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Generalized uncertainty principle from long-range kernel effects: The case of the Hawking black hole temperature

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We prove the existence of an analogy between spatial long-range interactions, which are of the convolution-type introduced in non-relativistic quantum mechanics, and the generalized uncertainty principle predicted from quantum gravity theories. As an illustration, black hole temperature effects are discussed. It is observed that for specific choices of the moment's kernels, cold black holes may emerge in the theory.

Keywords: long-range kernel effects, generalized uncertainty principle

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Spatial nonlocality has largely been studied in the literature based on various phenomenological approaches.[1,2] It plays a fundamental role in quantum entanglement, where a measurement on one of the entangled particles separated by certain unlimited distances instantaneously modifies the quantum state of the other particle. Theoretically, this modification is due to the change of the density matrix to a pure state.[3] Such a quantum paradox has been solved using the framework of the "many-worlds interpretation". Although the nonlocal aspect of quantum mechanics has been observed in the well-known Aharonov–Bohm effect through the emergence of global topological features,[4] it is considered one of the strangest properties of the quantum world. Several research papers have been published claiming to solve this fundamental problem, and to prove that quantum nonlocality does not exist (being one of the relics of the Copenhagen interpretation) in which each observed quantity has precisely one value at any instant.[5–8] No matter what the future will reveal for us, we believe that nonlocality is of particular significance in quantum mechanics and should be incorporated into the strange world of quantum theory. From the analytical side, nonlocal effects are usually accompanied with higher-order derivative (HOD) terms in the dynamical equations of motion, e.g., the Schrödinger equation.[9,10] In general, the incorporation of HOD terms in the Schrödinger equation may be performed without passing by the nonlocal approach, e.g., using the Doebner–Goldin modification,[11–14] which is limited to the 4$^{\text{th}}$-order term. Nevertheless, several interesting implications and outcomes have been obtained by studying the consequential 4$^{\text{th}}$-order Schrödinger equation.[15–19]

The aims of the present study are: first, to construct a quantum diffusion equation by taking into account the long-range kernel effects which are considered in reaction-diffusion theory used in biological systems[20–23] and neutron diffusion processes[24] among others;[25–27] and second, to show that such a kernel approach is associated with the generalized uncertainty principle (GUP), which was introduced to account for the existence of minimal and maximal lengths in nature.[28–36] The presence of a minimum length suggests, in general, that space–time may be fundamentally discrete, and hence, Planck-scale corrections to the basic Heisenberg's uncertainty principle must be introduced. In fact, one of the main outcomes of the GUP is the modification of the commutation relation between the position and its conjugate momentum, which results in shifting of the energy levels. These modifications create extra terms for all quantum mechanical systems, which could be detected by observers. In general, the GUP reads as[28]

$$\Delta p_i \Delta x_i \geq \frac{\hbar}{2}\left(1+\beta\left((\Delta p)^2+\langle p\rangle^2\right)\right. \\ \left.+2\beta\left((\Delta p_i)^2+\langle p_i\rangle^2\right)\right), \quad i=1,2,3, \tag{1}$$

where x and p are the position and conjugate momentum of a particle with the corresponding quantum observables denoted by $\hat{x}$ and $\hat{p}\beta = \beta_0 l_{\text{P}}^2/\hbar^2$, l_{P} is the Planck's length, $\hbar$ is the Planck's constant, β_0 is a constant and, for any operator $\hat{O}$, we have $\Delta O^2 \equiv \langle \hat{O}^2\rangle - \langle \hat{O}\rangle^2$ (mean square deviation) and the bracket denotes averaging over a certain quantum state $|\varphi\rangle$. Recall that, in general, the position and momentum are represented as multiplication or differentiation operators acting on square integrable position- or momentum-space wave functions, $\varphi(x) := \langle x|\varphi\rangle$or $\varphi(p) := \langle p|\varphi\rangle$. The Heisenberg algebra

