Superposition formulas of multi-solution to a reduced $(3+1)$-dimensional nonlinear evolution equation

Hangbing Shao(邵杭兵) and Bilige Sudao(苏道毕力格)[†]

Department of Mathemaitc, Inner Mongolia University of Technology, Hohhote 010051, *China*

(Received 20 October 2022; revised manuscript received 3 December 2022; accepted manuscript online 27 December 2022)

We gave the localized solutions, the interaction solutions and the mixed solutions to a reduced $(3+1)$-dimensional nonlinear evolution equation. These solutions were characterized by superposition formulas of positive quadratic functions, the exponential and hyperbolic functions. According to the known lump solution in the outset, we obtained the superposition formulas of positive quadratic functions by plausible reasoning. Next, we constructed the interaction solutions between the localized solutions and the exponential function solutions with the similar theory. These two kinds of solutions contained superposition formulas of positive quadratic functions, which were turned into general ternary quadratic functions, the coefficients of which were all rational operation of vector inner product. Then we obtained linear superposition formulas of exponential and hyperbolic function solutions. Finally, for aforementioned various solutions, their dynamic properties were showed by choosing specific values for parameters. From concrete plots, we observed wave characteristics of three kinds of solutions. Especially, we could observe distinct generation and separation situations when the localized wave and the stripe wave interacted at different time points.

Keywords: localized solutions, mixed solutions, Hirota bilinear method, linear superposition formulas

PACS: 02.30.Ik, 02.30.Jr **DOI:** 10.1088/1674-1056/acae7d

1. Introduction

Nonlinear evolution equations (NLEEs) are very popular equations in the fields of mathematics. The discovery of rogue wave solutions of NLEEs in other scientific fields, such as optics,[1] atmosphere,[2] oceanography,[3] and finance[4] has pushed them to a more important position. So finding exact solutions for NLEEs is significant. In nonlinear science, scholars have utilized various methods to obtain various solutions of NLEEs, such as the Darboux transformation,[5–7] the Hirota bilinear method,[8] Lie symmetry method,[9] the homogeneous balance method,[10] the tanh-coth method,[11] Bäcklund transformation,[12] consistent Riccati expansion method,[13,14] etc. It is quite efficient to use these methods to derive some solutions for NLEEs, such as variable separation solutions,[15] positon solutions,[16] algebraic-geometrical solutions,[17] N-soliton,[18] localized excitations,[19] dark solitary wave solutions,[20,21] etc. These methods and solutions are sometimes applied to other equations, such as differential-difference equations.[22–24]

In recent years, for some specific $(3+1)$-dimensional NLEEs, many researchers have given many kinds of solutions, such as soliton solutions, rational solutions and their interaction solutions.[25–29] The bilinear method supplies a good way for us to obtain various solutions. Interaction solutions between lump and stripe solitons,[30,31] high-order lump-type solutions,[32] general periodic solutions,[33] multiple soliton solutions,[34] multiple localized wave solutions (breather, rogue wave solutions, kink solitons, etc.)[35–37] and other exact analytical solutions[38] can be obtained via taking advantage of the Hirota bilinear method. Besides, Shi *et al.*[39] used this method and KP hierarchy reduction method to get the general high-order rogue waves. Asma Issasfa *et al.*[40] obtained the generalized bilinear form through the generalized Hirota bilinear method to get new lump solutions and interaction solutions. Lou converted the bilinear forms of some NLEEs into the full reversal symmetric forms and obtained multiple soliton solutions.[41] Based on the bilinear Bäcklund transformation, novel soliton molecules[42] and new interaction solutions (and they contained quadratic, hyperbolic and trigonometric functions)[43] can be obtained. By using the Hirota bilinear method and the analysis of the N-soliton solutions, many kinds of localized wave solutions can be obtained.[44]

In this paper, one of $(3+1)$-dimensional NLEEs is studied

$$-3u_{xz}+(2u_t+2uu_x-u_{xxx})_y+2(u_x\partial_x^{-1}u_y)_x=0. \tag{1}$$

The bilinear form of Eq. (1) and multi-solution has been obtained.[45] Zheng *et al.*[46] gave a more general form of the solutions to a certain evolution equation. We start from these results in order to obtain the superposition formulas of solutions in Sections 2 and 3.

First, through the variable transformation $u=-3(\ln f)_{xx}$, then taking $z=x$, Eq. (1) is transformed into

$$\left(2D_tD_y-D_x^3D_y-3D_x^2\right)(f\cdot f)=0,$$

where $f=f(x,y,t)$ is the real function and D_x, D_y, D_t are the

[†]Corresponding author. E-mail: inmathematica@126.com

© 2023 Chinese Physical Society and IOP Publishing Ltd

Hirota operators which are defined by

$$\begin{aligned} &D_x^m D_y^n (f(x,y)\cdot g(x,y)) \\ &= \frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial t^n} f(x+s,y+t)\cdot g(x-s,y-t)|_{s=0,t=0}, \\ &m,n = 0,1,2,3,\ldots. \end{aligned}$$

The reduced bilinear equation is equivalent to

$$\begin{aligned} &2f_{yt}f - 2f_t f_y - f_{xxxy}f + 3f_{xxy}f_x \\ &-3f_{xx}f_{xy} + f_{xxx}f_y - 3f_{xx}f + 3f_x^2 = 0. \end{aligned} \tag{2}$$

We suppose that Eq. (2) has the novel expression of exact solutions

$$f = a_1\sum_{i=1}^{m_1} f_i^2 + a_2\sum_{i=1}^{m_2} k_i\exp(g_i) + a_3\sum_{i=1}^{m_3} l_i\cosh(h_i) + c, \tag{3}$$

where f_i, g_i, and h_i, $i = 1,2,3,\ldots$ are linear functions with regard to x, y, and t, respectively. a_1, a_2, and a_3 are equal to 0 or 1, and c is an undetermined constant. We require that a_1, a_2, and a_3 are not all equal to zero, and at least one of m_1, m_2, and m_3 is arbitrary positive integer, for the sake of obtaining more general solutions. Such as taking $a_1 = 1$, $a_2 = a_3 = 0$ in Section 2, we have the superposition formula of positive quadratic function solutions. Besides, the others two cases discussed in this paper are respectively $a_1 = 1$, $a_2 = m_2 = 1, a_3 = 0$ in Section 3 and $a_1 = 0$, $a_2 = a_3 = 1$, $c = 0$ in Section 4.

