Unified entropy entanglement with tighter constraints on multipartite systems

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Monogamy and polygamy relations characterize the distributions of entanglement in multipartite systems. We provide a characterization of multiqubit entanglement constraints in terms of unified-(q,s) entropy. A class of tighter monogamy inequalities of multiqubit entanglement based on the α-th power of unified-(q,s) entanglement for $\alpha \geq 1$ and a class of polygamy inequalities in terms of the β-th power of unified-(q,s) entanglement of assistance are established in this paper. Our results present a general class of the monogamy and polygamy relations for bipartite entanglement measures based on unified-(q,s) entropy, which are tighter than the existing ones. What is more, some usual monogamy and polygamy relations, such as monogamy and polygamy relations based on entanglement of formation, Renyi-q entanglement of assistance and Tsallis-q entanglement of assistance, can be obtained from these results by choosing appropriate parameters (q,s) in unified-(q,s) entropy entanglement. Typical examples are also presented for illustration.

Keywords: entanglement, monogamy relations, polygamy relations

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1. Introduction

Quantum entanglement[1–8] has been considered as one of the most crucial resources for quantum communication, which has been shown to be perfectly secure against any interior or exterior eavesdropper. Whereas entanglement in bipartite quantum systems has been intensively studied with rich understanding, the situation becomes far more difficult in the case of multipartite quantum systems, and very few are known for its characterization and quantification. It is a fundamental difference, which regarded as an important property between quantum and classical world, that entanglement is restricted shareability in multiparty quantum systems. Such restriction of entanglement shareability among multiparty systems is known as monogamy of entanglement.[9,10] Moreover, the monogamy property has emerged as an ingredient in the security analysis of quantum key distribution.[11] Thus, it is an important and necessary task to characterize monogamy of entanglement to understand the whole picture of quantum entanglement in multipartite systems, as well as its possible applications in quantum information theory.

Qualitatively, monogamy of entanglement measures the shareability of entanglement in a composite quantum system, *i.e.*, the more two subsystems are entangled the less this pair has entanglement with the rest of the system. Taking use of the entanglement measurements to quantify entanglement between subsystems, monogamy inequalities can be constructed in a composite system. As we all know, the Coffman–Kundu–Wootters (CKW) inequality[9] is described by a kind of monogamy of bipartite quantum entanglement measured by concurrence[9] in a three-qubit quantum system. On the other hand, the assisted entanglement, which is a dual amount to bipartite entanglement measures, is also known to have a dually monogamous property in multiparty quantum systems, namely, polygamy of entanglement. Polygamy of entanglement is also mathematically characterized as polygamy inequality, $E_{aA|BC} \leq E_{aAB} + E_{aAC}$ for a three-party quantum state ρ_{ABC}, where E_{aABC} is the assisted entanglement.[13] The polygamy inequality was first proposed in three-qubit systems using tangle of assistance[13] and generalized into multiqubit systems in terms of various assisted entanglements.[14–16]

Recently, in terms of non-negative power of entanglement measures and assisted entanglements the monogamy and polygamy inequalities of multiqubit entanglement were generalized. It is shown that in the case of $x \geq \sqrt{2}$ and $x \geq 2$ the x-th power of entanglement of formation and concurrence can be used to establish multiqubit monogamy inequalities.[17] Later, tight classes of monogamy and polygamy inequalities of multiqubit entanglement using the non-negative power of various entanglement measures were also proposed,[18–22] and a monogamy and polygamy relations of multiqubit entanglement in terms of unified entropy were given.[23,24]

In this paper, we provide a full characterization of multiqubit entanglement monogamy and polygamy constraints in terms of non-negative power of entanglement measures based on unified entropy.[25,26] We devote to completing a class of tight monogamy inequalities of multiqubit entanglement

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based on the α-th power of unified-(q,s) entanglement for $\alpha \geq 1$. Respectively, in the case of $0 \leq \beta \leq 1$, a class of tight polygamy inequalities of multiqubit entanglement in terms of the β-th power of unified-(q,s) entanglement of assistance[27] is given.

2. Preliminaries for unified entropy monogamy relations

In this section, we recall some concept will use in this paper. Firstly, the unified-(q,s) entropy for a quantum state $\rho \in \mathcal{H}$ is defined as[25,26]

$$S_{q,s}(\rho) = \frac{1}{(1-q)s}\left((\mathrm{Tr}\rho^q)^s - 1\right) \tag{1}$$

with $q,s \geq 0$, $q \neq 1$, and $s \neq 0$. The unified-(q,s) entropy has a broadly physical meaning that it finds applications, such as, from characterizing quantum channels, studying of complexity beyond scrambling to witnessing entanglement, and so on.[28–31] Basing unified-(q,s) entropy, the entanglement for a bipartite pure state $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$ named unified-(q,s) entanglement (UE) is given as[16]

$$E_{q,s}(|\psi\rangle_{AB}) = S_{q,s}(\rho_A) \tag{2}$$

with $q,s \geq 0$ and $\rho_A = \mathrm{Tr}_B|\psi\rangle_{AB}\langle\psi|$ is the reduced density matrix onto subsystem A. Taking use of convex-roof extension, we can get the UE for bipartite mixed state $\rho_{AB} \in \mathcal{H}_{AB}$ based on its pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$ and $\sum_i p_i = 1$ that

$$E_{q,s}(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle_{AB}\}} \sum_i p_i E_{q,s}(|\psi_i\rangle_{AB}), \tag{3}$$

where the minimum is taken over all possible decompositions.

