu[\rho_B], \tag{25a}$$

$$\mu[\rho_{AB,\rm rG}]=\mu[\rho_{A,\rm rG}]\mu[\rho_{B,\rm rG}], \tag{25b}$$

$$\kappa[\rho_{AB},\rho_{AB,\rm rG}]=\kappa[\rho_A,\rho_{A,\rm rG}]\kappa[\rho_B,\rho_{B,\rm rG}]. \tag{25c}$$

Therefore, since ρ_B is a Gaussian state, we have $\kappa[\rho_B,\rho_{B,\rm rG}]=\mu[\rho_{B,\rm rG}]=\mu[\rho_B]$. Finally, we get

$$\delta_{\rm X}[\rho_{AB}]=1-\frac{2\kappa[\rho_A,\rho_{A,\rm rG}]\kappa[\rho_B,\rho_{B,\rm rG}]}{\mu[\rho_A]\mu[\rho_B]+\mu[\rho_{A,\rm rG}]\mu[\rho_{B,\rm rG}]}=\delta_{\rm X}[\rho_A]. \tag{26}$$

In addition, we see that the non-Gaussianity $\delta_{\rm X}[\rho]$ is not additive (nor multiplicative) with respect to a tensor product state $\rho=\bigotimes_{j=1}^{\infty}\rho_j$, i.e. $\delta_{\rm X}[\rho]\neq\sum_{j=1}^{\infty}\delta_{\rm X}[\rho_j]$.

2.3. Comparison with Genion's measure

Before ending this section, we return to compare our measure with Genoni's measure.[6] Our non-Gaussianity measure has the following advantages. First, our measure is symmetric and has the limiting value of unity for any quantum state, while for Genoni's measure, different quantum state has different limiting value of non-Gaussianity. For instance, the limiting value for Fock states is $1/2$, whereas for the Bell states $|\phi\rangle=(1/\sqrt{2})(|01\rangle\pm|10\rangle)$, the maximum amount of non-Gaussianity is equal to $2/3$. Second, it can be seen in the next section that our measure can effectively assess the non-Gaussianity of some quantum states that are not quantified by using Genion's measure. Third, in the aspect of choosing reference Gaussian state, our scheme is more simple and direct than Genoni's scheme since it does not need to optimize the parameters of reference Gaussian state such as the average number of photon, squeezing degree, and displace amplitude, thus freeing us from the computational difficulties that the optimization arises, which, in turn, leads to that it is much more convenient to extend to a case of multi-mode continuous-variable states. Finally, although our measure and Genion's measure are based on the same Hilbert–Schmidt distance, the normalization is different from each other. The normalization of Genion's measure is given by $2\mu[\rho]$, whereas it is given by $\mu[\rho]+\mu[\rho_{\rm rG}]$ in our measure $\delta_{\rm X}$. Furthermore, we obtain their ration as

$$\gamma=\frac{\delta_{\rm X}}{\delta_{\rm A}}=\frac{2\mu[\rho]}{\mu[\rho]+\mu[\rho_{\rm rG}]}. \tag{27}$$

Thus, we see that for a pure quantum state, we have $\gamma\geq 1$, namely, $\delta_{\rm X}\geq\delta_{\rm A}$. That is to say, it is underestimated the non-Gaussianity of all pure non-Gaussian states for Genoni's measure, but it is more complicated for a mixed quantum state since such a ratio may be less or greater than one,[26] which will be illustrated in the following section.

3. Examples

In the preceding section we have proposed an improved version of non-Gaussianity measure and a strategy for choosing the closet reference Gaussian state. Now we use this measure to evaluate the non-Gaussianity of several important classes of non-Gaussian states, namely, single-mode Schrödinger-cat-like states, single-mode statistical mixture of vacuum and Fock states, single-mode m-photon-added thermal states, multi-mode entangled-coherent-like states, and multi-mode photon-number entangled states (PNESs). At the same time, we compare our non-Gaussianity measure with Genoni's non-Gaussianity measure using the parameter γ.

3.1. Schrödinger-cat-like states

At first, we come back to the case of single-mode Schrödinger-cat-like states (3) and calculate their non-Gaussianity amount. Repeated the same steps above and after

some algebra, we can obtain the purity of the reference Gaussian state as

$$\mu[\rho_{\rm rG}] = \frac{N^3}{\sqrt{G_1(\alpha,\phi)G_2(\alpha,\phi)}}, \tag{28}$$

where $G_1(\alpha,\phi) = 2\alpha^2\cos^2(2\phi) - 2\alpha^2N^2 - N^4$ and $G_2(\alpha,\phi) = 2\alpha^2\,\mathrm{e}^{-2\alpha^2}\sin(2\phi) - N^2$, and the overlap between ρ in Eq. (3) and its reference Gaussian state $\rho_{\rm rG}$ is

$$\begin{aligned}\kappa[\rho,\rho_{\rm rG}] = \frac{1}{N^2\mu[\rho_{\rm rG}]}\Bigg\{&\cos^2(\phi)\exp\left[-\frac{(q-p)^2}{A-2B}\right]\\ &+\sin^2(\phi)\exp\left[-\frac{(q+p)^2}{A-2B}\right]+\frac{1}{2}\mathrm{e}^{-2\alpha^2}\sin(2\phi)\\ &\times\exp\left[-\frac{q^2(A-2B)-p^2(A+2B)}{A^2-4B^2}\right]\Bigg\},\end{aligned} \tag{29}$$