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is therefore modified and takes the form

$$[x_i, p_j] = \mathrm{i}\hbar\left(\delta_{ij} + \beta\delta_{ij}p^2 + 2\beta_{ij}p_i p_j\right), \tag{2}$$

where $p^2 = \sum_j p_j p_j$ and, besides, $[x_i, x_j] = [p_i, p_j] = 0$.[37] This version of the GUP could be tested by considering its physical effects on several phenomenological events in nature.[38–40] Assuming for simplicity that off-diagonal terms vanish, i.e., $\beta_{ij} = 0$, $i \neq j$, $\beta_{xx} = \beta_{yy} = \beta_{zz} = \beta$ (short-distance isotropic space) and hence $p_x^2 = p_y^2 = p_z^2 = p^2$, then the momentum operator takes the deformed form $\tilde{p}_i = (1 + 3\beta p^2)p_i$. The minimum length is therefore $(\Delta x)_{\min} \propto \hbar\sqrt{\beta}$. In that case, one may check that the Schrödinger Hamiltonian operator is extended to the first order in the correction term to $\hat{H}_{\mathrm{s}} = \frac{1}{2m}p^2 + \frac{3}{2m}\beta p^4 + V$, where V is the potential. We will prove that such an extension may be associated with a long-range kernel expansion.

To start, we introduce the following nonlocal Schrödinger integro-differential equation:[22,26,41,42]

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} &= -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} K\left(x - x'\right)\psi(x',t)\,\mathrm{d}x' \\ &\quad + V(x)\psi(x,t), \quad t > 0, \end{aligned} \tag{3}$$

$$\psi(x,0) = \psi_0(x), \quad x \in \boldsymbol{R}, \tag{4}$$

where $\psi(x,t)$ is the wave function of the particle and $K(x - x') : \boldsymbol{R} \to \boldsymbol{R}$ is a symmetric kernel measuring the influence of surrounding neighbors such that

$$K(x - x') = K(x' - x), \tag{5}$$

$$K(x - x') \overset{|x-x'|\to\infty}{\longrightarrow} 0. \tag{6}$$

Since the kernel is assumed to be symmetric, then the following integral holds:

$$\begin{aligned} &\int_{-\infty}^{+\infty} K\left(x - x'\right)\psi(x',t)\,\mathrm{d}x' \\ &\underset{y=x-x'}{=} \int_{-\infty}^{+\infty} K(y)\,\psi(x-y,t)\,\mathrm{d}y \\ &= \int_{-\infty}^{+\infty} K(y)\sum_{k=0}^{\infty}(-1)^k\frac{y^k}{k!}\frac{\partial^k\psi(x,t)}{\partial x^k}\,\mathrm{d}y, \end{aligned} \tag{7}$$

$$\int_{-\infty}^{+\infty} y^{2M+1}K(y)\,\mathrm{d}y = 0, \quad M = 0,1,2,\ldots \tag{8}$$

Introducing the moments by

$$M_{2m} = \frac{1}{(2M)!}\int_{-\infty}^{+\infty} y^{2M}K(y)\,\mathrm{d}y, \quad M = 0,1,2,\ldots, \tag{9}$$

we can hence write Eq. (3) as

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} &= -\frac{\hbar^2}{2m}\left(M_2\Delta + M_4\Delta^2 + \cdots\right)\psi(\boldsymbol{x},t) \\ &\quad + W(\boldsymbol{x})\psi(\boldsymbol{x},t), \quad t > 0, \end{aligned} \tag{10}$$

which may be written as

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial\tilde{\psi}(\boldsymbol{x},\tilde{t})}{\partial\tilde{t}} &= \left(\frac{p^2}{2m} - \frac{M_4}{M_2}\frac{p^4}{2m\hbar^2} + \cdots\right)\tilde{\psi}(\boldsymbol{x},\tilde{t}) \\ &\quad + R(\boldsymbol{x})\tilde{\psi}(\boldsymbol{x},\tilde{t}), \quad \tilde{t} > 0, \end{aligned} \tag{11}$$

where $W(\boldsymbol{x}) = M_0 - V(\boldsymbol{x})$, $R(\boldsymbol{x}) = W(\boldsymbol{x})/M_2$ is the effective potential, $\tilde{t} = M_2 t$ and $\tilde{\psi}(\boldsymbol{x},t) = M_2\psi(\boldsymbol{x},t)$. The effective Schrödinger Hamiltonian of the theory is $\hat{H}_{\mathrm{s}} = \frac{1}{2m}p^2 + \frac{3}{2m}\tilde{\beta}p^4 + R$, where $\tilde{\beta} = -M_4/(\hbar^2 M_2)$. Hence, one is required to have $M_2 > 0$ and $M_4 < 0$. In general, $|M_4| < |M_2|$ and is a measure of the long-range effects. It is therefore expected that $(\Delta x)_{\min} \propto \sqrt{-M_4/M_2}$ and

$$[x_i, p_j] = \mathrm{i}\hbar\left(\delta_{ij} - \frac{M_4}{\hbar^2 M_2}\delta_{ij}p^2 - \frac{M_4}{\hbar^2 M_2}p_i p_j\right). \tag{12}$$