This paper is arranged as follows. In Section 2, the localized solutions corresponding to the superposition formulas of positive quadratic function type will be given based on the analogy and induction. In Section 3, we will acquire the interaction solutions between the localized solutions and the stripe solitons. In Section 4, we will obtain linear superposition formulas of exponential and hyperbolic function solutions. In each section, we utilize the function plots to reflect different types of waves. In the final section, we make the conclusion of this paper.

2. The localized solutions corresponding to the positive quadratic function types

2.1. The localized solutions in the form of superposition

For the sake of getting the localized solutions for the reduced Eq. (1), the solution of the bilinear Eq. (2) is assumed as follows:

$$f = \sum_{i=1}^{m} f_i^2 + c, \tag{4}$$

where $f_i = a_{i1}x + a_{i2}y + a_{i3}t + a_{i4}$, $i = 1,2,3,\ldots,$ m, $c =$ constant > 0. If $m = 2$, the parameter relations are obtained by Chen *et al.*[45]

$$\begin{aligned} a_{13} &= \frac{3(a_{11}^2a_{12} + 2a_{11}a_{21}a_{22} - a_{12}a_{21}^2)}{2(a_{12}^2 + a_{22}^2)}, \\ a_{23} &= \frac{3(a_{22}a_{21}^2 + 2a_{11}a_{12}a_{21} - a_{11}^2a_{22})}{2(a_{12}^2 + a_{22}^2)}, \\ c &= -(a_{11}^3a_{12} + 2a_{11}^3a_{21}a_{22} - a_{11}^2a_{12}^2a_{21}a_{22} + a_{11}^2a_{21}a_{22}^3 \\ &\quad + a_{11}a_{12}^3a_{21}^2 + a_{11}a_{12}a_{21}^2a_{22}^2 + a_{12}^2a_{21}^3a_{22} \\ &\quad + a_{21}^3a_{22}^3)/(a_{11}a_{22} - a_{12}a_{21})^2, \\ &a_{12}a_{22} \neq 0, \quad a_{11}a_{22} \neq a_{12}a_{21}, \quad a_{11}a_{12} + a_{21}a_{22} < 0. \end{aligned}$$

And a_{13}, a_{23} can be written as

$$\begin{aligned} a_{13} &= \frac{3}{2}\cdot\frac{a_{11}T_2^{(1)} + a_{21}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}}{T_3^{(1)}} \\ &= \frac{3}{2}\cdot\frac{2a_{11}T_2^{(1)} - a_{12}T_1^{(1)}}{T_3^{(1)}}, \\ a_{23} &= \frac{3}{2}\cdot\frac{a_{21}T_2^{(1)} + a_{11}\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12}\end{vmatrix}}{T_3^{(1)}} \\ &= \frac{3}{2}\cdot\frac{2a_{11}T_2^{(1)} - a_{12}T_1^{(1)}}{T_3^{(1)}}, \end{aligned} \tag{5}$$

where $T_1{}^{(1)} = \boldsymbol{X}_1\cdot\boldsymbol{X}_1$, $T_2{}^{(1)} = \boldsymbol{X}_1\cdot\boldsymbol{Y}_1$, $T_3{}^{(1)} = \boldsymbol{Y}_1\cdot\boldsymbol{Y}_1$, and $\boldsymbol{X}_1 = (a_{11},\ a_{21})$, $\boldsymbol{Y}_1 = (a_{12},\ a_{22})$ represent vectors formed by coefficients of independent variables of f_i, $i = 1,2$ respectively, so $T_1{}^{(1)} = a_{11}^2 + a_{21}^2$, $T_2{}^{(1)} = a_{11}a_{12} + a_{21}a_{22}$, $T_3{}^{(1)} = a_{12}^2 + a_{22}^2$.

By observing formula (5), we can find that they are written as rational operation of the determinant and the inner product of vectors. And when $f = \sum_{i=1}^{3}(a_{i1}x + a_{i2}y + a_{i3}t + a_{i4})^2 + c$ solves the Eq. (2), we can analogize the relationships of coefficients

$$\begin{aligned} a_{i3} &= \frac{3}{2}\cdot\frac{a_{i1}T_2^{(2)} + \sum\limits_{k=1}^{3} a_{k1}\begin{vmatrix} a_{i1} & a_{i2} \\ a_{k1} & a_{k2}\end{vmatrix}}{T_3^{(2)}} \\ &= \frac{3}{2}\cdot\frac{2a_{i1}T_2^{(2)} - a_{i2}T_1^{(2)}}{T_3^{(2)}}, \quad i = 1,2,3, \\ &T_1{}^{(2)} = \boldsymbol{X}_2\cdot\boldsymbol{X}_2, \ T_2{}^{(2)} = \boldsymbol{X}_2\cdot\boldsymbol{Y}_2, \ T_3{}^{(2)} = \boldsymbol{Y}_2\cdot\boldsymbol{Y}_2, \end{aligned} \tag{6}$$