where $q=\alpha/\sqrt{2}$, $p=\alpha\cos(2\phi)/\sqrt{2}N^2$, and

$$A = 1+\frac{\alpha^2[1-\sin(2\phi)\,\mathrm{e}^{-2\alpha^2}]}{N^2}-\frac{\alpha^2\cos^2(2\phi)}{N^4}, \tag{30a}$$

$$B = \frac{\alpha^2[1+\sin(2\phi)]\,\mathrm{e}^{-2a^2}}{2N^2}-\frac{\alpha^2\cos^2(2\phi)}{2N^4}. \tag{30b}$$

With the derivations above, we can now proceed to numerically evaluate the non-Gaussianity of single-mode Schrödinger-cat-like states (3) using the formula (9) and study the non-Gaussian ration (27), as shown in Fig. 2.

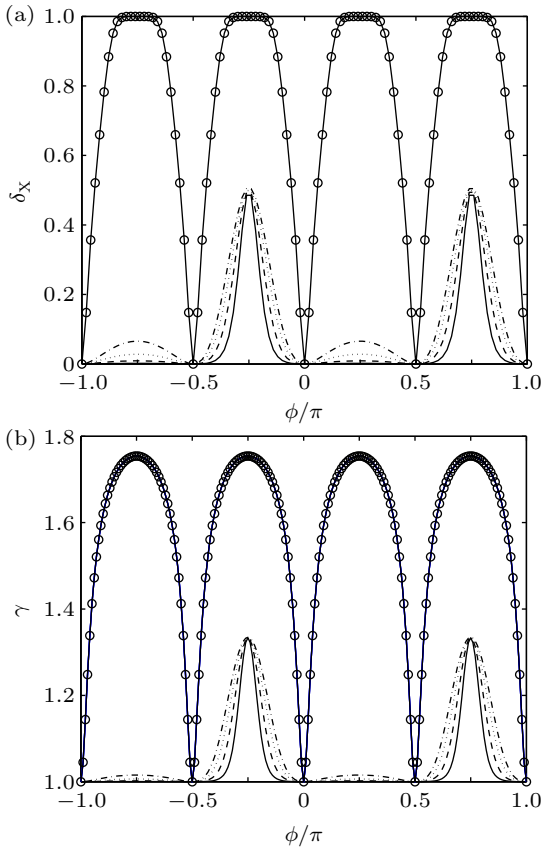


Fig. 2. (a) Non-Gaussianity $\delta_{\rm X}$ and (b) non-Gaussian ration γ for the single-mode SCLSs as a function of the parameter ϕ and for different values of α. From bottom to top, $\alpha=\{0.2,0.3,0.4,0.5,5\}$.

It is interesting to see from Fig. 2(a) that using our measure, the non-Gaussianity for even Schrödinger-cat-like states is not always equal to zero for small values of the amplitudes α. It means that our measure has more superiority and validity than Genoni's measure.[6] In addition, one can observe that the $\delta_{\rm X}$ can attain a local maximal value at $\phi=\pm(2n+1)(\pi/4)$, $n\in\mathbb{Z}$, as long as $\alpha\neq 0$, showing that the non-Gaussianity of the even and odd cat coherent states is much stronger than that of other Schrödinger-cat-like states. One also observes that the non-Gaussianity can be strengthened with the increasing of the amplitude α. For enough large α values, the $\delta_{\rm X}$ is bounded by one, instead of $1/2$, for every value of ϕ. Of course, our non-Gaussianity measure also shares some properties in common with Genoni's non-Gaussianity measure. For example, we can obtain that the $\delta_{\rm X}=0$ for the case of $\phi=n\pi$ and $\phi=[n+(1/2)]\pi$, $n\in\mathbb{Z}$, at which single-mode Schrödinger-cat-like states become pure coherent states, belonging to Gaussian states. For the case of $\alpha\to 0$, we note that the non-Gaussianity of even Schrödinger-cat-like states goes to zero, whereas that of odd Schrödinger-cat-like states is still large. This is due to the fact that for very small coherent amplitude, even Schrödinger-cat-like states are only slightly different from the vacuum one $|0\rangle$, while odd Schrödinger-cat-like states are very near to the Fock state $|1\rangle$. However, for the higher values of coherent amplitudes α, the non-Gaussianity $\delta_{\rm X}$ of all Schrödinger-cat-like states is a periodic function of phase ϕ with a period $\pi/2$ and has similar non-Gaussian behaviors. In Fig. 2(b) we compare $\delta_{\rm X}$ with $\delta_{\rm A}$ by introducing the non-Gaussian ration γ. One observes that $\delta_{\rm X}$ is always greater than or equal to $\delta_{\rm A}$, i.e., $\gamma\geq 1$, irrespective of α and ϕ.

3.2. Mixture of Fock states

As a second example, let us consider a special family of single-mode non-Gaussian mixed states, statistical mixture of the vacuum state and n-photon Fock state is defined by

$$\rho = (1-p)|0\rangle\langle 0| + p|n\rangle\langle n|, \tag{31}$$

where $p\in[0,1]$ and $n\geq 1$.

