# Genuine Einstein–Podolsky–Rosen steering of generalized three-qubit states via unsharp measurements

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We aim to explore all possible scenarios of $(1 \to 2)$ (where one wing is untrusted and the others two wings are trusted) and $(2 \to 1)$ (where two wings are untrusted, and one wing is trusted) genuine tripartite Einstein–Podolsky–Rosen (EPR) steering. The generalized Greenberger–Horne–Zeilinger (GHZ) state is shared between three spatially separated parties, Alice, Bob and Charlie. In both $(1 \to 2)$ and $(2 \to 1)$, we discuss the untrusted party and trusted party performing a sequence of unsharp measurements, respectively. For each scenario, we deduce an upper bound on the number of sequential observers who can demonstrate genuine EPR steering through the quantum violation of tripartite steering inequality. The results show that the maximum number of observers for the generalized GHZ states can be the same with that of the maximally GHZ state in a certain range of state parameters. Moreover, both the sharpness parameters range and the state parameters range in the scenario of $(1 \to 2)$ steering are larger than those in the scenario of $(2 \to 1)$ steering.

**Keywords:** quantum information, quantum steering, generalized Greenberger–Horne–Zeilinger (GHZ) state

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## 1. Introduction

Since entanglement is an important resource in quantum communication tasks, characterizing entanglement becomes one of the key problems for information theoretical applications. There are three common ways to characterize entanglement. The first one is entanglement witnesses,[1–3] but most of entanglement witnesses need to know the state information in advance. Since entanglement is a necessary condition for Bell nonlocality which is the strongest quantum correlation.[4,5] The violations of Bell-type inequalities can also be used to verify entanglement.[6–8] However, detecting Bell nonlocality requires high detector efficiencies and low level of noise, and thus their implementation is limited. Einstein–Podolsky–Rosen (EPR) steering[9,10] is a form of quantum correlation that lies between entanglement and Bell-nonlocality. In the process of entanglement detection, EPR steering has more noise resistant than Bell nonlocality and requires less experimental requirements than entanglement witnesses.[5]

The EPR steering is important in quantum network applications. The EPR steering can establish a link with quantum secret sharing protocols.[11–13] Suppose Alice can steer Bob, which means that multiple Alice can share a secret message with a single Bob. In 1935, Schrödinger proposed EPR steering: two parties, Alice and Bob, share a two-qubit entangled state. Alice (untrusted party) steers Bob's subsystem (trusted party) into eigenstate of position or momentum by choosing the measurement.[14] In 2007, Wiseman *et al*. presented an operational definition of steering through the mathematical formulation,[5] which proved that the steerable states are a strict subset of entangled states. Since then, the EPR steering theory has been rapidly developed.[15–26] Many studies of EPR steering theory were attributed to the solution of semidefinite programming (SDP). Skrzypczyk *et al*. summarized the application of SDP in EPR steering.[15] This is a powerful technique that can be used to study many problems related to steering, including steering detection,[18] steering quantification,[19] and cloning of steering.[27] A series of steering inequalities[20–23] and steering witnesses[24–26] have been proposed for quantifying steering.

In various information processing tasks, detecting genuine entanglement sequentially can be used to recycle genuine multipartite entangled resources.[28] In 2015, Silva *et al*. first explored sequential scenario: a two-qubit entangled state is shared between two spatially separated parties, Alice and Bob. Alice performs projective measurement on the first wing and multiple Bob perform unsharp measurements on the second wing by choosing the same sharpness parameters.[29] The EPR steering is a widely studied concept that captures sequence of measurements.[30–33] For two-qubit states, EPR steering had been deeply and intensively discussed in Ref. [30]. For a single Alice steers multiple Bob via unsharp measurements, at most 2 Bob can violate the Clauser–Horne–Shimony–Holt-

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like inequality with a single Alice and at most 3 Bob can violate the three-settings linear steering inequality with a single Alice. For the standard Greenberger–Horne–Zeilinger (GHZ) state, when the untrusted party, Alice, performs a sequence of measurements, at most 3 Alice can violate the steering inequality with a single Bob and a single Charlie; when the trusted party, Charlie, performs a sequence of measurements, there are at most 6 Charlie can violate the steering inequality with a single Alice and a single Bob.[31]

Since using states of low entanglement is necessary for low detection efficiencies in real experimental scenarios,[34] we study the genuine tripartite EPR steering scenarios for generalized GHZ state in this paper, including a one-sided device independent $(1 \to 2)$ scenario (one wing is untrusted and other two wings are trusted) and a two-sided device independent $(2 \to 1)$ scenario (one wing is trusted and other two wings are untrusted). Two fixed observers in two wings perform projective measurements and a sequence of multiple observers in the third wing perform unsharp measurements. Here, we explore untrusted parties and trusted parties perform a sequence of $m$ $(m = 0, 1, 2, 3, \ldots)$ measurements to demonstrate genuine EPR steering based on the generalized GHZ state, respectively. According to the sequence of the former $m$ observers violating the genuine EPR steering inequality with the others two wings, we obtain an upper bound on the sharpness parameters and a lower bound on the GHZ state parameters. According to the fact that the $(m+1)$th observer cannot violate the genuine EPR steering inequality, a lower bound on the sharpness parameters and an upper bound of the GHZ state parameter are derived. We find that within a certain range of sharpness parameters, the non-maximally GHZ states can reach the same number of observers in the sequence as the maximally GHZ state. Additionally, $(1 \to 2)$ steering applies to large ranges of sharpness parameters and state parameters compared to $(2 \to 1)$ steering.

## 2. Recapitulating tripartite steering

The EPR steering exhibits an inherent asymmetric feature. That is, different observers may have different steerability.[35–37] Consider two parties, Alice and Bob, who share a two-qubit entangled state. In some cases, the ability of Bob to steer Alice's state is not equal to the ability of Alice to steer Bob's state. Here, we consider $(n \to m)$ genuine tripartite steering ($n$ observers are untrusted and other $m$ observers are trusted), where $m, n \in \{1, 2\}$. Because the untrusted party can be regarded as a black box, that is, her measurements and output results are unknown, $(n \to m)$ genuine tripartite steering is also called an $n$-sided device independent scenario.[31] In this paper, we use the violations of genuine tripartite EPR steering inequalities to prove genuine EPR steering among Alice, Bob, and Charlie.

### 2.1. A one-sided device-independent scenario $(1 \to 2)$

Three particles of a generalized genuine GHZ state are assigned to three distant parties, Alice, Bob and Charlie, respectively. Alice performs measurement on the first particle, Bob performs measurement on the second particle and Charlie performs measurement on the third particle. In this case, we first consider $(1 \to 2)$ genuine tripartite entanglement (Alice is untrusted, Bob and Charlie are trusted), which includes two scenarios.

**Scenario A** Multiple Alice attempt to steer a single Bob and a single Charlie. Sequential measurements are performed on the first particle, and Alice$^1$ sends the un-normalized post-measurement state to Alice$^2$, who in turn sends her state to Alice$^3$ after unsharp measurement, and so on. Multiple Alice in the middle perform unsharp measurements, and the last Alice performs projective measurement. The measurements are independent of each other, that is, Alice$^m$ ignores the choices of the former $m-1$ Alices's inputs (outputs). Bob and Charlie perform projective measurements. The measurements of Alice$^m$, Bob and Charlie are respectively denoted as $A_x^m$, $B_y$, $C_z$, where $x,\ y,\ z \in \{1, 2, 3\}$, $m \in \{1, 2, 3, \ldots\}$. The measurement results are denoted as $a$, $b$, $c$, where $a,\ b,\ c \in \{+1, -1\}$, as shown in Fig. 1. The dotted lines indicate that the GHZ states are transmitted. The blue box indicates the untrusted part, and the white box indicates the trusted part. This paper has solved that how many Alice can genuinely steer Bob–Charlie.

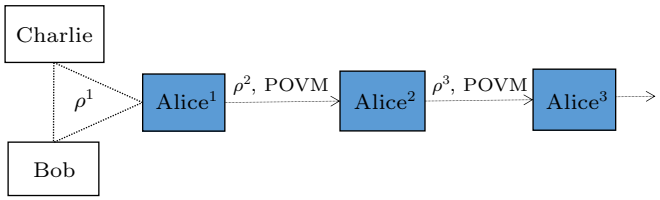



**Fig. 1.** In the sequential scenario where generalized genuine tripartite EPR steering from multiple Alice to a single Bob and a single Charlie.