where $\boldsymbol{X_2} = (a_{11},a_{21},a_{31})$, $\boldsymbol{Y_2} = (a_{12},a_{22},a_{32})$. If we substitute f with coefficient relations (6) into Eq. (2), we can obtain

$$c = -\frac{A_1}{B_1}, \tag{7}$$

where

$$\begin{aligned} A_1 &= T_1{}^{(2)}T_2{}^{(2)}T_3{}^{(2)} + T_1{}^{(2)}T_3{}^{(2)}T_6{}^{(2)} + 2\cdot T_2{}^{(2)}T_4{}^{(2)}T_5{}^{(2)} \\ &\quad - T_1{}^{(2)}(T_5{}^{(2)})^2 - T_3{}^{(2)}(T_4{}^{(2)})^2 - T_6{}^{(2)}(T_2{}^{(2)})^2, \\ B_1 &= T_1{}^{(2)}T_3{}^{(2)} - (T_2{}^{(2)})^2, \\ &T_4{}^{(2)} = \boldsymbol{X}_2\cdot\boldsymbol{C}_2, \ T_5{}^{(2)} = \boldsymbol{Y}_2\cdot\boldsymbol{C}_2, \ T_6{}^{(2)} = \boldsymbol{C}_2\cdot\boldsymbol{C}_2, \\ &\boldsymbol{C}_2 = (a_{14},a_{24},a_{34}). \end{aligned}$$

On the basis of Eqs. (5)–(7), we can infer solution (4) of Eq. (2), and the relationships of parameters are

$$a_{i3}=\frac{3}{2}\cdot\frac{2a_{i1}T_2-a_{i2}T_1}{T_3},\ \ i=1,2,3,\ldots,\ m,\ \ c=-\frac{A}{B},\quad(8)$$

where

$$\begin{aligned}
&A=T_1T_2T_3+T_1T_3T_6+2\cdot T_2T_4T_5-T_1(T_5)^2\\
&\quad-T_3(T_4)^2-T_6(T_2)^2,\\
&B=T_1T_3-(T_2)^2,\\
&T_1=\boldsymbol{X}\cdot\boldsymbol{X},\ \ T_2=\boldsymbol{X}\cdot\boldsymbol{Y},\ \ T_3=\boldsymbol{Y}\cdot\boldsymbol{Y},\\
&T_4=\boldsymbol{X}\cdot\boldsymbol{C},\ \ T_5=\boldsymbol{Y}\cdot\boldsymbol{C},\ \ T_6=\boldsymbol{C}\cdot\boldsymbol{C},\\
&\boldsymbol{X}=(a_{11},a_{21},\cdots,a_{m1}),\ \ \boldsymbol{Y}=(a_{12},a_{22},\cdots,a_{m2}),\\
&\boldsymbol{C}=(a_{14},a_{24},\cdots,a_{m4}),
\end{aligned}$$

and the conditions must satisfy $m\geq 2$, $T_3\neq 0$, $A<0$, $B\neq 0$. Under the parameter relationships in Eq. (8), f in Eq. (4) can be deduced as

$$\begin{aligned}
f&=\sum_i(a_{i1}x+a_{i2}y+a_{i3}t+a_{i4})^2+c\\
&=\sum_i\big(a_{i1}^2x^2+a_{i2}^2y^2+a_{i3}^2t^2+2a_{i1}a_{i2}xy+2a_{i1}a_{i3}xt\\
&\quad+2a_{i2}a_{i3}yt+2a_{i1}a_{i4}x+2a_{i2}a_{i4}y+2a_{i3}a_{i4}t+a_{i4}^2\big)+c\\
&=T_1x^2+T_3y^2+\frac{9}{4}\cdot\frac{(T_1)^2}{T_3}t^2+2T_2xy+3\cdot\frac{T_1T_2}{T_3}xt\\
&\quad+3\cdot\frac{2(T_2)^2-T_1T_3}{T_3}yt+2T_4x+2T_5y\\
&\quad+3\cdot\frac{2T_4T_2-T_5T_1}{T_3}t+T_6-\frac{A}{B}.
\end{aligned}\quad(9)$$

By substituting f and its derivatives into the left side of Eq. (2), the result is zero. Ultimately, the localized solution of Eq. (1) is

$$u=\frac{-6T_1f+12\left(T_1x+T_2y+\frac{3}{2}\cdot\frac{T_1T_2}{T_3}t+T_4\right)^2}{f^2}.\quad(10)$$

By increasing the number of positive quadratic functions, new localized solutions of some NLEEs can be obtained, such as the lump-type solutions containing the superposition of three functions[47] and solutions containing the superposition of five functions.[48] And we can achieve these by increasing the dimensions of the vectors (8).

2.2. Dynamic properties of the localized solutions

For solution functions (10), we regard t as a constant and then the extreme points of function solutions (10) include (x_1,y_1), (x_2,y_2), and (x_3,y_3), where