We can obtain the purities of state (31) and its reference Gaussian state as

$$\mu[\rho] = 1-2p+2p^2, \tag{32a}$$

$$\mu[\rho_{\rm rG}] = \frac{1}{1+2np}, \tag{32b}$$

as well as their trace overlap is

$$\kappa[\rho,\rho_{\rm rG}] = \frac{1}{1+np}\left[1-p+p\left(\frac{np}{1+np}\right)^n\right]. \tag{33}$$

By using the derived results (32) and (33), we plot in Fig. 3 the corresponding non-Gaussianity of SMVFSs and the non-Gaussian ration γ as function of the parameter p for several values of the photon number n. By direct inspection, one

sees that the non-Gaussianity δ_X naturally vanishes for $p = 0$, while the limiting value of the non-Gaussianity of SMVFSs for infinite number of photons n is given by $\lim_{n\to\infty} \delta_X = 1$. In other cases, the non-Gaussianity is bounded by $0 < \delta_X < 1$. It shows that for a given number photon n, the pure Fock states possess much more non-Gaussianity than statistical mixtures of the vacuum state and n-photon Fock state. One also sees that the non-Gaussianity increases monotonically with the photon number n, namely, the larger the photon number is, the more the non-Gaussianity of SMVFSs becomes. These behaviors are in agreement with the result reported in Ref. [6]. In Fig. 3(b), we can see that when $n > 1$, the non-Gaussian ration $\gamma > 1$ and hence $\delta_X > \delta_A$. However, when taking $n = 1$ and $0 < p < 0.5$, we visualize that $\gamma < 1$, namely, $\delta_X < \delta_A$. It has been shown in Ref. [27] that for quantum state $\rho = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$ with $p < 0.5$, it has a positive Wigner function which is not just a mixture of Gaussian states. Motivated by this, we can obtain an inequality of the photon number n and the parameter p: $2np^2 - (2n-1)p + n - 1 < 0$. If such an inequality holds, then the statistical mixture of the n-photon and vacuum optical states has the positive Wigner function; otherwise, its Wigner function takes on negative values, which is usually used as an indicator of non-classicality of quantum state. Recently, several works have tried to establish a connection between this feature and other quantum corrections like quantum discord[28] and quantum contextuality.[29]

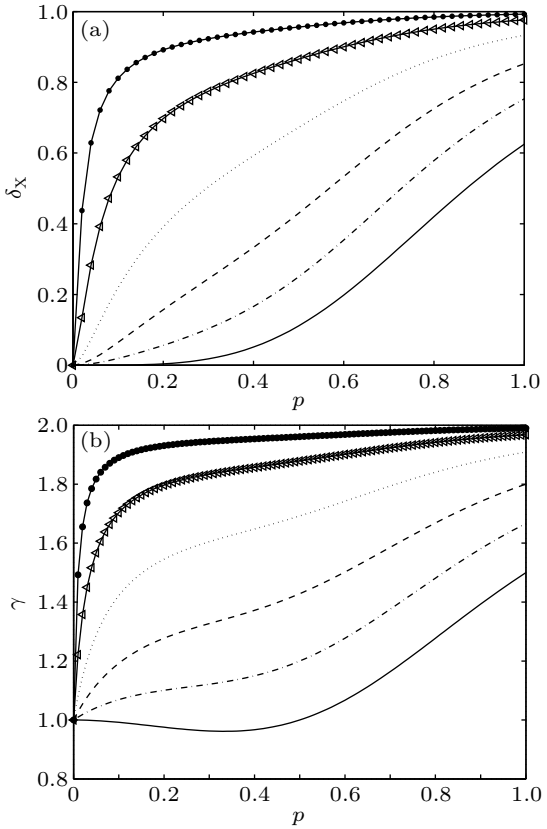


Fig. 3. (a) Non-Gaussianity δ_X and (b) non-Gaussian ration γ of SMVFSs as a function of p and for different values of photon numbers n. From bottom to top, $n = \{1, 2, 4, 10, 30, 100\}$.

As is well known, a non-Gaussian state may be classical or nonclassical, but our example tells us that a quantum state with the negative Wigner function is bound to be non-Gaussian. In other words, the negative value of Wigner function is a sufficient condition that quantum state is non-Gaussian.

3.3. Photon-added thermal states

The photon addition and subtraction are an important technique for generating quantum non-Gaussian states. Here we consider another family of single-mode non-Gaussian mixed states, m-photon-added thermal state, which is defined as[10,11]

$$\rho_{ad} = \frac{1}{m!(1+\bar{n})^m} \hat{a}^{+m} \rho_{th} \hat{a}^m, \tag{34}$$

where m is the added photon number and ρ_{th} is the density operator of thermal state with mean number of photon $\bar{n}$, which can be expressed, in the Fock state basis, as

$$\rho_{th} = (1-x) \sum_{n=0}^{\infty} x^n |n\rangle\langle n|, \tag{35}$$

in which the thermal parameter $x = \bar{n}/(1+\bar{n})$, $0 \le x \le 1$. Thus, we see that mPATS is parameterized by two parameters: x and m. The limit $x \to 0$ corresponds to Fock state $|m\rangle$, where the limit $m \to 0$ corresponds to the thermal state.