Hence, the moments of the kernels are associated with the $\tilde{\beta}$ parameter in the framework of the GUP. To illustrate, we select the following Gaussian kernel, although various kernels may be selected:

$$K(x) = l_{\mathrm{P}}^{-3}\left(\alpha_1\,\mathrm{e}^{-(x/\beta_1 l_{\mathrm{P}})^2} - \alpha_2\,\mathrm{e}^{-(x/\beta_{2l_{\mathrm{P}}})^2}\right), \tag{13}$$

with $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \boldsymbol{R}^+$. In fact, we have considered a non-compactly supported Gaussian symmetric kernel of the form $K(x) \propto \mathrm{e}^{-x^2/b^2}$, where $0 < a \ll 1$ and $0 < b \ll 1$, i.e., the long-range effects are weak and $K(x)$ tends to zero quickly. In many situations the restriction to the Gaussian kernel represents a simple approximation; yet, different kernels may be used which are not necessarily symmetric. In general, a Gaussian kernel function is used to describe quantum nonlocal diffusion processes.[26] Recall that equation (3) is comparable to a nonlocal diffusion equation with imaginary time. If the kernel has a second moment, then the Cauchy condition in 1D $\int_{\boldsymbol{R}}|x|^2K(x)\,\mathrm{d}x < +\infty$ holds; yet, by symmetry, the associated first moment is zero. The Fourier transform of the kernel $\hat{K}(w) = \mathrm{F.T.}(K(x))$ accordingly takes the asymptotic form $(w \to 0)$ and the form $\hat{K}(w) \approx 1 - |w|^2 + O(|w|^2)$. Hence, the nonlocal Schrödinger diffusion equation obeys the Gaussian law $K(x) = \mathrm{e}^{-x^2}$. If, for instance, we consider a more general expansion of $\hat{K}(w)$ of the form $\hat{K}(w) \approx 1 - c\,|w|^\alpha + O(|w|^\alpha)$, $c > 0$, $0 < \alpha < 2$. This special form leads, in general, to a nonlocal fractional Laplace parabolic equation.[43] If we choose $\hat{K}(w) \approx 1 + cw^2\ln w$, $c > 0$, then it has been proven in[43] that the asymptotic behavior is still given by a solution of the diffusion equation, yet is viewed in a different time scale. Although asymmetric kernels are very common in nonlocal diffusion and dispersal processes,[44] they deserve to be considered in quantum mechanics. This will be discussed in a forthcoming work.

After simple integration, we find

$$M_2 = \frac{\sqrt{\pi}}{4}\left(\alpha_1\beta_1^3 - \alpha_2\beta_2^3\right), \tag{14}$$

$$M_4 = \frac{\sqrt{\pi}}{32} l_{\rm P}^2 \left(\alpha_1\beta_1^5 - \alpha_2\beta_2^5\right). \tag{15}$$

The constraints $M_2 > 0$ and $M_4 < 0$ are verified if

$$\alpha_1 \in \boldsymbol{R}^+, \quad \beta_1 \in \boldsymbol{R}^+, \quad 0 < \alpha_2 < \alpha_1, \tag{16}$$

and

$$(-1)^{2/5}\sqrt[5]{\frac{\alpha_1}{\alpha_2}}\beta_1 < \beta_2 < (-1)^{2/3}\sqrt[3]{\frac{\alpha_1}{\alpha_2}}\beta_1. \tag{17}$$

We set, for illustration reasons, $\alpha_1 = 1$, $\alpha_2 = 0.01$, $\beta_1 = 1$ and $\beta_2 = 3$, which give $M_2 \approx 0.32$ and $M_4 \approx -0.08 l_{\rm P}^2$, i.e., $\tilde{\beta} \approx l_{\rm P}^2/4\hbar^2 = \beta_0 l_{\rm P}^2/\hbar^2$ and $\beta_0 = 1/4$. Therefore, an association between the long-range kernel effects and the GUP is obtained, which leads to a modified commutation relation between the position coordinates and momenta. This unquestionably would affect a host of quantum phenomena in various fields, ranging from atomic physics to condensed matter and statistical physics. From the relativistic side, it is natural to expect that the dispersion relation is modified and takes the form $E^2 = p^2c^2(1 - M_4/\hbar^2 M_2)^2 + m^2c^4$, where c is the celerity of light[45]. It depends on the higher-order moments/long-range kernel effects. As an illustration, we discuss the implications of the present approach on black hole temperature effects.