**Scenario B** A single Alice attempt to steer a single Bob and multiple Charlie (as shown in Fig. 2). We have solved that how many Charlie, along with a single Bob, can be genuinely steered by a single Alice.

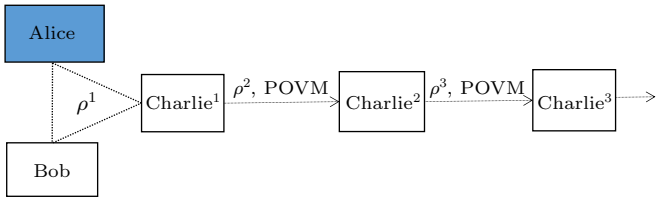



**Fig. 2.** In the sequential scenario where generalized genuine tripartite EPR steering from a single Alice to a single Bob and multiple Charlie.

In order to obtain genuine tripartite steering witnesses, Cavalcanti *et al*. designed several steering inequalities which detect genuine steering of GHZ in the scenario of $(1 \to 2)$ and $(2 \to 1)$.[38] If the generalized GHZ state is shared by three parties (a sequence of $n$ Alice or a sequence of $m$ Charlie perform

unsharp measurements and other two parties perform projective measurements), the scenarios of genuine tripartite EPR steering from Alice to Bob–Charlie ($1 \to 2$) can be proved by the following inequality violation:

$$G_1 = 1+0.1547\langle Z_B Z_C\rangle - \frac{1}{3}\big(\langle A_3 Z_B\rangle + \langle A_3 Z_C\rangle + \langle A_1 X_B X_C\rangle - \langle A_1 Y_B Y_C\rangle - \langle A_2 X_B Y_C\rangle - \langle A_2 Y_B X_C\rangle\big) \geqslant 0. \quad (1)$$

Since Alice in the $(1 \to 2)$ genuine tripartite steering is untrusted, she can be regarded as a black box and Alice can choose measurement $A_x$, $x \in \{1,2,3\}$. Bob and Charlie are completely trusted, so the measurements that performed by Bob and Charlie in the joint probability distribution of the inequality are known, and $X$, $Y$, and $Z$ represent Pauli operators. In the quantum system, the maximum quantum violation value is $-0.845$.[38] This quantum bound can be obtained from the following standard GHZ state:

$$|\psi_{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad (2)$$

and measurements

$$\begin{aligned} A_1 &= \sigma_x, \ A_2 = \sigma_y, \ A_3 = \sigma_z, \\ B_1 &= \sigma_x, \ B_2 = \sigma_y, \ B_3 = \sigma_z, \\ C_1 &= \sigma_x, \ C_2 = \sigma_y, \ C_3 = \sigma_z, \end{aligned} \quad (3)$$

we also denote $\sigma_x \equiv X$, $\sigma_y \equiv Y$, $\sigma_z \equiv Z$.

### 2.2. A two-sided device-independent scenario ($2 \to 1$)

For a generalized three-particle state, we consider $(2 \to 1)$ genuine tripartite steering (Alice and Bob are untrusted, Charlie is trusted), which includes two cases.

**Scenario C** Multiple Alice and a single Bob attempt to steer a single Charlie. Sequential measurements are performed on the first particle, and Alice$^1$ sends the un-normalized post-measurement state to Alice$^2$, who in turn sends her state to Alice$^3$ after unsharp measurement, and so on. The middle Alice performs unsharp measurements, and the last Alice performs projective measurement. The measurements are independent of each other. The measurements of Alice$^k$, Bob and Charlie are respectively denoted as $A_x^k$, $B_y$, $C_z$, where $k \in \{1,2,3,\ldots\}$. The measurement results are denoted as $a$, $b$, $c$, where $a$, $b$, $c \in \{+1,-1\}$ (as shown in Fig. 3). This paper has solved that how many Alice and a single Bob can genuinely steer a single Charlie.

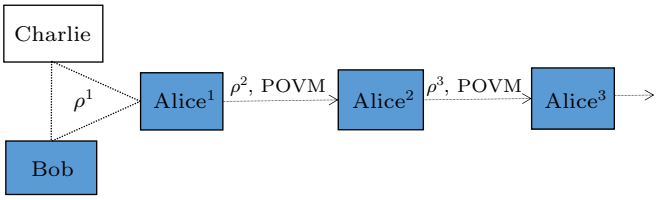



**Fig. 3.** In the sequential scenario where generalized genuine tripartite EPR steering from multiple Alice and a single Bob to a single Charlie.

**Scenario D** A single Alice and a single Bob attempt to steer multiple Charlie (as shown in Fig. 4). We have solved that how many Charlie can be genuinely steered by a single Alice and a single Bob.

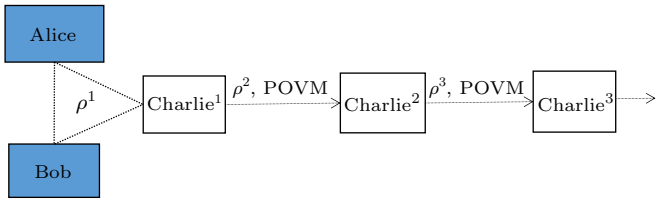



**Fig. 4.** In the sequential scenario where generalized genuine tripartite EPR steering from a single Alice and a single Bob to multiple Charlie.

If the generalized GHZ state is shared by three parties (a sequence of $n$ Alice or a sequence of $m$ Charlie perform unsharp measurements and other two parties perform projective measurements), the scenarios of genuine tripartite EPR steering from Alice–Bob to Charlie $(2 \to 1)$ can be proved by the following inequality:

$$G_2 = 1-0.183\big(\langle A_3 B_3\rangle + \langle A_3 Z\rangle + \langle B_3 Z\rangle\big) - 0.258\big(\langle A_1 B_1 X\rangle - \langle A_1 B_2 Y\rangle - \langle A_2 B_1 Y\rangle - \langle A_2 B_2 X\rangle\big) \geqslant 0. \quad (4)$$

Because Alice and Bob in $(2 \to 1)$ genuine tripartite steering are untrusted, they can be regarded as two black boxes. Alice and Bob can choose measurements $A_x$, $B_y$, respectively, where $x$, $y \in \{1,2,3\}$. Charlie is completely credible, so the measurements performed by Charlie in the joint probability involving the inequality are known. In the quantum system, the maximum quantum violation value is $-0.582$.[38] This quantum bound can also be obtained from the standard GHZ state and measurements in Eqs. (2) and (3).

## 3. Steering by multiple observers

For the four different steering scenarios, we deduce the maximum number of sequential observers who can detect genuine steering sequentially with the others two parties in the following.

### 3.1. Alice-sided performing sequential measurements

We consider multiple Alice (Alice$^1$, Alice$^2$, Alice$^3$,$\ldots$) to perform unsharp measurements focusing on the trade-off between the degree of disturbance and the amount of information we gain about the system.[29] We demonstrate that secret messages can be shared with the maximum number of Alice using only one copy of a generalized three-qubit GHZ state

$$|\psi_{{\rm GHZ}_\alpha}\rangle = \cos\alpha|000\rangle + \sin\alpha|111\rangle, \quad (5)$$

where $0 < \alpha \leq \pi/4$. When $\alpha$ gets $\pi/4$, it is a three-particle maximally entangled state. $\{|0\rangle, |1\rangle\}$ is the orthogonal basis of $C^2$ space. We can prove the genuine tripartite EPR steering

from the Alice to Bob–Charlie ($1 \to 2$) with multiple Alice by the violation of inequality (1), and the genuine tripartite EPR steering from Alice–Bob to Charlie ($2 \to 1$) with multiple Alice by the violation of inequality (4). The operator of inequality (1) and inequality (4) is given by

$$\begin{aligned} g_1 &= 1+0.1547Z_{\rm B}Z_{\rm C}-\frac{1}{3}\big(A_3Z_{\rm B}+A_3Z_{\rm C}+A_1X_{\rm B}X_{\rm C} \\ &\quad -A_1Y_{\rm B}Y_{\rm C}-A_2X_{\rm B}Y_{\rm C}-A_2Y_{\rm B}X_{\rm C}\big), \qquad (6) \\ g_2 &= 1-0.183\,(A_3B_3+A_3Z+B_3Z)-0.258\big(A_1B_1X \\ &\quad -A_1B_2Y-A_2B_1Y-A_2B_2X\big). \qquad (7) \end{aligned}$$