The mean photon number of mPATSs is given by

$$\overline{N} = \mathrm{Tr}[\hat{a}^+ \hat{a} \rho_{ad}] = \bar{n}(m+1) + m. \tag{36}$$

Making use of Eqs. (34)–(36), we get the purity of mPATS as

$$\mu[\rho_{ad}] = \frac{(-1)^m}{(1+2\bar{n})^{m+1}} P_m(z), \tag{37}$$

where $z = -\frac{2\bar{n}^2+2\bar{n}+1}{1+2\bar{n}}$ and $P_m(z)$ is the Legendre polynomial of m order.

The purity of the corresponding reference Gaussian state is given by

$$\mu[\rho_{rG}] = \frac{1}{1+2\overline{N}}, \tag{38}$$

and their trace overlap is given by

$$\kappa[\rho_{ad}, \rho_{rG}] = \frac{\overline{N}^m}{[1+\bar{n}+\overline{N}]^{m+1}}. \tag{39}$$

By carefully examining Eqs. (37)–(39), we see that when $m = 0$ (the case of no photon addition), we have $\mu[\rho_{ad}] = \mu[\rho_{rG}] = \kappa[\rho_{ad}, \rho_{rG}]$ and then $\delta_X = 0$, while for the case of $\bar{n} = 0$ (or $x \to 0$), the non-Gaussianity of m-photon-added thermal state is exactly equal to that of Fock state $|m\rangle\langle m|$.

In Fig. 4, we plot the non-Gaussianity and non-Gaussian ration of mPATSs as a function of the thermal parameter x for different photon-added number $m = 1, 3, 5, 10$. From Fig. 4(a),

one can see that like Genoni's non-Gaussianity measure, our non-Gaussianity decreases monotonically with the thermal parameter x and increases with the photon-addition number m. But, it is completely different from the result reported in Ref. [10], where the non-Gaussianity of such states is quantified by the Wehrl-entropy measure $N(\rho)$ and is found to be the same as that of the Fock state $|m\rangle\langle m|$, i.e., $N(\rho_{\rm ad}) = N(\rho_{|m\rangle\langle m|})$, independent of the thermal mean occupancy $\bar{n}$. One also observes that a difference appears when considering the upper bound on the non-Gaussianity in the case of m tending to infinity. The non-Gaussianity of mPATSs by employing our measure saturates at the value 1, where it is 0.5 by the Genoni's measure[11] and infinity via the Wher-entropy measure introduced by Solomom's measure.[10] In Fig. 4(b), we see that $\delta_{\rm X}$ is always greater than or equal to $\delta_{\rm A}$, irrespective of x and m.

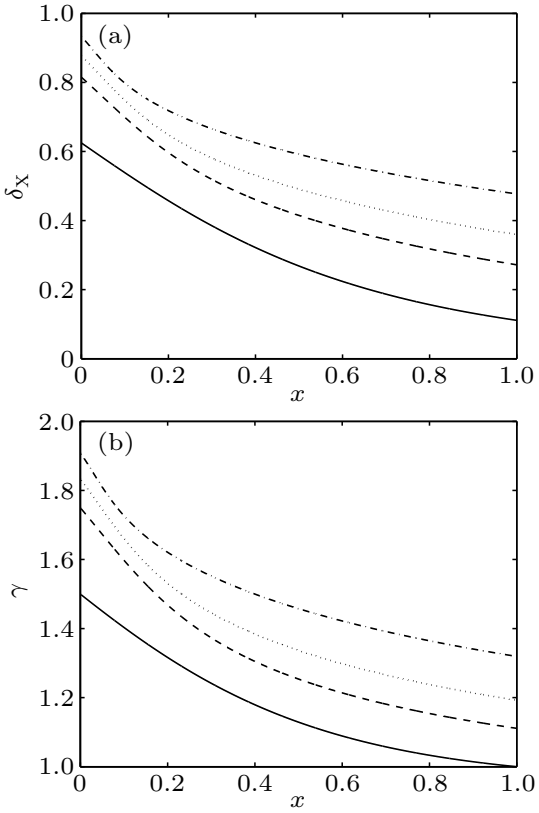


Fig. 4. (a) Non-Gaussianity $\delta_{\rm X}$ and (b) non-Gaussian ration γ of mPATSs as a function of the thermal parameter x and for different photon-addition number m. From bottom to top, $m = \{1,3,5,10\}$.

3.4. Multi-mode entangled-coherent-like states

Multipartite quantum states are an essential resource in quantum network and quantum communication. However, little attention is focused on the evaluation of non-Gaussianity of multi-mode non-Gaussian entangled states. Therefore, here we will consider n-mode entangled-coherent-like states,[30,31] which are given by

$$|\phi\rangle = \frac{1}{N_n}\left[\cos(\theta)|\alpha\rangle^{\otimes n} + \sin(\theta)|-\alpha\rangle^{\otimes n}\right], \tag{40}$$

where $|\alpha\rangle^{\otimes n} \equiv \underbrace{|\alpha\rangle\otimes|\alpha\rangle\cdots\otimes|\alpha\rangle}_{n}$ with n denoting the mode number and $N_n = \sqrt{1+\sin(2\theta)\,{\rm e}^{-2n|\alpha|^2}}$ is the normalized factor. For simplicity's sake, we assume α to be real.