$$\begin{aligned}
&x_1=vt+\frac{E_1}{B},\ \ x_2=vt+\frac{E_1+\sqrt{-3(\boldsymbol{X}\cdot\boldsymbol{Y})(\boldsymbol{Y}\cdot\boldsymbol{Y})B}}{B},\\
&x_3=vt+\frac{E_1-\sqrt{-3(\boldsymbol{X}\cdot\boldsymbol{Y})(\boldsymbol{Y}\cdot\boldsymbol{Y})B}}{B},\\
&y_1=y_2=y_3=ht+\frac{E_2}{B},\\
&v=-3\cdot\frac{\boldsymbol{X}\cdot\boldsymbol{Y}}{\boldsymbol{Y}\cdot\boldsymbol{Y}},\ \ h=\frac{3}{2}\cdot\frac{\boldsymbol{X}\cdot\boldsymbol{X}}{\boldsymbol{Y}\cdot\boldsymbol{Y}},\\
&E_1=(\boldsymbol{X}\cdot\boldsymbol{Y})(\boldsymbol{Y}\cdot\boldsymbol{C})-(\boldsymbol{Y}\cdot\boldsymbol{Y})(\boldsymbol{X}\cdot\boldsymbol{C}),\\
&E_2=(\boldsymbol{X}\cdot\boldsymbol{Y})(\boldsymbol{X}\cdot\boldsymbol{C})-(\boldsymbol{X}\cdot\boldsymbol{X})(\boldsymbol{Y}\cdot\boldsymbol{C}),\\
&B=(\boldsymbol{X}\cdot\boldsymbol{X})(\boldsymbol{Y}\cdot\boldsymbol{Y})-(\boldsymbol{X}\cdot\boldsymbol{Y})^2.
\end{aligned}$$

The corresponding extremum of (x_1,y_1) is $u_1=6B/(\boldsymbol{X}\cdot\boldsymbol{Y})(\boldsymbol{Y}\cdot\boldsymbol{Y})$, and corresponding extremums of (x_2,y_2) and (x_3,y_3) are both $u_2=-3B/4(\boldsymbol{X}\cdot\boldsymbol{Y})(\boldsymbol{Y}\cdot\boldsymbol{Y})$. Therefore, the localized wave of solution (10) exists the obvious lowest trough located at (x_1,y_1), the value of which is u_1. Two equally obvious crests are located at (x_2,y_2) and (x_3,y_3) respectively, the values of which are both u_2. In addition, the wave velocity is (v,h) and the amplitude is $|u_1|$.

Take $m=4$ as an example, and set the parameters $\boldsymbol{X}=(2,-2,6,1)$, $\boldsymbol{Y}=(-2,-2,-1,-1)$, $\boldsymbol{C}=(0,0,0,0)$. Then we can obtain

$$u=\frac{-270f_1+3(f_1)_x^2}{{f_1}^2},\quad(11)$$

where $f_1=45x^2+10y^2+\frac{3645}{8}t^2-14xy-\frac{189}{2}xt-\frac{528}{5}yt+\frac{3051}{401}$. At this time, the trough is located at $(21t/10,27t/4)$ and the amplitude is $1203/35$. Figure 1 shows that the localized wave keeps its shape and size during its propagation.

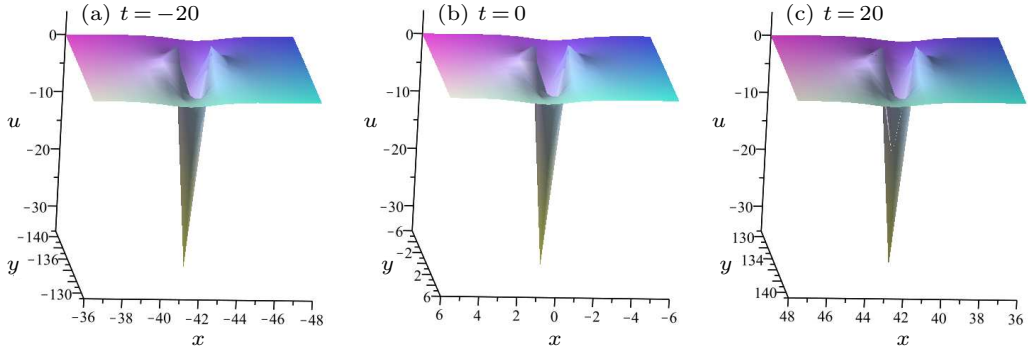


Fig. 1. 3D plots of the localized solution u Eq. (11) to Eq. (1).

3. Interaction solutions between localized solutions and one stripe soliton

For some special $(2+1)$-dimensional NLEEs, some researchers have obtained the interaction solutions containing the summation of arbitrary number of exponential functions.[49] To make a difference, the interaction solutions of Eq. (1) contain the summation of m $(m \geq 2)$ positive quadratic functions in this section. We firstly set that the form of Eq. (2) solutions is

$$\begin{aligned} f &= \sum_{i=1}^{m} f_i^2 + \exp(l_1) + c \\ &= \sum_{i=1}^{m} (d_{i1}x + d_{i2}y + d_{i3}t + d_{i4})^2 \\ &\quad + \exp(e_{11}x + e_{12}y + e_{13}t + e_{14}) + c, \end{aligned} \tag{12}$$

which need to satisfy $c = \text{constant} > 0$. When $m = 2$, the parameters satisfy

$$\begin{aligned} &d_{12} = -\frac{d_{11}e_{12}^2(d_{11}^2 + d_{21}^2)}{4d_{21}^2}, \quad d_{13} = -\frac{6d_{11}d_{21}^2(d_{11}^2 - 3d_{21}^2)}{e_{12}^2(d_{11}^2 + d_{21}^2)^2}, \\ &d_{14} = 0, \quad d_{22} = \frac{e_{12}^2(d_{11}^2 + d_{21}^2)}{4d_{21}}, \\ &d_{23} = -\frac{6d_{21}^3(3d_{11}^2 - d_{21}^2)}{e_{12}^2(d_{11}^2 + d_{21}^2)^2}, \quad d_{24} = 0, \\ &e_{11} = \frac{-4d_{21}^2}{e_{12}(d_{11}^2 + d_{21}^2)}, \quad e_{13} = \frac{8d_{21}^4(3d_{11}^2 - d_{21}^2)}{e_{12}^3(d_{11}^2 + d_{21}^2)^3}, \\ &c = \frac{e_{12}^2(d_{11}^2 + d_{21}^2)^3(d_{11}^2 - d_{21}^2)}{16d_{21}^4 d_{11}^2}, \\ &d_{11}d_{21}e_{12} \neq 0, \quad d_{11}^2 - d_{21}^2 > 0. \end{aligned} \tag{13}$$