In fact, it has been argued that, at large scales, the GUP takes the subsequent form[46]

$$\Delta p_i \Delta x_i \geq \frac{\hbar}{2}\left(1 + \alpha\frac{l_{\rm P}^2(\Delta p)^2}{\hbar^2}\right). \tag{18}$$

This leads to a generalized Hawking's temperature for evaporating black holes, which is given by

$$\begin{aligned} T &= \frac{Mc^2}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha\hbar c}{GM^2}}\right) \\ &= \frac{Mc^2 l_{\rm P}^2}{4\hbar^2\tilde{\beta}\pi}\left(1 - \sqrt{1 - \frac{\hbar^3\tilde{\beta}c}{GM^2 l_{\rm P}^2}}\right). \end{aligned} \tag{19}$$

Here, M is the black hole mass and G is the gravitational constant. During the evaporation process, the volume of the black hole shrinks and stops when $M \to M_{\rm P} = \sqrt{\hbar c/G}$ (Planck's mass) for $\tilde{\beta} \sim O(1)$.[46] The importance of the GUP in the black hole evaporation process comes from the avoidance of the temperature's blow up when $M \to 0$. Based on the nonlocal range interactions introduced in this study, we find

$$\begin{aligned} T &= -\frac{Mc^2 l_{\rm P}^2 M_2}{4M_4\pi}\left(1 - \sqrt{1 + \frac{M_4\hbar c}{M_2 GM^2 l_{\rm P}^2}}\right) \\ &= -\frac{Mc^2 l_{\rm P}^2 M_2}{4\pi M_4}\left(1 - \sqrt{1 + \frac{M_4 M_{\rm P}^2}{M_2 M^2 l_{\rm P}^2}}\right). \end{aligned} \tag{20}$$

We observe that, for $M_4/M_2 = -\gamma^2 l_{\rm P}^2$, where γ is a real parameter, equation (20) is reduced to

$$T = \frac{Mc^2}{4\pi\gamma^2}\left(1 - \sqrt{1 - \gamma^2\frac{M_{\rm P}^2}{M^2}}\right), \tag{21}$$

and therefore, for $M \geq \gamma M_{\rm P}$, we get $T \approx M_{\rm P}c^2/4\pi\gamma$. For high values of the parameter γ, the temperature of the black hole is less than $M_{\rm P}c^2$, e.g., exotic cold black holes.[47] For the Gaussian kernel (13) with $M_2 \approx 0.32$ and $M_4 \approx -0.08 l_{\rm P}^2$, we find

$$\begin{aligned} T &= \frac{Mc^2}{\pi}\left(1 - \sqrt{1 - \frac{\hbar c}{4GM^2}}\right) \\ &= \frac{Mc^2}{\pi}\left(1 - \sqrt{1 - \frac{M_{\rm P}^2}{4M^2}}\right), \end{aligned} \tag{22}$$

and hence, for $M \geq M_{\rm P}/2$, $T < Mc^2$. This is the temperature of the black hole remnants.[48,49] One may select different numerical values, which may give rise to lower black hole temperatures.

To conclude, we have proved that the quantum diffusion equation obtained through the framework of "long-range kernel effects" may be correlated to the Generalized Uncertainty Principle predicted from quantum gravity effects. This is in agreement with the approach introduced in Refs. [50,51], where quantum gravity states may remain coherent at very large scales and they could play a critical role in the understanding of macroscopic gravitational effects. This correlation between the GUP and long-range effects may have interesting consequences in several fields, including higher-dimensional black holes,[52] cosmology,[53] dark energy[54,55] and quantum geometry[56,57] among others.[58–64] A variable family of nonnegative integrable kernels may be chosen which are not necessarily symmetric, which may give rise to various phenomenological quantum effects. Work in these directions is under construction.

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