After some algebra, we can obtain the first-order moments of the position and momentum operators for the j-th mode as

$$\langle\hat{x}_j\rangle = \frac{\sqrt{2}\alpha\cos(2\theta)}{N_n^2},\ \ \langle\hat{p}_j\rangle = 0, \tag{41}$$

and the covariance matrix of the n-mode entangled-coherent-like states (40) is

$$\boldsymbol{\sigma}[\rho] = \begin{pmatrix} \boldsymbol{\sigma}_{11} & \boldsymbol{\sigma}_{12} & \cdots & \boldsymbol{\sigma}_{1n} \\ \boldsymbol{\sigma}_{12}^{\rm T} & \boldsymbol{\sigma}_{22} & \cdots & \boldsymbol{\sigma}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\sigma}_{1n}^{\rm T} & \boldsymbol{\sigma}_{2n}^{\rm T} & \cdots & \boldsymbol{\sigma}_{nn} \end{pmatrix}, \tag{42}$$

where $\boldsymbol{\sigma}_{jj} = {\rm diag}(h,f)$ is a 2×2 local covariance matrix of mode j with the matrix elements $h = \frac{N_n^4+4\alpha^2[N_n^2-\cos^2(2\theta)]}{2N_n^4}$ and $f = \frac{N_n^2-4\alpha^2{\rm e}^{-2n\alpha^2}\sin(2\theta)}{2N_n^2}$, and $\boldsymbol{\sigma}_{jk} = {\rm diag}(c,d)$ is the mutual correlation matrix with the elements $c = \frac{\alpha^2\left[N_n^2-2\cos^2(2\theta)\right]}{N_n^4}$ and $d = -\frac{\alpha^2{\rm e}^{-2n\alpha^2}\sin(2\theta)}{N_n^2}$, respectively.

With use of Eq. (42), the purity of the corresponding reference Gaussian state is given by

$$\begin{aligned}\mu[\rho_{\rm rG}] \ =\ & [(2nB-nA-n+1)(2A-4B+2nB-nA+n-1) \\ & \times(2A-4B-1)]^{-1/2},\end{aligned} \tag{43}$$

where A and B have the same express as Eq. (30), the only difference being that ${\rm e}^{-2\alpha^2}$ is replaced by ${\rm e}^{-2n\alpha^2}$ and N is replaced by N_n.

The trace overlap is given by

$$\begin{aligned}\kappa[|\phi\rangle\langle\phi|,\rho_{\rm rG}] \ =\ & \frac{\cos^2(\phi)\,{\rm e}^{-n(q-p)^2}+\sin^2(\phi)\,{\rm e}^{-n(q+p)^2}}{N_n^2\mu[\rho_{\rm rG}]G_n} \\ & +\frac{\sin(2\phi)\,{\rm e}^{-2n\alpha^2}}{2N_n^2\mu[\rho_{\rm rG}]}\exp(H_n),\end{aligned} \tag{44}$$

where

$$H_n = \frac{n\left[q^2G_n+p^2F_n\right]}{[A+2(n-1)B]^2-[(n-1)A+2B-n+1]^2}, \tag{45a}$$

$$G_n = (2-n)A+(2n-4)B+n-1, \tag{45b}$$

$$F_n = n(1-A-2B)-1. \tag{45c}$$

Using our introduced method, we may evaluate the non-Gaussianity $\delta_{\rm X}$ of n-mode entangled-coherent-like states with respect with to ϕ and for several values of mode numbers n and the amplitudes α, which is reported in Fig. 5. It can be seen from the plots that the non-Gaussianity $\delta_{\rm X}$ is a function of n, α, and ϕ. As expected, we can obtain that $\delta_{\rm X} = 0$ for

the case of $\phi = m\pi$ and $\phi = [m+(1/2)]\pi$, $m \in \mathbb{Z}$, at which multi-mode entangled-coherent-like states become a Gaussian product state. For a case of $\phi = [(2m+1)\pi]/4$, the higher values of the δ_X of multi-mode entangled-coherent-like states are achieved. That is because such states become the even entangled coherent ones for $\phi = \pi/4$ and odd entangled coherent ones for $\phi = -\pi/4$. They have highly non-Gaussian features. For larger values of α, the δ_X becomes an even function of ϕ, approaching to unity at $\phi = [(2m+1)\pi]/4$, $m \in \mathbb{Z}$. We also see that the δ_X is an increasing function of the field modes n. When the mode numbers, n, of the field increase, the δ_X dramatically increases. In addition, we compare Fig. 2(a) with Fig. 1(a) and can see that for the small amplitudes of α, the δ_X is not equal to null in the regime of $m\pi < \phi < [m+(1/2)]\pi$, $(m \in \mathbb{Z})$, showing again that the δ_X has much advantages than the δ_A in quantifying the non-Gaussianity of quantum states.