According to the tips of Section 2, we directly give the solution in the form of Eq. (12) in this section. Equation (1) has interaction solutions, and accordingly in Eq. (12) the relationships of solutions parameters to Eq. (2) satisfy

$$\begin{aligned} &d_{i2} = -\frac{d_{i1}e_{12}^2 H}{4d_{m1}^2}, \quad d_{i3} = -\frac{6d_{i1}d_{m1}^2(H - 4d_{m1}^2)}{e_{12}^2 H^2}, \quad d_{i4} = 0, \\ &(i = 1, 2, 3, \ldots, m-1) \\ &d_{m2} = \frac{e_{12}^2 H}{4d_{m1}}, \quad d_{m3} = -\frac{6d_{m1}^3(3H - 4d_{m1}^2)}{e_{12}^2 H^2}, \quad d_{m4} = 0, \\ &e_{11} = \frac{-4d_{m1}^2}{e_{12}H}, \quad e_{13} = \frac{8d_{m1}^4(3H - 4d_{m1}^2)}{e_{12}^3 H^3}, \\ &c = \frac{e_{12}^2 H^3(H - 2d_{m1}^2)}{16d_{m1}^4(H - d_{m1}^2)}, \end{aligned} \tag{14}$$

where $H = \boldsymbol{X} \cdot \boldsymbol{X}$, $\boldsymbol{X} = (d_{11}, d_{21}, \ldots, d_{m1})$ and the relationships of parameters in Eq. (14) satisfy $m \geq 2$, $e_{12}d_{m1}(H - d_{m1}^2) \neq 0$, $H - 2d_{m1}^2 > 0$. Expressions (14) contain the case (13) that the vector is two-dimensional. We can set $f = w + v$, where $w = \sum_{i=1}^{m} f_i^2 + c$, $v = \exp(l_1)$, and under the parameter relationships of expressions (14), we have

$$\begin{aligned} v &= \exp\left(\frac{-4d_{m1}^2}{e_{12}H}x + e_{12}y + \frac{8d_{m1}^4(3H - 4d_{m1}^2)}{e_{12}^3 H^3}t + e_{14}\right), \\ w &= Hx^2 + \frac{e_{12}^4 H^3}{16d_{m1}^4}y^2 + \frac{36d_{m1}^4}{e_{12}^4 H}t^2 + \frac{e_{12}^2 H(2d_{m1}^2 - H)}{2d_{m1}^2}xy \\ &\quad + \frac{12d_{m1}^2(2d_{m1}^2 - H)}{e_{12}^2 H}xt + \frac{3(H^2 - 8d_{m1}^2 H + 8d_{m1}^4)}{H}yt \\ &\quad + \frac{e_{12}^2 H^3(H - 2d_{m1}^2)}{16d_{m1}^4(H - d_{m1}^2)}. \end{aligned} \tag{15}$$

Finally, substituting $f = w + v$ with above relationship expressions (15) into Eq. (2), we get 0, so $f = w + v$ is an alternative form of the expression (12) with relations (14). It is worth noting that we get the new localized solutions $u = -3(\ln w)_{xx}$. By using $u = -3(\ln f)_{xx}$, the interaction solutions of Eq. (1) are

$$u = -3(\ln f)_{xx} = \frac{-6\left((\boldsymbol{X} \cdot \boldsymbol{X})x + \dfrac{8d_{m1}^4}{e_{12}^2(\boldsymbol{X} \cdot \boldsymbol{X})^2}v\right)f + 3f_x^2}{f^2}. \tag{16}$$

We take $m = 5$ as an example. Setting the special values of the following parameters are $\boldsymbol{X} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, 1, 1)$, $e_{12} = -2$, $e_{14} = 0$, then we substitute them into Eq. (16) to procure

$$u = -3 \cdot \frac{(w_1 + v_1)_{xx}(w_1 + v_1) - (w_1 + v_1)_x^2}{(w_1 + v_1)^2}, \tag{17}$$

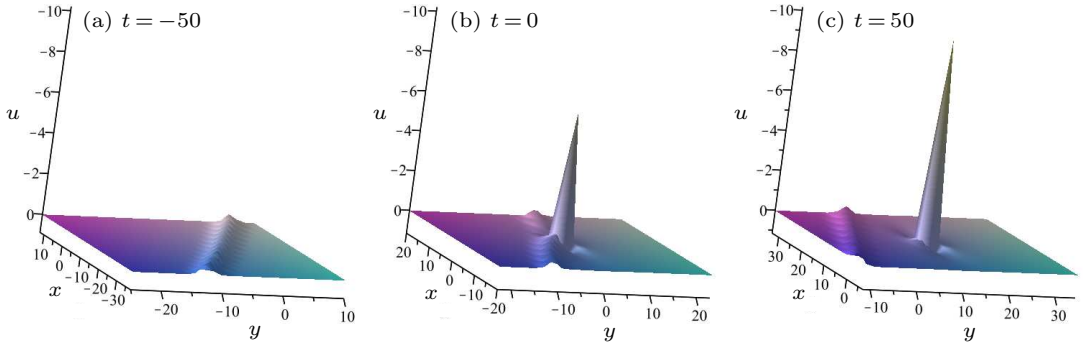


Fig. 2. 3D plots of the interaction solution u Eq. (17) of Eq. (1).