It is obviously to found that UE is one of the most general classes of bipartite entanglement measures which can cover the Renyi-q entanglement (RE)[32,33] as s tends to 0 and converges to Tsallis-q entanglement (TE)[15,34] when s tends to 1, what is more, with any nonnegative s, converges to entanglement of formation (EoF) as q tends to 1,

$$\lim_{q\to 1} E_{q,s}(\rho_{AB}) = E_{\mathrm{f}}(\rho_{AB}). \tag{4}$$

That is to say, RE, TE, and EoF are special classes of UE.[16]

As a dual quantity to entanglement, the entanglement of assistance (EoA) is also an important quantity to entanglement. We consider the unified-(q,s) entanglement of assistance (UEoA) that[27]

$$E^a_{q,s}(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle_{AB}\}} \sum_i p_i E_{q,s}(|\psi_i\rangle_{AB}) \tag{5}$$

for each $q,s \geq 0$, where the maximum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi\rangle_{AB}\langle\psi|$ and $\sum_i p_i = 1$. Similar to UE in Eq. (3), the UEoA is continuous for the parameters q and s assures that UEoA reduces to Renyi-q entanglement of assistance (REoA)[33] and Tsallis-q entanglement of assistance (TEoA)[15] when s tends to 0 or 1, respectively. For any nonnegative s, with q tends to 1, UEoA reduces to entanglement of assistance (EoA)[27]

$$\lim_{q\to 1} E^a_{q,s}(\rho_{AB}) = E^a(\rho_{AB}). \tag{6}$$

For any N-qubit state $\rho_{A_1A_2\cdots A_N}$ and its two-qubit reduced density matrices $\rho_{A_1A_i}$ with $i = 2,\ldots,N$, we can establish a monogamy inequality in multiqubit systems based on UE,

$$E_{q,s}(\rho_{A_1|A_2\cdots A_N}) \geq \sum_{i=2}^{N} E_{q,s}(\rho_{A_1A_i}), \tag{7}$$

for $q \geq 2$, $0 \leq s \leq 1$, and $qs \leq 3$.[16] It is also shown that UEoA can be used to characterize the polygamy of multiqubit entanglement as[27]

$$E^a_{q,s}(\rho_{A_1|A_2\cdots A_N}) \leq \sum_{i=2}^{N} E^a_{q,s}(\rho_{A_1A_i}) \tag{8}$$

for $1 \leq q \leq 2$ and $-q^2+4q-3 \leq s \leq 1$.

3. Tighter monogamy relations of multiqubit for unified entropy

Monogamy relation is one of the essential properties of quantum entanglement, which characterizes the distribution of entanglement in a multipartite system. Recently, it has become one of the hot issues in the study of quantum information and quantum computation. As is known, monogamy relation is closely related to the security of quantum cryptographic protocols based on entanglement.[11] Thus, optimized monogamy relations give rise to finer characterizations of the entanglement distributions, and then limits the amount of correlations that an eavesdropper can have with the honest parties. In this paper, we provide a class of tighter monogamy relations for multiqubit entanglement based on α-th power of UE. Before we present our main results, we first give a lemma.

Lemma 1 Suppose that k is a real number satisfying $0 < k \leq 1$. Then, for any real numbers x and t, $0 \leq t \leq k$, we have $(1+t)^x \geq 1+\mathcal{K}_x t^x$ for $x \geq 1$, where $\mathcal{K}_x = [(1+k)^x - 1]/k^x$.

From Lemma 1, we have the following theorem, which states that a class of monogamy inequalities of multipartite entanglement can be established.

Theorem 1 For any $2\otimes 2\otimes 2$ mixed state $\rho \in H_A \otimes H_B \otimes H_C$, if $kE_{q,s}(\rho_{AB}) \geq E_{q,s}(\rho_{AC})$ with $0 < k \leq 1$, we have

$$E^\alpha_{q,s}(\rho_{A|BC}) \geq E^\alpha_{q,s}(\rho_{AB}) + \mathcal{K}_\alpha E^\alpha_{q,s}(\rho_{AC}), \tag{9}$$

for $\alpha \geq 1$, $q \geq 2$, $0 \leq s \leq 1$, $qs \leq 3$ and $\mathcal{K}_\alpha = [(1+k)^\alpha - 1]/k^\alpha$.