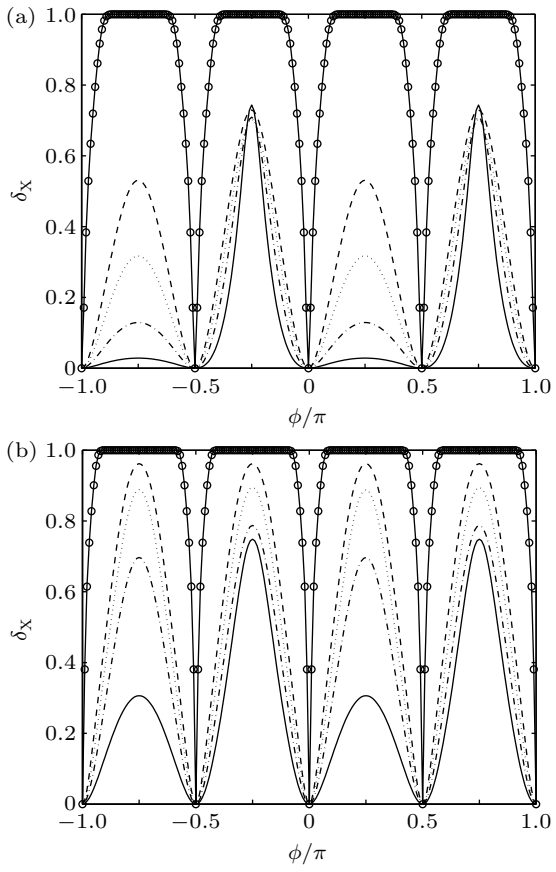


Fig. 5. Non-Gaussianity δ_X of n-mode entangled-coherent-like states as a function of ϕ for two choices of n: (a) $n = 5$ and (b) $n = 20$, and different values of α. From bottom to top, $\alpha = \{0.2, 0.3, 0.4, 0.5, 5\}$.

3.5. Photon-number entangled states

As our final example, we want to consider a specific family of multi-mode non-Gaussian entangled states: n-mode photon-number entangled states,[32] which can be expressed as

$$|\Psi\rangle = \cos\phi|0\rangle^{\otimes n} + \sin\phi|M\rangle^{\otimes n}, \tag{46}$$

where $|M\rangle^{\otimes n} \equiv \underbrace{|M\rangle \otimes |M\rangle \cdots \otimes |M\rangle}_{n}$. It is clear to see that when $\phi = m\pi$, $m \in \mathbb{Z}$, the states (46) correspond to a n-mode factorized vacuum state, belonging to Gaussian state, while for $\phi = [m+(1/2)]\pi$, $m \in \mathbb{Z}$, such states reduce to a n-mode factorized Fock state, which is high non-Gaussian; when $\phi = [m+(1/4)]\pi$, $m \in \mathbb{Z}$, they become the maximally entangled states.

After a little algebra, we can obtain the covariance matrix of the states (46) as

$$\boldsymbol{\sigma}[\rho] = \begin{pmatrix} (1-a)\boldsymbol{I}_2 & b\delta_{M-1,0}\boldsymbol{\sigma}_z & \cdots & b\delta_{M-1,0}\boldsymbol{\sigma}_z \\ b\delta_{M-1,0}\boldsymbol{\sigma}_z & (1-a)\boldsymbol{I}_2 & \cdots & b\delta_{M-1,0}\boldsymbol{\sigma}_z \\ \vdots & \vdots & \ddots & \vdots \\ b\delta_{M-1,0}\boldsymbol{\sigma}_z & b\delta_{M-1,0}\boldsymbol{\sigma}_z & \cdots & (1-a)\boldsymbol{I}_2 \end{pmatrix}, \tag{47}$$

with $a = (1/2)\cos(2\phi)$, $b = (1/2)\sin(2\phi)$, and $\boldsymbol{\sigma}_z = \mathrm{diag}(1,-1)$, while all the first-order moments are null. From Eq. (47), we see that its covariance matrix is a diagonal block matrix for $M \geq 2$.

Proceeding as above and after a simple calculation, we can easily obtain the purity of the reference Gaussian state and their trace overlap, which are not written here for reason of space. In Fig. 6, we plot the non-Gaussianities δ_X and δ_A of multi-mode photon-number entangled states as a function of ϕ for different photon numbers M and for different mode numbers n. It is seen from Figs. 6(a) and 6(c) that the non-Gaussianity δ_X exhibits the periodic oscillations with the phase ϕ, regardless of n. And the non-Gaussianity can be enhanced by increasing the Fock number M and the mode number n of quantum states (46). As expect, for $\phi = 0$, the non-Gaussianity is equal to zero, while the maximum amount of non-Gaussianity is achieved for $\phi = \pm(1/2)\pi$. For this case, the states (46) correspond to the n-mode factorized Fock states, proving again that Fock states have very highly non-Gaussian features. On the other hand, it is interesting to see that when $\phi = \pm\pi/4$ and $\pm 3\pi/4$, multi-mode photon-number entangled states are maximally entangled, but the non-Gaussianity is not maximal, which is completely different from the result obtained by the non-Gaussianity measure δ_A. This point can be seen from Figs. 6(b) and 6(d) that for the non-Gaussianity δ_A, it likes a saddle on the interval: $\phi \in [(2m+1)(\pi/4), (2m+1)(3\pi/4)]$, $m = 0, \pm 1, \ldots$. More specifically, for given ϕ and n, the maximum amount of non-Gaussianity δ_A can be obtained at $\phi = \pm\pi/4$ and $\pm 3\pi/4$, while a local minimum of Gaussianity δ_A is observable in $\phi = \pm\pi/2$, while our measure has a convex behavior. Finally, we see that for n-mode photon-number entangled states, the limiting value of δ_X is equal to 1, while the limiting value of the δ_A varies within the range $[0.5, 1)$ conditioning on the numbers of field mode.