Alice$^m$ performs the unsharp measurement with sharpness parameter $\lambda_m$ along the spin direction $\hat{\boldsymbol{x}}_i^m$, with $i \in \{1,2,3\}$. The unsharp measurement is given by

$$E_{a^m|\hat{x}_i^m}^{\lambda_m} = \lambda_m \frac{\mathbb{I}_2 + a^m \hat{\boldsymbol{x}}_i^m \cdot \boldsymbol{\sigma}}{2} + (1-\lambda_m)\frac{\mathbb{I}_2}{2}, \qquad (8)$$

$$\hat{\boldsymbol{x}}_i^m = \sin\theta_i^{x^m}\cos\phi_i^{x^m}\hat{X} + \sin\theta_i^{x^m}\sin\phi_i^{x^m}\hat{Y} + \cos\theta_i^{x^m}\hat{Z}, \qquad (9)$$

where $i \in \{0,1,2\}$, $0 \leqslant \theta_i^{x^m} \leqslant \pi$, $0 \leqslant \phi_i^{x^m} \leqslant 2\pi$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. $\sin\theta_i^{x^m}\cos\phi_i^{x^m}$, $\sin\theta_i^{x^m}\sin\phi_i^{x^m}$, and $\cos\theta_i^{x^m}$ represent the measured components on the three Pauli operators, respectively. The output result of Alice$^m$ is represented by $a^m \in \{+1,-1\}$. These can also be viewed as noise implementations of projective measurements. The reason for using this measurement is that if we use projective measurements, only the first observer of the sequence can exhibit steering with the others two parties since the disturbance will prevent all the following observers from doing so. The use of sharpness parameters provides a trade-off between the disturbance and information gain. We require the last observer to perform projective measurement, so that he can obtain more information and exhibit steering with the others two parties. When $\lambda_m = 1$, the measurement of Alice$^m$ is a projective measurement. We consider that the measurement settings of Alice$^m$ are unbiased.

Bob and Charlie perform projective measurement $\Pi_{b|\hat{\boldsymbol{y}}_j} = (\mathbb{I}_2 + b\hat{\boldsymbol{y}}_j \cdot \boldsymbol{\sigma})/2$ and $\Pi_{c|\hat{\boldsymbol{z}}_k} = \mathbb{I}_2 + c\hat{\boldsymbol{z}}_k \cdot \boldsymbol{\sigma}/2$ along the spin directions $\hat{\boldsymbol{y}}_j$ and $\hat{\boldsymbol{z}}_k$ (with $j,\ k \in \{1,2,3\}$ ), respectively. $a,\ b \in \{+1,-1\}$ present the projective measurement results of Bob and Charlie. The direction $\hat{\boldsymbol{y}}_j$ and $\hat{\boldsymbol{z}}_k$ can be written as

$$\hat{\boldsymbol{y}}_j = \sin\theta_j^y\cos\phi_j^y\hat{X} + \sin\theta_j^y\sin\phi_j^y\hat{Y} + \cos\theta_j^y\hat{Z}, \qquad (10)$$

$$\hat{\boldsymbol{z}}_k = \sin\theta_k^z\cos\phi_k^z\hat{X} + \sin\theta_k^z\sin\phi_k^z\hat{Y} + \cos\theta_k^z\hat{Z}, \qquad (11)$$

where $j,\ k \in \{1,2,3\}$, $0 \leqslant \theta_i^y,\ \theta_k^z \leqslant \pi$, $0 \leqslant \phi_j^y,\ \phi_k^z \leqslant 2\pi$. We assume that the choice of measurement settings of Alice$^m$ are $(\theta_1^{x_m}, \phi_1^{x_m}; \theta_2^{x_m}, \phi_2^{x_m}; \theta_3^{x_m}, \phi_3^{x_m}) = (\pi/2, 0; \pi/2, \pi/2; 0, 0)$, with sharpness parameter $\lambda_m$. Alice$^1$ randomly chooses an unsharp measurement $A_i^1$, $i \in \{1,2,3\}$, and sends the postmeasurement state to Alice$^2$. By using the Lüders update rule,[39] the postmeasurement state, after Alice performs the unsharp measurement along the spin direction $\hat{\boldsymbol{x}}_i^1$ with sharpness parameter $\lambda_1$, is given by

$$\begin{aligned} \rho_{\rm un}^2 &= \frac{1}{3}\sum_{a^1|\hat{\boldsymbol{x}}_i^{\lambda_1}} \Bigg[ \left\{ \sqrt{E_{a^1|\hat{\boldsymbol{x}}_i^{\lambda_1}}} \otimes I_2 \otimes I_2 \right\} \\ &\quad \times \rho^1 \left\{ \sqrt{E_{a^1|\hat{\boldsymbol{x}}_i^{\lambda_1}}} \otimes I_2 \otimes I_2 \right\} \Bigg], \qquad (12) \end{aligned}$$

where $\rho^1 = |\psi_{{\rm GHZ}_\alpha}\rangle\langle\psi_{{\rm GHZ}_\alpha}|$. Similarly, we can get the postmeasurement state $\rho_{\rm un}^m$ that Alice$^{m-1}$ performs the unsharp measurement along the spin direction $\hat{\boldsymbol{x}}_i^{m-1}$, $i \in \{1,2,3\}$ with sharpness parameter $\lambda_{m-1}$

$$\begin{aligned} \rho_{\rm un}^m &= \frac{1}{3}\sum_{a^{m-1}|\hat{\boldsymbol{x}}_i^{\lambda_{m-1}}} \Bigg[ \left\{ \sqrt{E_{a^{m-1}|\hat{\boldsymbol{x}}_i^{\lambda_{m-1}}}} \otimes I_2 \otimes I_2 \right\} \\ &\quad \times \rho_{\rm un}^{m-1} \left\{ \sqrt{E_{a^{m-1}|\hat{\boldsymbol{x}}_i^{\lambda_{m-1}}}} \otimes I_2 \otimes I_2 \right\} \Bigg]. \qquad (13) \end{aligned}$$

According to the measurement expression of inequalities (1) and (4) and the postmeasurement state $\rho_{\rm un}^m$, we calculate the correlation function among Alice$^m$, Bob and Charlie ($m \in \{0,1,2,3,\ldots\}$). Then the expressions on the left side of inequalities (1) and (4) are calculated by $G_1^m = {\rm Tr}(\rho_{\rm un}^m g_1)$ and $G_2^m = {\rm Tr}(\rho_{\rm un}^m g_2)$.

For Scenario A, the measurements of Bob and Charlie are known in the correlation function on the left side of inequality (1). After detailed calculation, we can get the following conclusion, which is proved in Appendix A.

**Theorem 1** For genuine tripartite EPR steering from Alice$^m$ to Bob–Charlie ($1 \to 2$) on the generalized three-qubit GHZ state (5), the expected violation value of inequality (1) is given by $G_1^m$ and the number of Alice, the range of parameter $\lambda_m$, and $\alpha$ are summarized in Table 1

$$G_1^m = 1+0.1547-\lambda_m\frac{2+4\sin 2\alpha}{3}\prod_{j=1}^{m-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right). \qquad (14)$$

For scenario C, the measurements of Charlie are known in the correlation function on the left side of inequality (4). After detailed calculation, we can get the following conclusion, which is proved in Appendix B.