where $w_1=\frac{41}{16}x^2+\frac{68921}{4096}y^2+\frac{36}{41}t^2-\frac{369}{128}xy-\frac{27}{41}xt-\frac{4557}{656}yt+\frac{620289}{409600}$, $v_1=\exp(\frac{32}{41}x-2y-\frac{15104}{68921}t)$. We can also have the localized wave $u_1=-3(\ln w_1)_{xx}$ alone, the lowest trough of which is localized at $(x_4,y_4)=(\frac{432t}{1681},\frac{384t}{1681})$ and the corresponding value is $-51200/5043$. If we substitute $x=x_4$ and $y=y_4$ into the expression (17), $\lim_{t\to+\infty}u=-51200/5043$ will be obtained. It means that the localized wave is separated from the stripe wave after a period of time and exists stably. Observing Fig. 2, these plots show physical characteristic of a interaction between the localized wave and the stripe wave, and it can be seen that only one stripe soliton exists at $t=-50$, the localized wave generates from the inside of stripe wave at $t=0$, and they have separated at $t=50$. This interaction is similar to the interaction between the lump and stripe wave in Ref. [50].

4. Mixed solutions corresponding to linear superposition of exponential and hyperbolic function types

In this section we will give solutions of Eq. (2) in the form of the linear superposition of exponential and hyperbolic function types. Setting the solution of Eq. (2) is

$$\begin{aligned} f \;=\; & d_1\sum_{i=1}^{m}k_i\cosh(b_{i1}x+b_{i2}y+b_{i3}t) \\ & +d_2\sum_{j=1}^{n}l_j\exp(h_{j1}x+h_{j2}y+h_{j3}t), \end{aligned} \tag{18}$$

which needs to satisfy $d_1,d_2=0$ or 1, ${d_1}^2+{d_2}^2\neq 0$, $k_i,l_j>0$, $i=1,2,3,\ldots,\ m,\ j=1,2,3,\ldots,n$. For nonlinear differential equations, finding solutions that can be linearly superposed, such as the special linear superposition solutions in Ref. [51], is quite difficult. Therefore, we believe it is very meaningful to obtain the linear superposition solutions (18) of Eq. (2).

In order to obtain parameter relations, we firstly take $d_1=1$, $d_2=0$, $m=2$, and then substitute the expression (18) into Eq. (2). We collect the terms with regard to sinh, cosh functions and independent variables x, y, and t. Then coefficients are made equal to zero, which results in algebraic equations. These equations can be solved by Maple: $b_{11}=-1/b_{12}$, $b_{13}=-1/2b_{12}^3$, $b_{22}=-1/b_{21}$, $b_{23}=b_{21}^3/2$, which can be written as

$$b_{11}=-\frac{1}{b_{12}},\ \ b_{13}=-\frac{1}{2b_{12}^3},\ \ b_{21}=-\frac{1}{b_{11}},\ \ b_{23}=\frac{b_{21}^3}{2}. \tag{19}$$

Expression (19) has a significantly obvious regularity, so we directly infer that parameter relations of solutions with the form of Eq. (18) satisfy

$$b_{i2}=-\frac{1}{b_{i1}},\ \ b_{i3}=\frac{b_{i1}^3}{2},\ \ i=1,2,3,\ldots,m,$$

$$h_{j2}=-\frac{1}{h_{j1}},\ \ h_{j3}=\frac{h_{j1}^3}{2},\ \ j=1,2,3,\ldots,n, \tag{20}$$

and they must satisfy $b_{i1}\neq 0$, $h_{j1}\neq 0$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$. The following proves that the expression (18) with parameters (20) can be used to solve Eq. (2).

Proof To start with, the expression (18) with parameters (20) can be written as

$$\begin{aligned} & f=\sum_{i=1}^{n}f_i, \\ & f_i=k_i\cosh(b_{i1}x-\frac{1}{b_{i1}}y+\frac{b_{i1}^3}{2}t)+l_i\exp(h_{i1}x-\frac{1}{h_{i1}}y+\frac{h_{i1}^3}{2}), \\ & k_i,l_i\geq 0,\ \ b_{i1},h_{i1}\neq 0,\ \ i=1,2,3,\ldots,n,\ \ \sum_{i=1}^{n}(k_i^2+l_i^2)\neq 0. \end{aligned} \tag{21}$$

It is not difficult to verify that f Eq. (21) is a solution of Eq. (2) for $n=1$, $n=2$. It is assumed that f Eq. (21) solves Eq. (2) for $n-1$, so we need to prove the case for n. Let us set $f=p+q$, $p=\sum_{i=1}^{n-1}f_i$, and $q=f_n$. Then $p+q$ is substituted into Eq. (2), and due to the inductive hypothesis, it is necessary to prove the result of

$$\begin{aligned} & 2(p_{yt}q+q_{yt}p)-2(p_tq_y+q_tp_y)-(p_{xxxy}q+q_{xxxy}p) \\ & +3(p_{xxy}q_x+q_{xxy}p_x)-3(p_{xx}q_{xy}+q_{xx}p_{xy}) \\ & +3(p_{xxx}q_y+q_{xxx}p_y)-3(p_{xx}q+q_{xx}p)+6p_xq_x \end{aligned} \tag{22}$$

is zero. The expression (22) can be turned into

$$\begin{aligned} & \sum_{i=1}^{n-1}\Big[2(f_i)_{yt}(f_n)+2(f_n)_{yt}(f_i)-2(f_i)_t(f_n)_y-2(f_n)_t(f_i)_y \\ & -(f_n)_{xxxy}(f_i)-(f_i)_{xxxy}(f_n)+3(f_n)_{xxy}(f_i)_x+3(f_i)_{xxy}(f_n)_x \\ & -3(f_n)_{xx}(f_i)_{xy}-3(f_i)_{xx}(f_n)_{xy}+3(f_n)_{xxx}(f_i)_y+3(f_i)_{xxx}(f_n)_y \\ & -3(f_n)_{xx}(f_i)-3(f_i)_{xx}(f_n)+6(f_n)_x^2(f_i)_x^2\Big]. \end{aligned} \tag{23}$$

For arbitrary $i=1,2,3,\ldots,n-1$, if $p=f_i$, $q=f_n$, it will be easy verified that the result of the expression (22) is zero. That is to say, the formula in the summation symbol in the expression (23) results in zero, and this completes the proof.