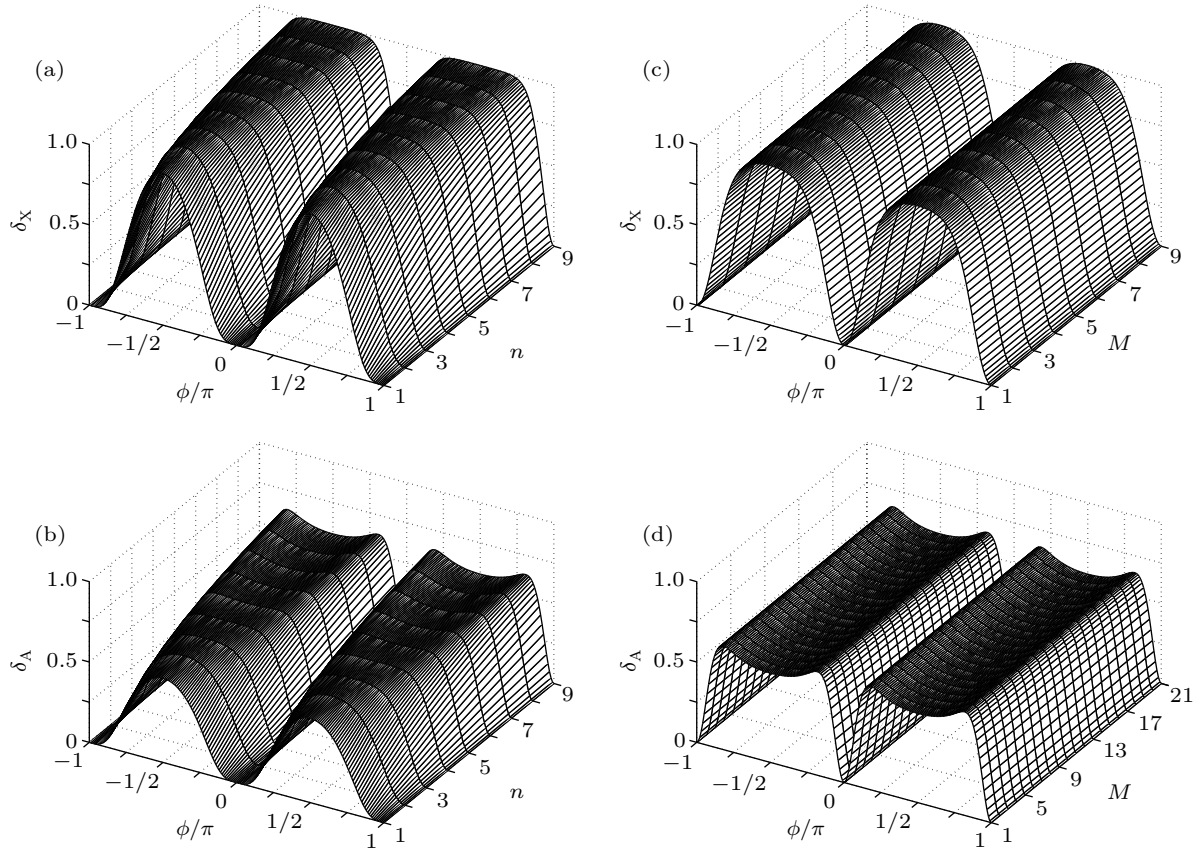


Fig. 6. Non-Gaussianities δ_X (topmost panel) and δ_A (lower panel) of GHZ-like PNES as a function of (a)–(b) ϕ and n for $M = 4$, and of (c)–(d) ϕ and M for $n = 10$.

4. Gaussianity versus non-Gaussianity

The Gaussianity of a quantum state is an extremely important and open issue in quantum information theory. Recently, Mandilara and Cerf introduced a degree of Gaussianity of state ρ, which is defined as[17]

$$g \equiv \frac{\mathrm{Tr}(\rho\rho_{\mathrm{rG}})}{\mathrm{Tr}(\rho_{\mathrm{rG}}\rho_{\mathrm{rG}})} = \frac{\kappa[\rho,\rho_{\mathrm{rG}}]}{\mu[\rho_{\mathrm{rG}}]}. \tag{48}$$

We see that this measure becomes unity for all Gaussian states, but is equal to 1 for some non-Gaussian states, e.g., the state $\rho_{|2\rangle\langle 2|} = \frac{1}{2\sqrt{2}}|2\rangle\langle 2| + (1 - \frac{1}{2\sqrt{2}})|0\rangle\langle 0|$.[33] Simultaneously, it is not a normalizing quantity and is varied in a range between $2/e$ and 2, that is , $2/e \leq g \leq 2$, implying that their introduced measure is not a good measure of Gaussianity of a quantum state.