**Theorem 2** For genuine tripartite EPR steering from Alice$^n$–Bob to Charlie ($2 \to 1$) on generalized three-qubit GHZ state (5), the expression on the left side of inequality (4) is given by $G_2^n$ and the number of Charlie, the range of parameters $\lambda_n$, and $\alpha$ are summarized in Table 2

$$G_2^n = 0.817-\lambda_n(0.366+1.032\sin 2\beta)\prod_{j=1}^{n-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right). \qquad (15)$$

**Table 1.** The permissible ranges of the sharpness parameter $\lambda_m$ (where $0<\lambda_m\leq 1$) of Alice$^m$ and state parameter $\alpha$ of the GHZ state are used to demonstrate genuine tripartite EPR steering from Alice$^m$ to Bob–Charlie.

| Alice$^m$ | Alice$^0$ | Alice$^1$ | Alice$^2$ | Alice$^3$ |
|---|---|---|---|---|
| $\alpha$ | $0<\alpha\leq 0.187$ | $0.187<\alpha\leq 0.343$ | $0.343<\alpha\leq 0.503$ | $0.503<\alpha\leq \pi/4$ |
| $\lambda_1$ | – | $0.764\leq\lambda_1\leq 1$ | $0.644\leq\lambda_1<0.764$ | $0.577\leq\lambda_1<0.644$ |
| $\lambda_2$ | – | – | $0.764\leq\lambda_2\leq 1$ | $0.658\leq\lambda_2<0.764$ |
| $\lambda_3$ | – | – | – | $0.787\leq\lambda_3\leq 1$ |
| $\lambda_4$ | – | – | – | – |

**Table 2.** The permissible ranges of the sharpness parameter $\lambda_n$ (where $0<\lambda_n\leq 1$) of Alice$^n$ and state parameter $\alpha$ of the GHZ state are used to demonstrate genuine tripartite EPR steering from Alice$^n$–Bob to Charlie.

| Alice$^n$ | Alice$^0$ | Alice$^1$ | Alice$^2$ | Alice$^3$ |
|---|---|---|---|---|
| $\alpha$ | $0<\alpha\leq 0.226$ | $0.226<\alpha\leq 0.375$ | $0.375<\alpha\leq 0.532$ | $0.532<\alpha\leq \pi/4$ |
| $\lambda_1$ | – | $0.764\leq\lambda_1\leq 1$ | $0.644\leq\lambda_1<0.764$ | $0.584\leq\lambda_1<0.644$ |
| $\lambda_2$ | – | – | $0.764\leq\lambda_2\leq 1$ | $0.668\leq\lambda_2<0.764$ |
| $\lambda_3$ | – | – | – | $0.805\leq\lambda_3\leq 1$ |
| $\lambda_4$ | – | – | – | – |

### 3.2. Charlie-sided performing sequential measurements

We consider multiple Charlie (Charlie$^1$, Charlie$^2$, Charlie$^3$,…) to perform unsharp measurements focusing on the trade-off between the degree of disturbance and the amount of information we gain about the system.[29] We demonstrate that secret messages can be shared with the maximum number of Charlie using only one copy of a generalized three-qubit GHZ state (5). The choose of measurement settings of Charlie$^k$ are $(\theta_1^{z_k},\phi_1^{z_k};\ \theta_2^{z_k},\phi_2^{z_k};\ \theta_3^{z_k},\phi_3^{z_k})=(\pi/2,0;\pi/2,\pi/2;0,0)$, with sharpness parameter $\lambda_k$. Charlie$^1$ randomly chooses an unsharp measurement $C_i^1$, $i\in\{1,2,3\}$, and sends the postmeasurement state to Charlie$^2$. By using the Lüders update rule,[39] the postmeasurement state $\rho_{\rm un}^k$, after Charlie performs the unsharp measurement along the spin direction $\hat{z}_i^{k-1}$ with sharpness parameter $\lambda_{k-1}$, is given by

$$\rho_{\rm un}^k=\frac{1}{3}\sum_{c^{k-1}|\hat{z}_i^{\lambda_{k-1}}}\left[\left\{\sqrt{E_{c^{k-1}|\hat{z}_i^{\lambda_{k-1}}}}\otimes I_2\otimes I_2\right\}\times\rho_{\rm un}^{k-1}\left\{\sqrt{E_{c^{k-1}|\hat{z}_i^{\lambda_{k-1}}}}\otimes I_2\otimes I_2\right\}\right],\tag{16}$$

where

$$\sqrt{E_{c^{k-1}|\hat{z}_i^{k-1}}^{\lambda_{k-1}}}=\sqrt{\frac{1+\lambda_{k-1}}{2}}\left(\frac{I_2+c^{k-1}\hat{z}_i^{\lambda_{k-1}}\boldsymbol{\sigma}}{2}\right)+\sqrt{\frac{1-\lambda_{k-1}}{2}}\left(\frac{I_2-c^{k-1}\hat{z}_i^{\lambda_{k-1}}\boldsymbol{\sigma}}{2}\right).\tag{17}$$

According to the inequalities (1) and (4) and the postmeasurement state $\rho_{\rm un}^k$, we calculate the correlation function among Alice, Bob, and Charlie$^k$ ($k\in\{0,1,2,3,\ldots\}$). Finally, the expression for the average of the left-hand sides of inequalities (1) and (4) are calculated by $G_1^k=\mathrm{Tr}(\rho_{\rm un}^k g_1)$ and $G_2^k=\mathrm{Tr}(\rho_{\rm un}^k g_2)$.

For Scenario B, the measurements of Bob and Charlie$^k$ are given in the correlation function on the left side of the inequality (1). After detailed calculation, we have the following conclusions, which is proved in Appendix C.

**Theorem 3** For genuine tripartite EPR steering from Alice to Bob–Charlie$^k$ ($1\rightarrow 2$) on the three-qubit generalized three-qubit GHZ state (5), the expected violation value of inequality (1) is given by $G_1^k$ and the range of parameters $\lambda_k$ and $\alpha$ are summarized in Table 3

$$G_1^k=\frac{2}{3}+\lambda_k\left(0.1547-\frac{1}{3}-\frac{4}{3}\sin 2\alpha\right)\times\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right).\tag{18}$$

For Scenario D, the measurements of Charlie$^k$ are known in the correlation function on the left side of inequality (4). After detailed calculation, we get the following conclusion, which is proved in Appendix D.

**Theorem 4** For genuine tripartite EPR steering from Alice–Bob to Charlie$^t$ ($2\rightarrow 1$) simultaneously on the generalized three-qubit GHZ state (5), the expected violation value of inequality (4) is given by $G_2^t$ and the range of parameters $\lambda_t$ and $\alpha$ are summarized in Table 4

$$G_2^t=0.817-\lambda_t(0.366+1.032\sin 2\beta)\times\prod_{j=1}^{t-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right).\tag{19}$$

**Table 3.** The permissible ranges of the sharpness parameter $\lambda_k$ (where $0 < \lambda_k \leq 1$) of Charlie$^k$ and state parameter $\alpha$ of the GHZ state are used to demonstrate genuine tripartite EPR steering from Alice to Bob–Charlie$^k$.

| Charlie$^k$ | Charlie$^0$ | Charlie$^1$ | Charlie$^2$ | Charlie$^3$ | Charlie$^4$ | Charlie$^5$ | Charlie$^6$ | Charlie$^7$ |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $0 < \alpha \leq 0.187$ | $0.187 < \alpha \leq 0.274$ | $0.274 < \alpha \leq 0.349$ | $0.349 < \alpha \leq 0.421$ | $0.421 < \alpha \leq 0.497$ | $0.497 < \alpha \leq 0.587$ | $0.587 < \alpha \leq \pi/4$ | – |
| $\lambda_1$ | – | $0.764 \leq \lambda_1 \leq 1$ | $0.644 \leq \lambda_1 < 0.764$ | $0.568 \leq \lambda_1 < 0.644$ | $0.514 \leq \lambda_1 < 0.568$ | $0.473 \leq \lambda_1 < 0.514$ | $0.441 \leq \lambda_1 < 0.473$ | – |
| $\lambda_2$ | – | – | $0.764 \leq \lambda_2 \leq 1$ | $0.644 \leq \lambda_2 < 0.764$ | $0.568 \leq \lambda_2 \leq 0.644$ | $0.514 \leq \lambda_2 \leq 0.568$ | $0.473 \leq \lambda_2 \leq 0.514$ | – |
| $\lambda_3$ | – | – | – | $0.764 \leq \lambda_3 \leq 1$ | $0.644 \leq \lambda_3 \leq 0.764$ | $0.568 \leq \lambda_3 \leq 0.644$ | $0.514 \leq \lambda_3 \leq 0.568$ | – |
| $\lambda_4$ | – | – | – | – | $0.764 \leq \lambda_4 \leq 1$ | $0.644 \leq \lambda_4 \leq 0.764$ | $0.568 \leq \lambda_4 \leq 0.644$ | – |
| $\lambda_5$ | – | – | – | – | – | $0.764 \leq \lambda_5 \leq 1$ | $0.644 \leq \lambda_5 \leq 0.764$ | – |
| $\lambda_6$ | – | – | – | – | – | – | $0.764 \leq \lambda_6 \leq 1$ | – |
| $\lambda_7$ | – | – | – | – | – | – | – | – |