Based on the solutions (18) with parameter relations (20) of Eq. (2), we obtain solutions of Eq. (1)

$$u=\frac{-3\cdot f_{xx}f+3f_x^2}{f^2}, \tag{24}$$

where

$$\begin{aligned} f \;=\; & d_1\sum_{i=1}^{m}k_i\cosh\left(b_{i1}x-\frac{1}{b_{i1}}y+\frac{b_{i1}^3}{2}t\right) \\ & +d_2\sum_{j=1}^{n}l_j\exp\left(h_{ji}x-\frac{1}{h_{j1}}y+\frac{h_{j1}^3}{2}\right). \end{aligned}$$

The above group of formulas shows the propagation of strip waves. For instance, we take $m=3$, $n=1$, $b_{11}=1$,

$b_{21}=2$, $b_{31}=4$, $h_{11}=3$, $d_1=d_2=k_1=k_2=k_3=l_1=1$, and then the solution of Eq. (1) is

$$u=\frac{-3(f_2)_{xx}f_2+3(f_2)_x^2}{(f_2)^2}, \tag{25}$$

where $f_2=\cosh(x-y+\frac{t}{2})+\cosh(2x-\frac{y}{2}+4t)+\cosh(4x-\frac{y}{4}+32t)+\exp(3x-\frac{y}{3}+\frac{27t}{2})$. From Fig. 3, it can be seen that multiple stripe waves merge into a pair of stripe waves with the cross shape at $t=0$, while they present bifurcate shapes at the rest of the time.

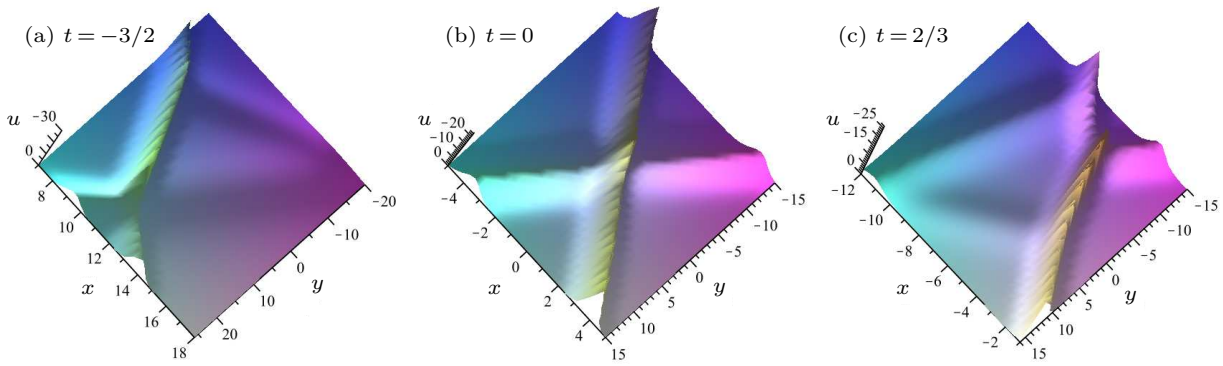


Fig. 3. 3D plots of the solution u Eq. (25) of Eq. (1).

5. Conclusion

By mainly using plausible reasoning and mathematical software Maple, we derived the localized solutions, the interaction solutions and the mixed solutions of Eq. (1) corresponding to the superposition formulas. In Section 2, in regard to the simple parameter expressions of lump solutions, we transformed them into forms of determinants or vectors, then the parameter relations of lump-type solutions were given by analogy in expressions (6). Based on the additive form of two or three quadratic functions, the general forms with the summation of m $(m\geq 2)$ positive quadratic functions were summed up directly in expressions (8). We expressed the solutions of Eq. (2) as the summation of m $(m\geq 2)$ positive quadratic functions, the coefficients of which were the rational operation of vector inner products in Eq. (9). And we finished up the localized solutions of Eq. (1) by $u=-3(\ln f)_{xx}$. Moreover, we gave mathematical properties and dynamic properties of solutions (10). Figure 1 showed the dynamic properties of the localized wave corresponding to the summation of four positive quadratic functions. Section 2 enlightened us that the parameters relations belonging to the part of quadratic functions can be expressed as the rational operation of vector product when we derive the interaction solutions. So in Section 3, based on the simple parameter relations in the form of two quadratic functions added with a exponential function, we directly gave solutions of Eq. (2) that contained the summation of m $(m\geq 2)$ positive quadratic functions. Interaction solutions (16) of Eq. (1) contained localized solutions $-3(\ln w)_{xx}$ and w was from Eq. (15). From the concrete solution (17), we found that the localized wave would get rid of influence of the stripe wave and remain steady after a period of time. Figure 2 showed a separation process of the localized wave and the stripe wave. In Section 4, utilizing Maple and induction, we obtained the mixed solution expressions (24) corresponding to the summation of arbitrary number of exponential and hyperbolic cosine function types. Figure 3 showed the dynamic properties of several stripe waves.

For partial differential equations that could be transformed into Hirota bilinear forms, as long as their plain forms of lump solutions or interaction solutions possess good symmetry and regularity, the method is applicable.

Acknowledgments

Project supported by the National Natural Science Foundation of China (Grant No. 12061054) and Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region of China (Grant No. NJYT-20-A06).