According to the previous statement, we can redefine the Gaussianity of state (48) and introduce its improved version as

$$\delta_{\mathrm{G}} \equiv \frac{2\kappa[\rho,\rho_{\mathrm{rG}}]}{\mu[\rho] + \mu[\rho_{\mathrm{rG}}]}. \tag{49}$$

After the reformulation, the quantify δ_{G} is obviously symmetric under the permutation of the two quantum states. We also see that $\delta_{\mathrm{G}} = 1$ is attained if and only if $\rho = \rho_{\mathrm{rG}}$, whereas we can obtain $\delta_{\mathrm{G}} \in [0,1)$ for any pure or mixed non-Gaussian state. For instance, by using the improved Gaussianity measure, the Gaussianity for the state $\rho_{|2\rangle\langle 2|}$ is given by 0.8655, instead of 1. This treatment removes the blockade of quantifying the Gaussianity of quantum state and can be proven as follows. Firstly we investigate the upper bound on the Gaussianity δ_{G} of a quantum state ρ. According to the definition of the Hilbert–Schmidt distance, we have $d = \mathrm{Tr}(\rho - \rho_{\mathrm{rG}})^2 = \mathrm{Tr}(\rho^2) + \mathrm{Tr}(\rho_{\mathrm{rG}}^2) - 2\mathrm{Tr}(\rho\rho_{\mathrm{rG}}) = \mu[\rho] + \mu[\rho_{\mathrm{rG}}] - 2\kappa[\rho,\rho_{\mathrm{rG}}]$. We see that $d = 0$ if $\rho = \rho_{\mathrm{rG}}$; in other cases, $d > 0$. Consequently, we can get $2\kappa[\rho,\rho_{\mathrm{rG}}] < \mu[\rho] + \mu[\rho_{\mathrm{rG}}]$ for a case of $\rho \neq \rho_{\mathrm{rG}}$, that is, $\delta_{\mathrm{G}} < 1$. Secondly, its lower bound $\delta_{\mathrm{G}} = 0$ can be attained if they are completely discernible.

Here, we emphasize that by using our improved measures, we can arrive at an interesting and important expression

$$\delta_{\mathrm{X}} + \delta_{\mathrm{G}} = 1, \tag{50}$$

which suggests that for any quantum state, its total amount of Gaussianity and non-Gaussianity is conserved, that is, the increase of non-Gaussianity gives rise to the decrease of Gaussianity, or vice versa. This is easily understood from the two facts. One is that according to the phase-space theory, any continuous-variable quantum state has not only the lower-order cumulants but also the higher-order cumulants, that is, it contains both Gaussian and non-Gaussian components.[34] Such a Gaussian part may be treated using the cumulant generating function of the first- and second-order cumulants of a

quantum state,[10] which is only responsible for the quantification of Gaussianity. Analogously, a non-Gaussian part may be treated using the cumulant generating function of higher-order cumulants, which becomes a cornerstone of non-Gaussianity measures. Consequently from this point, the Gaussianity and non-Gaussianity of a quantum state are both oppose and complementary to each other and fuse into a resource theory of quantum state. The other is that the trace overlap between two density matrices is a key ingredient in the Hilbert–Schmidt distance measurement and can be widely interpreted as a measure of how close two quantum states are to each other. When they are the same, the trace overlap is 1 with a perfect overlap; while when they are orthogonal to each other, it is zero. In addition, from the perspective of measurement, such a trace overlap can be interpreted as the probability $p_{\rm G}$ of measuring a given state ρ to be in its reference Gaussian state $\rho_{\rm rG}$, i.e., $p_{\rm G} = {\rm Tr}[\rho\rho_{\rm rG}]$, while in the event of failure, it is directly equivalent to the probability $p_{\rm NG}$ of non-Gaussian measuring on the state ρ, then the relation between them can be given by $p_{\rm NG} = 1 - p_{\rm G}$. We note that several previous works have already focused on Gaussian measurement of quantum systems and have developed many schemes of quantum metrology based on the Gaussian positive operator-valued measures (POVMs).[35–38] Therefore, we hope that our study could pave the way for the further experimental development of source theory of quantum state.

5. Conclusion

In this paper, we have proposed an improved measure of non-Gaussianity of a quantum state, which can help us to precisely evaluate the non-Gaussianity of a wide range of non-Gaussian quantum states including a mixture of vacuum and arbitrary-photon Fock states, Schrödinger-cat-like states of small amplitudes, and multi-mode non-Gaussian entangled states. Besides these, we also have introduced a procedure of constructing the reference Gaussian state from a given quantum state. It turns out to be a more efficient and intuitive method. Finally, we have also examined the existing Gaussianity measure of a quantum state and presented an improved version to overcome its drawbacks. Particularly, we have found out that there exists a subtle connection between the Gaussianity and non-Gaussianity of a quantum state.

The characterization and quantification of non-Gaussianity of a quantum state are an interesting and open issue. Different measures may yield different orderings of non-Gaussianity. Our work is only a start. We expect the present study might motivate deeper analysis of the relationship between non-Gaussianity and Gaussianity using other measures based on the quantum relative entropy, uncertainty relation, and fidelity, as well as the non-Gaussianity-bounded uncertain relations.

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