**Table 4.** The permissible ranges of the sharpness parameter $\lambda_t$ (where $0 < \lambda_t \leq 1$) of Charlie$^t$ and state parameter $\alpha$ of the GHZ state are used to demonstrate genuine tripartite EPR steering from Alice–Bob to Charlie$^t$.

| Charlie$^t$ | Charlie$^0$ | Charlie$^1$ | Charlie$^2$ | Charlie$^3$ |
|---|---|---|---|---|
| $\alpha$ | $0 < \alpha \leq 0.226$ | $0.226 < \alpha \leq 0.375$ | $0.375 < \alpha \leq 0.532$ | $0.532 < \alpha \leq \pi/4$ |
| $\lambda_1$ | – | $0.764 \leq \lambda_1 \leq 1$ | $0.644 \leq \lambda_1 < 0.764$ | $0.584 \leq \lambda_1 < 0.644$ |
| $\lambda_2$ | – | – | $0.764 \leq \lambda_2 \leq 1$ | $0.668 \leq \lambda_2 < 0.764$ |
| $\lambda_3$ | – | – | – | $0.805 \leq \lambda_3 \leq 1$ |
| $\lambda_4$ | – | – | – | – |

## 4. Summary and discussion

The EPR steering is one of the key resources for information processing tasks in quantum networks.[9,10] The EPR steering of two-qubit quantum states and three-qubit maximally entangled states has been well studied.[30–33] However, when the experiment has low detection efficiencies, using maximally entangled state will limit the experimental results.[34] The EPR steering of non-maximally entanglement states has great research significance. In this paper, we have carried out detailed analysis for non-maximally GHZ states.

Due to the asymmetry of EPR steering,[35] the scenario of genuine tripartite EPR steering from Alice to Bob–Charlie ($1 \to 2$) and the scenario of genuine tripartite EPR steering from Alice–Bob to Charlie ($2 \to 1$) are analyzed, respectively. These two scenarios can be further divided according to sequence of measurements performed by trusted party or untrusted party. We have obtained the maximum number of multiple observers in four scenarios, respectively. When Alice (untrusted party) performs a sequence of measurements, in the Alice steering Bob–Charlie scenario and the Alice–Bob steering Charlie scenario, the number of multiple observers is the same. There are at most 3 Alice who can detect genuine tripartite EPR steering with a single Bob and a single Charlie. When Charlie (trusted party) performs a sequence of measurements, we can get up to 7 Charlie in the scenario of Alice steering Bob–Charlie and 3 Charlie in the scenario of Alice–Bob steering Charlie. For all the four scenarios, we have given the range of sharpness parameter and state parameter during which the three wings can demonstrate genuine EPR steering through the quantum violation of tripartite steering inequality. The results show that for a certain range of non-maximally GHZ states, the upper bound of the sequential observers is the same as that in the situation for maximally GHZ state, which is counter-intuitive to some extent. In addition, we have to admit that the steering inequality and the maximum quantum violation value in this paper are both most suitable for the maximally GHZ state. Finding the appropriate steering inequality for the non-maximally GHZ state is still a problem. We will consider adding parameters to solve this problem in future works, which may lead to more precise results for steering of generalized GHZ states.

## Appendix A: Proof of Theorem 1

Considering that the maximum quantum violation value of inequation (1) is $-0.845$, we adopt the measurement settings of Alice$^k$ as $(\theta_1^{x_k}, \phi_1^{x_k}; \theta_2^{x_k}, \phi_2^{x_k}; \theta_3^{x_k}, \phi_3^{x_k}) = (\pi/2, 0; \pi/2, \pi/2; 0, 0)$. From Eq. (9) we have

$$A^k_{0|1} = \frac{I_2 + \lambda_k \sigma_x}{2}, \quad A^k_{1|1} = \frac{I_2 - \lambda_k \sigma_x}{2},$$
$$A^k_{0|2} = \frac{I_2 + \lambda_k \sigma_y}{2}, \quad A^k_{1|2} = \frac{I_2 - \lambda_k \sigma_y}{2},$$
$$A^k_{0|3} = \frac{I_2 + \lambda_k \sigma_z}{2}, \quad A^k_{1|3} = \frac{I_2 - \lambda_k \sigma_z}{2},$$

next we have

$$A^k_1 = A^k_{0|1} - A^k_{1|1} = \lambda_k \sigma_x,$$
$$A^k_2 = A^k_{0|2} - A^k_{1|2} = \lambda_k \sigma_y,$$
$$A^k_3 = A^k_{0|3} - A^k_{1|3} = \lambda_k \sigma_z,$$

then

$$G^k_1 = \mathrm{Tr}\left(\rho^k_{\mathrm{un}} g_1\right)$$

$$
\begin{aligned}
&= 1+0.1547\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k} I_2\otimes\sigma_z\otimes\sigma_z\right]\\
&-\frac{\lambda_k}{3}\Big\{\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_z\otimes\sigma_z\otimes I_2\right]+\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_z\otimes I_2\otimes\sigma_z\right]\\
&+\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_x\otimes\sigma_x\otimes\sigma_x\right]-\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_x\otimes\sigma_y\otimes\sigma_y\right]\\
&-\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_y\otimes\sigma_x\otimes\sigma_y\right]-\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_y\otimes\sigma_y\otimes\sigma_x\right]\Big\}, \qquad \text{(A1)}
\end{aligned}
$$

when $k=1$, we can calculate the inequality among Alice[1], Bob and Charlie

$$
G_1^1 = 1+0.1547-\lambda_1\frac{(2+4\sin 2\alpha)}{3}, \qquad \text{(A2)}
$$

where we have used $\rho^1=|\psi_{\mathrm{GHZ}_\alpha}\rangle\langle\psi_{\mathrm{GHZ}_\alpha}|$.

In order to get the relationship between $\rho^1$ and $\rho_{\mathrm{un}}^k$, we get

$$
\begin{aligned}
\rho_{\mathrm{un}}^{k} &= \frac{1}{3}\sum_{a^{k-1}|\hat{\boldsymbol{x}}_i^{\lambda_{i-1}}}\left[\left\{\sqrt{E_{a^{k-1}|\hat{\boldsymbol{x}}_i^{\lambda_{i-1}}}}\otimes I_2\otimes I_2\right\}\right.\\
&\left.\times\rho_{\mathrm{un}}^{k-1}\left\{\sqrt{E_{a^{k-1}|\hat{\boldsymbol{x}}_i^{\lambda_{i-1}}}}\otimes I_2\otimes I_2\right\}\right]\\
&= \frac{1+\sqrt{1-\lambda_{k-1}^2}}{2}\rho_{\mathrm{un}}^{k-1}+\frac{1-\sqrt{1-\lambda_{k-1}^2}}{6}\\
&\times\Big[(\sigma_x\otimes I_2\otimes I_2)\,\rho_{\mathrm{un}}^{k-1}\,(\sigma_x\otimes I_2\otimes I_2)\\
&+(\sigma_y\otimes I_2\otimes I_2)\,\rho_{\mathrm{un}}^{k-1}\,(\sigma_y\otimes I_2\otimes I_2)\\
&+(\sigma_z\otimes I_2\otimes I_2)\,\rho_{\mathrm{un}}^{k-1}\,(\sigma_z\otimes I_2\otimes I_2)\Big], \qquad \text{(A3)}
\end{aligned}
$$