References

[1] Solli D R, Ropers C, Koonath P and Jalali B 2014 *Nature* **450** 1054
[2] Stenflo L and Marklund M 2009 *J. Plasma Phys.* **76** 293
[3] Benetazzo A, Ardhuin F, Bergamasco F, Guimarães P V, Schwendeman M, Sclavo M, Thomson J and Torselo A 2017 *Nature* **7** 8276
[4] Liu B, Zhang X E, Wang B and Lü X 2022 *Mod. Phys. Lett. B* **36** 2250057
[5] Imai K and Nozaki K 1996 *Prog. Theor. Phys.* **96** 521
[6] Pan C C, Baronio F and Chen S H 2020 *Acta Phys. Sin.* **69** 010504 (in Chinese)
[7] Song C Q and Zhu Z N 2020 *Acta Phys. Sin.* **69** 010204 (in Chinese)
[8] Wazwaz A M 2008 *Appl. Math. Comput.* **201** 489
[9] Hu H C and Liu F Y 2020 *Chin. Phys. B* **29** 40201
[10] Abdel Rady A S, Osman E S and Khalfallah M 2010 *Appl. Math. Comput.* **217** 1385
[11] Wazwaz A M 2007 *Appl. Math. Comput.* **190** 633
[12] Fu W, Zhang D J and Zhou R G 2009 *Chin. Phys. Lett.* **31** 090202
[13] Chen J C, Zheng Y M and Hu Y H 2017 *Chin. Phys. Lett.* **34** 010201
[14] Liu P, Xu H R and Yang J R 2020 *Acta Phys. Sin.* **69** 010203 (in Chinese)
[15] Zhao Y W, Xia J W and Xing Lü 2022 *Nonlinear Dyn.* **108** 4195
[16] Yang X Y, Zhang Z and Li B 2009 *Chin. Phys. B* **29** 100501
[17] Geng X G 2003 *J. Phys. A: Math. Gen.* **36** 2289
[18] Geng X G and Ma Y L 2007 *Phys. Lett. A* **369** 285

[19] Huang L L and Chen Y 2019 *Commun. Nonlinear Sci. Numer. Simul.* **67** 237
[20] Xu D H and Lou S Y 2020 *Acta Phys. Sin.* **69** 014208 (in Chinese)
[21] Li M, Wang B T, Xu T and Shui J J 2020 *Acta Phys. Sin.* **69** 010502 (in Chinese)
[22] Zhang Y and Zhou R G 2017 *Chin. Phys. Lett.* **31** 110203
[23] Chen K and Zhang D J 2016 *Chin. Phys. Lett.* **33** 100201
[24] Wen X Y and Wang H T 2020 *Acta Phys. Sin.* **69** 010205 (in Chinese)
[25] Zhang H Q and Ma W X 2017 *Comput. Math. Appl.* **73** 2339
[26] Tang Y N and Liang Z J 2022 *Part. Differ. Equ. Appl. Math.* **5** 110326
[27] Lü J Q, Bilige S and Chaolu T 2018 *Nonlinear Dyn.* **91** 1669
[28] Raza S T R, Bibi I, Younis M and Bekir A 2021 *Chin. Phys. B* **30** 010502
[29] Liu Y K and Li B 2017 *Chin. Phys. Lett.* **34** 10202
[30] Tang Y N, Tao S Q, Zhou M L and Guan Q 2017 *Nonlinear Dyn.* **89** 429
[31] Pu J C and Hu H C 2018 *Appl. Math. Lett.* **85** 77
[32] Fang T, Wang H, Wang Y H and Ma W X 2019 *Commun. Theor. Phys.* **71** 927
[33] Liu Y K and Li B 2017 *Chin. Phys. Lett.* **34** 10202
[34] Lou S Y 2020 *Chin. Phys. B* **29** 80502
[35] Huang L L, Yue Y F and Chen Y 2018 *Comput. Math. Appl.* **76** 831
[36] Wang H F and Zhang Y F 2020 *Chin. Phys. B* **29** 40501
[37] Qian C, Rao J G, Liu Y B and He J S 2016 *Chin. Phys. Lett.* **33** 110201
[38] Zhang R F and Sudao B 2019 *Nonlinear Dyn.* **95** 3041
[39] Shi Y B and Zhang Y 2017 *Commun. Nonlinear Sci. Numer. Simul.* **44** 120
[40] Issasfa A and Lin J 2020 *Commun. Theor. Phys.* **72** 125003
[41] Lou S Y 2020 *Acta Phys. Sin.* **69** 010503 (in Chinese)
[42] Tang X Y, Cui C J, Liang Z F and Ding W 2021 *Nonlinear Dyn.* **105** 2549
[43] Yin Y H, Lü X and Ma W X 2022 *Nonlinear Dyn.* **108** 4181
[44] Yue Y F, Huang L L and Chen Y 2019 *Appl. Math. Lett.* **89** 70
[45] Chen M D, Li X, Wang Y and Li B 2017 *Commun. Theor. Phys.* **67** 595
[46] Zheng P F and Jia M 2018 *Chin. Phys. B* **27** 120201
[47] Liu J G 2018 *Appl. Math. Lett.* **86** 36
[48] Zhang L L, Yu J P, Ma W X, Khalique C M and Sun Y L 2021 *Nonlinear Dyn.* **104** 4317
[49] Lü X and Chen S J 2021 *Nonlinear Dyn.* **103** 947
[50] Lü X and Chen S J 2021 *Commun. Nonlinear Sci. Numer. Simul.* **103** 105939
[51] Hao X Z and Lou S Y 2022 *Math. Method Appl. Sci.* **45** 5774