substituting Eq. (A3) into Eq. (A1), we get

$$
\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k} I_2\otimes\sigma_z\otimes\sigma_z\right] = \mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1} I_2\otimes\sigma_z\otimes\sigma_z\right]=\cdots=\mathrm{Tr}\left[\rho^1 I_2\otimes\sigma_z\otimes\sigma_z\right]=1, \qquad \text{(A4)}
$$

$$
\begin{aligned}
\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_z\otimes\sigma_z\otimes I_2\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_z\otimes\sigma_z\otimes I_2\right]=\cdots\\
&=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_z\otimes\sigma_z\otimes I_2\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right), \qquad \text{(A5)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_z\otimes I_2\otimes\sigma_z\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_z\otimes I_2\otimes\sigma_z\right]=\cdots\\
&=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_z\otimes I_2\otimes\sigma_z\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right), \qquad \text{(A6)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_x\otimes\sigma_x\otimes\sigma_x\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_x\otimes\sigma_x\otimes\sigma_x\right]=\cdots\\
&=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_x\otimes\sigma_x\otimes\sigma_x\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\sin 2\alpha, \qquad \text{(A7)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_x\otimes\sigma_y\otimes\sigma_y\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_x\otimes\sigma_y\otimes\sigma_y\right]=\cdots\\
&=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_x\otimes\sigma_y\otimes\sigma_y\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)(-\sin 2\alpha), \qquad \text{(A8)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_y\otimes\sigma_x\otimes\sigma_y\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_y\otimes\sigma_x\otimes\sigma_y\right]=\cdots\\
&=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_y\otimes\sigma_x\otimes\sigma_y\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)(-\sin 2\alpha), \qquad \text{(A9)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k}\sigma_y\otimes\sigma_y\otimes\sigma_x\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_y\otimes\sigma_y\otimes\sigma_x\right]=\cdots\\
&=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_y\otimes\sigma_y\otimes\sigma_x\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)(-\sin 2\alpha), \qquad \text{(A10)}
\end{aligned}
$$

substituting Eqs. (A4)–(A10) into Eq. (A1) and using Eq. (A2), we obtain Eq. (14). If we have $G_1^1=0$, $G_1^2=0$, $G_1^3=0,\ldots$, $G_1^k=0,\ldots$, then

$$\lambda_1=\frac{3.4641}{2+4\sin 2\alpha},\ \lambda_2=\frac{3\lambda_1}{1+2\sqrt{1-\lambda_1^2}},$$
$$\lambda_3=\frac{3\lambda_2}{1+2\sqrt{1-\lambda_2^2}},\ \ldots,\ \lambda_k=\frac{3\lambda_{k-1}}{1+2\sqrt{1-\lambda_{k-1}^2}},$$

where $\lambda_k\in(0,1]$, $\lambda_k>\lambda_{k-1}$ $(k>0)$. This sequence is monotonically increasing.

Let us first study the situation that no Alice can steer a single Bob and a single Charlie. Here, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1\in(0,1]$, Alice$^1$ cannot sequentially violate the inequality (1). At this time, we can get $0<\sin 2\alpha\le 0.366$, $0<\alpha\le 0.187$. Therefore, in the case of $0<\alpha\le 0.187$, even if all three observers, including Alice, Bob and Charlie, perform projective measurements, the correlation between their inputs and outputs cannot violate inequality (1). Because the state parameter $\alpha$ is very weak, any Alice cannot genuinely steer a single Bob and a single Charlie when the generalized entangled states is initially shared.

Next, we discuss only one Alice can steer a single Bob and a single Charlie. For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1\in(0,1]$, Alice$^1$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_2\in(0,1]$, Alice$^2$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_1$ takes the maximum value of 1, we get $0.366<\sin 2\alpha$. Secondly, when $\lambda_2$ takes the maximum value of 1, we get $\sin 2\alpha\le 0.634$, and then $0.187<\alpha\le 0.343$, $0.764\le\lambda_1\le 1$. At this time, at most one Alice and a single Bob, a single Charlie sequentially violate inequality (1) and Alice$^1$ performs unsharp measurement with the sharpness parameter $0.764\le\lambda_1\le 1$.

Two Alice can steer a single Bob and a single Charlie. Here, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1,\ \lambda_2\in(0,1]$, Alice$^1$ and Alice$^2$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_3\in(0,1]$, Alice$^3$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_2$ takes the maximum value of 1, we get $\lambda_1<0.764$, $0.634<\sin 2\alpha$. Secondly, when $\lambda_3$ takes the maximum value of 1, we get $0.764\le\lambda_2$, $0.644\le\lambda_1$, $\sin 2\alpha\le 0.845$, and then $0.343<\alpha\le 0.503$, $0.644\le\lambda_1<0.764$, $0.764\le\lambda_2\le 1$. At this time, at most two Alice and a single Bob, a single Charlie sequentially violate inequality (1) and Alice$^1$ and Alice$^2$ perform unsharp measurements with the sharpness parameters $0.644\le\lambda_1<0.764$ and $0.764\le\lambda_2\le 1$.

Three Alice can steer a single Bob and a single Charlie. In this case, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1,\ \lambda_2,\ \lambda_3\in(0,1]$, Alice$^1$, Alice$^2$, and Alice$^3$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_4\in(0,1]$, Alice$^4$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_3$ takes the maximum value of 1, we get $\lambda_2<0.764$, $\lambda_1<0.644$, $0.845<\sin 2\alpha$. Secondly, when $\lambda_4$ takes the maximum value of 1, we get $0.764\le\lambda_3$, $0.644\le\lambda_2$, $0.568\le\lambda_1$, $\sin 2\alpha\le 1.025$, for $0<\sin 2\alpha\le 1$, and then $0.787\le\lambda_3$, $0.658\le\lambda_2$, $0.577\le\lambda_1$. We have $0.845<\sin 2\alpha\le 1$, $0.503<\alpha\le\pi/4$, $0.577\le\lambda_1<0.644$, $0.658\le\lambda_2<0.764$, $0.787\le\lambda_3\le 1$. At this time, at most three Alice and a single Bob, a single Charlie sequentially violate inequality (1) and Alice$^1$, Alice$^2$, and Alice$^3$ perform unsharp measurements with the sharpness parameters $0.577\le\lambda_1<0.644$, $0.658\le\lambda_2<0.764$, and $0.787\le\lambda_3\le 1$.

Four Alice can steer a single Bob and a single Charlie. In this case, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4\in(0,1]$, Alice$^1$, Alice$^2$, Alice$^3$, and Alice$^4$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_5\in(0,1]$, Alice$^5$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_4$ takes the maximum value of 1, we get $\lambda_3<0.764$, $\lambda_2<0.644$, $\lambda_1<0.568$, $1.025<\sin 2\alpha$. This is contradict with $0<\sin 2\alpha\le 1$. Secondly, when $\sin 2\alpha\le 1$, $\lambda_i$ $(i=1,2,3)$ take the minimum value of $\lambda_3=0.787$, $\lambda_2=0.658$, $\lambda_1=0.577$, and $\lambda_4=1.057$. This is not true. Further, for $G_1^1=0$, $G_1^2=0$, $G_1^3=0$, and $\lambda_4=1$, we have $G_1^3=0.626$. Therefore, Alice$^1$, Alice$^2$, Alice$^3$, and Alice$^4$ cannot sequentially violate the inequality (1) with a single Bob and a single Charlie at the same time.

## Appendix B: Proof of Theorem 2

By the same method as proving Theory 1, we obtain at most $m$ $(m=0,1,2,3)$ Alice and a single Bob steering a single Charlie and give the range of entanglement parameter $\alpha$ and the range of sharpness parameters $\lambda_i$ $(i=1,2,3)$, as shown in Table 2.

## Appendix C: Proof of Theorem 3

Considering that the maximum quantum violation value of inequality (4) is $-0.582$, we adopt the measurement settings of Charlie$^k$ as $(\theta_1^{z_k},\phi_1^{z_k};\theta_2^{z_k},\phi_2^{z_k};\theta_3^{z_k},\phi_3^{z_k})=(\pi/2,0;\pi/2,\pi/2;0,0)$. From Eq. (11) we have

$$C_{0|1}^k=\frac{I_2+\lambda_k\sigma_x}{2},\ C_{1|1}^k=\frac{I_2-\lambda_k\sigma_x}{2},$$
$$C_{0|2}^k=\frac{I_2+\lambda_k\sigma_y}{2},\ C_{1|2}^k=\frac{I_2-\lambda_k\sigma_y}{2},$$
$$C_{0|3}^k=\frac{I_2+\lambda_k\sigma_z}{2},\ C_{1|3}^k=\frac{I_2-\lambda_k\sigma_z}{2},$$
$$C_1^k=C_{0|1}^k-C_{1|1}^k=\lambda_k\sigma_x,$$
$$C_2^k=C_{0|2}^k-C_{1|2}^k=\lambda_k\sigma_y,$$
$$C_3^k=C_{0|3}^k-C_{1|3}^k=\lambda_k\sigma_z,$$

then

$$\begin{aligned} G_1^k &= \mathrm{Tr}\left(\rho_{\mathrm{un}}^k g_1\right) \\ &= 1+0.1547\mathrm{Tr}\left[\rho_{\mathrm{un}}^k I_2\otimes\sigma_z\otimes\sigma_z\right] \\ &\quad -\frac{\lambda_k}{3}\Big\{\mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_z\otimes\sigma_z\otimes I_2\right]+\mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_z\otimes I_2\otimes\sigma_z\right] \\ &\quad +\mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_x\otimes\sigma_x\otimes\sigma_x\right]-\mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_x\otimes\sigma_y\otimes\sigma_y\right] \\ &\quad -\mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_y\otimes\sigma_x\otimes\sigma_y\right]-\mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_y\otimes\sigma_y\otimes\sigma_x\right]\Big\}, \quad \text{(C1)} \end{aligned}$$

when $k=1$, we can calculate the inequality between Alice, Bob, and Charlie[1]

$$G_1^1=\frac{2}{3}+\lambda_1\left(0.1547-\frac{1+4\sin 2\alpha}{3}\right), \quad \text{(C2)}$$

where we have used $\rho^1=|\psi_{\mathrm{GHZ}_\alpha}\rangle\langle\psi_{\mathrm{GHZ}_\alpha}|$.

In order to get the relationship between $\rho^1$ and $\rho_{\mathrm{un}}^k$, we get

$$\begin{aligned} \rho_{\mathrm{un}}^k &= \frac{1}{3}\sum_{c^{k-1}|\hat{\boldsymbol{z}}_i^{\lambda_{i-1}}}\Bigg[\left\{I_2\otimes I_2\otimes\sqrt{E_{c^{k-1}|\hat{\boldsymbol{z}}_i^{\lambda_{i-1}}}}\right\} \\ &\quad\times\rho_{\mathrm{un}}^{k-1}\left\{I_2\otimes I_2\otimes\sqrt{E_{c^{k-1}|\hat{\boldsymbol{z}}_i^{\lambda_{i-1}}}}\right\}\Bigg] \\ &= \frac{1+\sqrt{1-\lambda_{k-1}^2}}{2}\rho_{\mathrm{un}}^{k-1}+\frac{1-\sqrt{1-\lambda_{k-1}^2}}{6} \\ &\quad\times\Big[(I_2\otimes I_2\otimes\sigma_x)\rho_{\mathrm{un}}^{k-1}(I_2\otimes I_2\otimes\sigma_x) \\ &\quad+(I_2\otimes I_2\otimes\sigma_y)\rho_{\mathrm{un}}^{k-1}(I_2\otimes I_2\otimes\sigma_y) \\ &\quad+(I_2\otimes I_2\otimes\sigma_z)\rho_{\mathrm{un}}^{k-1}(I_2\otimes I_2\otimes\sigma_z)\Big], \quad \text{(C3)} \end{aligned}$$

substituting Eq. (C3) into Eq. (C1), we get

$$\begin{aligned} \mathrm{Tr}\left[\rho_{\mathrm{un}}^k I_2\otimes\sigma_z\otimes\sigma_z\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}I_2\otimes\sigma_z\otimes\sigma_z\right]=\cdots \\ &= \prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1 I_2\otimes\sigma_z\otimes\sigma_z\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right), \quad \text{(C4)} \\ \mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_z\otimes\sigma_z\otimes I_2\right] &= \mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_z\otimes\sigma_z\otimes I_2\right]=\cdots=\mathrm{Tr}\left[\rho^1\sigma_z\otimes\sigma_z\otimes I_2\right]=1, \quad \text{(C5)} \\ \mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_z\otimes I_2\otimes\sigma_z\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_z\otimes I_2\otimes\sigma_z\right]=\cdots \\ &= \prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_z\otimes I_2\otimes\sigma_z\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right), \quad \text{(C6)} \\ \mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_x\otimes\sigma_x\otimes\sigma_x\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_x\otimes\sigma_x\otimes\sigma_x\right]=\cdots \\ &= \prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_x\otimes\sigma_x\otimes\sigma_x\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\sin 2\alpha, \quad \text{(C7)} \\ \mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_x\otimes\sigma_y\otimes\sigma_y\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_x\otimes\sigma_y\otimes\sigma_y\right]=\cdots \\ &= \prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_x\otimes\sigma_y\otimes\sigma_y\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)(-\sin 2\alpha), \quad \text{(C8)} \\ \mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_y\otimes\sigma_x\otimes\sigma_y\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_y\otimes\sigma_x\otimes\sigma_y\right]=\cdots \\ &= \prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_y\otimes\sigma_x\otimes\sigma_y\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)(-\sin 2\alpha), \quad \text{(C9)} \\ \mathrm{Tr}\left[\rho_{\mathrm{un}}^k\sigma_y\otimes\sigma_y\otimes\sigma_x\right] &= \frac{1+2\sqrt{1-\lambda_{k-1}^2}}{3}\mathrm{Tr}\left[\rho_{\mathrm{un}}^{k-1}\sigma_y\otimes\sigma_y\otimes\sigma_x\right]=\cdots \\ &= \prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)\mathrm{Tr}\left[\rho^1\sigma_y\otimes\sigma_y\otimes\sigma_x\right]=\prod_{j=1}^{k-1}\left(\frac{1+2\sqrt{1-\lambda_j^2}}{3}\right)(-\sin 2\alpha), \quad \text{(C10)} \end{aligned}$$

substituting Eqs. (C4)–(C10) into Eq. (C1) and using Eq. (C2), we obtain Eq. (18). If we have $G_1^1=0$, $G_1^2=0$, $G_1^3=0,\ldots,G_1^k=0,\ldots$, then

$$\lambda_1=\frac{2/3}{(1+4\sin 2\alpha)/3-0.1547},\quad \lambda_2=\frac{3\lambda_1}{1+2\sqrt{1-\lambda_1^2}},$$

$$\lambda_3=\frac{3\lambda_2}{1+2\sqrt{1-\lambda_2^2}},\quad \ldots,\quad \lambda_k=\frac{3\lambda_{k-1}}{1+2\sqrt{1-\lambda_{k-1}^2}},$$

where $\lambda_k\in(0,1]$, $\lambda_k>\lambda_{k-1}$ $(k>0)$, this sequence is monotonically increasing.

Let us first study the situation that no Charlie can demonstrate genuine tripartite EPR steering with a single Alice and a single Bob. Here, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1\in(0,1]$, Charlie$^1$ cannot violate the inequality (1). At this time, we can get $0<\sin 2\alpha\leq 0.366$, $0<\alpha\leq 0.187$. Therefore, in the case of $0<\alpha\leq 0.187$, even if all three observers, including Alice, Bob and Charlie, perform projective measurements, the correlation between their inputs and outputs cannot violate inequality (1). Because the state parameter of quantum state is very weak, any Charlie cannot demostrate steering with Bob and Alice when the generalized entangled states are initially shared.

Next, we discuss only one Charlie can demonstrate genuine tripartite EPR steering with a single Alice and a single Bob. For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1\in(0,1]$, Charlie$^1$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_2\in(0,1]$, Charlie$^2$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_1$ takes the maximum value of 1, we get $0.366<\sin 2\alpha$. Secondly, when $\lambda_2$ takes the maximum value of 1, we get $\sin 2\alpha\leq 0.521$ and $0.764\leq\lambda_1$. At this time, at most a Charlie, a single Alice, and a single Bob violate inequality (1) and Charlie$^1$ performs unsharp measurement with the sharpness parameter $0.764\leq\lambda_1\leq 1$.

Furture, two Charlie can demonstrate genuine tripartite EPR steering with a single Alice and a single Bob. Here, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1,\ \lambda_2\in(0,1]$, Charlie$^1$ and Charlie$^2$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_3\in(0,1]$, Charlie$^3$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_2$ takes the maximum value of 1, we get $\lambda_1<0.764$, and $0.581<\sin 2\alpha$. Secondly, when $\lambda_3$ takes the maximum value of 1, we get $0.764\leq\lambda_2$, $0.644\leq\lambda_1$, and $\sin 2\alpha\leq 0.619$. At this time, at most two Charlie and a single Alice, a single Bob violate inequality (2) and Charlie$^1$ and Charlie$^2$ perform unsharp measurements with the sharpness parameters $0.644\leq\lambda_1<0.764$ and $0.764\leq\lambda_2\leq 1$.

Three Charlie can demonstrate genuine tripartite EPR steering with a single Alice and a single Bob. Here, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1,\ \lambda_2,\ \lambda_3\in(0,1]$, Charlie$^1$, Charlie$^2$, and Charlie$^3$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_4\in(0,1]$, Charlie$^4$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_3$ takes the maximum value of 1, we get $\lambda_2<0.764$, $\lambda_1<0.644$, and $0.619<\sin 2\alpha$. Secondly, when $\lambda_4$ takes the maximum value of 1, we get $0.764\leq\lambda_3$, $0.644\leq\lambda_2$, $0.568\leq\lambda_1$, and $\sin 2\alpha\leq 0.746$. At this time, at most three Charlie and a single Alice, a single Bob violate inequality (1) and Charlie$^1$, Charlie$^2$, and Charlie$^3$ perform unsharp measurements with the sharpness parameters $0.568\leq\lambda_1<0.644$, $0.644\leq\lambda_2<0.764$, and $0.764\leq\lambda_3\leq 1$.

Four Charlie can demonstrate genuine tripartite EPR steering with a single Alice and a single Bob. Here, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4\in(0,1]$, Charlie$^1$, Charlie$^2$, Charlie$^3$, and Charlie$^4$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_5\in(0,1]$, Charlie$^5$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_4$ takes the maximum value of 1, we get $\lambda_1<0.568$, $\lambda_2<0.644$, $\lambda_3<0.764$, and $0.746<\sin 2\alpha$. Secondly, when $\lambda_5$ takes the maximum value of 1, we get $0.764\leq\lambda_4$, $0.644\leq\lambda_3$, $0.568\leq\lambda_2$, $0.514\leq\lambda_1$, and $\sin 2\alpha\leq 0.839$. At this time, at most four Charlie and a single Alice, a single Bob violate inequality (1) and Charlie$^1$, Charlie$^2$, Charlie$^3$, and Charlie$^4$ perform unsharp measurements with the sharpness parameters $0.514\leq\lambda_1<0.568$, $0.568\leq\lambda_2<0.644$, $0.644\leq\lambda_3<0.764$, and $0.764\leq\lambda_4\leq 1$.

Five Charlie can demonstrate genuine tripartite EPR steering with a single Alice and a single Bob. Here, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5\in(0,1]$, Charlie$^1$, Charlie$^2$, Charlie$^3$, Charlie$^4$, and Charlie$^5$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_6\in(0,1]$, Charlie$^6$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_5$ takes the maximum value of 1, we get $\lambda_1<0.514$, $\lambda_2<0.568$, $\lambda_3<0.644$, $\lambda_4<0.764$, and $0.893<\sin 2\alpha$. Secondly, when $\lambda_6$ takes the maximum value of 1, we get $0.764\leq\lambda_5$, $0.644\leq\lambda_4$, $0.568\leq\lambda_3$, $0.514\leq\lambda_2$, $0.473\leq\lambda_1$, and $\sin 2\alpha\leq 0.923$. At this time, at most five Charlie and a single Alice, a single Bob violate inequality (1) and Charlie$^1$, Charlie$^2$, Charlie$^3$, Charlie$^4$, and Charlie$^5$ perform unsharp measurements with the sharpness parameters $0.473\leq\lambda_1<0.514$, $0.514\leq\lambda_2<0.568$, $0.568\leq\lambda_3<0.644$, $0.644\leq\lambda_4<0.764$, and $0.764\leq\lambda_5\leq 1$, respectively.

Six Charlie can demonstrate genuine tripartite EPR steering with a single Alice and a single Bob. Here, for choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6\in(0,1]$, Charlie$^1$, Charlie$^2$, Charlie$^3$, Charlie$^4$, Charlie$^5$, and Charlie$^6$ can sequentially violate the inequality (1). For choice of state parameter $\alpha\in(0,\pi/4]$, with $\lambda_7\in(0,1]$, Charlie$^7$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_6$ takes the maximum value of 1, we get $\lambda_1<0.473$, $\lambda_2<0.514$,

$\lambda_3 < 0.568$, $\lambda_4 < 0.644$, $\lambda_5 < 0.764$, and $0.923 < \sin 2\alpha$. Secondly, when $\lambda_7$ takes the maximum value of 1, we get $0.764 \leq \lambda_6$, $0.644 \leq \lambda_5$, $0.568 \leq \lambda_4$, $0.514 \leq \lambda_3$, $0.473 \leq \lambda_2$, $0.411 \leq \lambda_1$, and $\sin 2\alpha \leq 1$. At this time, at most six Charlie and a single Alice, a single Bob violate inequality (1) and Charlie$^1$, Charlie$^2$, Charlie$^3$, Charlie$^4$, Charlie$^5$, and Charlie$^6$ perform unsharp measurements with the sharpness parameters $0.441 \leq \lambda_1 < 0.473$, $0.473 \leq \lambda_2 < 0.514$, $0.514 \leq \lambda_3 < 0.568$, $0.568 \leq \lambda_4 < 0.644$, $0.644 \leq \lambda_5 < 0.764$, and $0.764 \leq \lambda_6 \leq 1$.

Seven Charlie can demonstrate genuine tripartite EPR steering with a single Alice and a single Bob. Here, for choice of state parameter $\alpha \in (0, \pi/4]$, with $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in (0,1]$, Charlie$^1$, Charlie$^2$, Charlie$^3$, Charlie$^4$, Charlie$^5$, Charlie$^6$, and Charlie$^7$ can sequentially violate the inequality (1). For choice of state parameter $\alpha \in (0, \pi/4]$, with $\lambda_8 \in (0,1]$, Charlie$^8$ cannot sequentially violate the inequality (1). Firstly, when $\lambda_7$ takes the maximum value of 1, we get $\lambda_1 < 0.441$, $\lambda_2 < 0.473$, $\lambda_3 < 0.514$, $\lambda_4 < 0.568$, $\lambda_5 < 0.644$, $\lambda_6 < 0.764$, and $1 < \sin 2\alpha$. Secondly, when $\lambda_8$ takes the maximum value of 1, we get $0.764 \leq \lambda_7$, $0.644 \leq \lambda_6$, $0.568 \leq \lambda_5$, $0.514 \leq \lambda_4$, $0.473 \leq \lambda_3$, $0.441 \leq \lambda_2$, $0.415 \leq \lambda_1$, and $\sin 2\alpha \leq 1.206$. For $0 < \sin 2\alpha \leq 1$, Therefore, Charlie$^1$, Charlie$^2$, Charlie$^3$, Charlie$^4$, Charlie$^5$, Charlie$^6$, and Charlie$^7$ cannot sequentially violate the inequality (1) with a single Alice and a single Bob.

## Appendix D: Proof of Theorem 4

By the same method as proving Theory 3, we obtain at most $m$ ($m = 0,1,2,3$) Charlie can be steered by Alice and Bob, and give the range of state parameters $\alpha$ and measured sharpness parameters $\lambda_i$ ($i = 1,2,3$), as shown in Table 4.

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