Proof It has been shown that $E_{q,s}(\rho_{A|BC}) \geq E_{q,s}(\rho_{AB}) + E_{q,s}(\rho_{AC})$ for arbitrary $2\otimes 2\otimes 2$ tripartite state ρ_{ABC}.[16] Then, if $kE_{q,s}(\rho_{AB}) \geq E_{q,s}(\rho_{AC})$, we have

$$\begin{aligned} E^{\alpha}_{q,s}(\rho_{A|BC}) &\geq (E_{q,s}(\rho_{AB}) + E_{q,s}(\rho_{AC}))^{\alpha} \\ &= E^{\alpha}_{q,s}(\rho_{AB})\left(1+\frac{E_{q,s}(\rho_{AC})}{E_{q,s}(\rho_{AB})}\right)^{\alpha} \\ &\geq E^{\alpha}_{q,s}(\rho_{AB})\left[1+\mathcal{K}_{\alpha}\left(\frac{E_{q,s}(\rho_{AC})}{E_{q,s}(\rho_{AB})}\right)^{\alpha}\right] \\ &= E^{\alpha}_{q,s}(\rho_{AB}) + \mathcal{K}_{\alpha}E^{\alpha}_{q,s}(\rho_{AC}), \end{aligned}$$

where the second inequality is due to Lemma 1.

Since the symmetry of subsystems A and B, we have assumed that $E_{q,s}(\rho_{AB}) \geq E_{q,s}(\rho_{AC})$ in Theorem 1. Moreover, If $E_{q,s}(\rho_{AB}) = 0$ and $E_{q,s}(\rho_{AB}) \geq E_{q,s}(\rho_{AC})$, then $E_{q,s}(\rho_{AB}) = E_{q,s}(\rho_{AC}) = 0$, the lower bound is trivially. So, without loss of generality, $E_{q,s}(\rho_{AB}) > 0$ is set. Furthermore, Theorem 1 can be generalized to multipartite qubit systems, we have the following theorem.

Theorem 2 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, if $kE_{q,s}(\rho_{AB_i}) \geq \sum_{j=i+1}^{N-1} E_{q,s}(\rho_{AB_j})$ for every $i = 1,\ldots,N-2$, we have

$$\begin{aligned} E^{\alpha}_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}) \geq\ & E^{\alpha}_{q,s}(\rho_{AB_1}) + \cdots + \mathcal{K}^{N-3}_{\alpha}E^{\alpha}_{q,s}(\rho_{AB_{N-2}}) \\ & + \mathcal{K}^{N-2}_{\alpha}E^{\alpha}_{q,s}(\rho_{AB_{N-1}}), \end{aligned} \tag{10}$$

for $\alpha \geq 1$, $q \geq 2$, $0 \leq s \leq 1$, $qs \leq 3$, and $\mathcal{K}_{\alpha} = [(1+k)^{\alpha} - 1]/k^{\alpha}$.

Proof From the inequality in Lemma 1, we have

$$\begin{aligned} & E^{\alpha}_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}) \\ \geq\ & E^{\alpha}_{q,s}(\rho_{AB_1}) + \mathcal{K}_{\alpha}\left(\sum_{j=2}^{N-1} E_{q,s}(\rho_{AB_j})\right)^{\alpha} \\ \geq\ & E^{\alpha}_{q,s}(\rho_{AB_1}) + \mathcal{K}_{\alpha}E^{\alpha}_{q,s}(\rho_{AB_2}) + \mathcal{K}^{2}_{\alpha}\left(\sum_{j=3}^{N-1} E_{q,s}(\rho_{AB_j})\right)^{\alpha} \\ \geq\ & \cdots \\ \geq\ & E^{\alpha}_{q,s}(\rho_{AB_1}) + \mathcal{K}_{\alpha}E^{\alpha}_{q,s}(\rho_{AB_2}) + \cdots + \mathcal{K}^{N-2}_{\alpha}E^{\alpha}_{q,s}(\rho_{AB_{N-1}}). \end{aligned} \tag{11}$$

Here, we point out some important facts that, on one hand, equation (7) is a special case of Theorem 2 which gives a new class of monogamy relations for multiqubit states since equation (10) reduces to Eq. (7) when $\alpha = 1$ and $k = 1$. On the other hand, comparing with another class of multiqubit monogamy inequalities in terms of α-th power of UE provided by Ref. [24], Theorem 2 may show its superiority. In fact, the authors in Ref. [24] showed any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$ satisfies

$$E^{\alpha}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}) \geq \sum_{j=1}^{N-1}(2^{\alpha}-1)^{j-1}E^{\alpha}_{q,s}(\rho_{AB_j}), \tag{12}$$

with $\alpha \geq 1$, $q \geq 2$, and $0 \leq s \leq 1$, $qs \leq 3$ and constraint $E_{q,s}(\rho_{AB_i}) \geq \sum_{j=i+1}^{N-1} E_{q,s}(\rho_{AB_j})$ for each $i = 1,\ldots,N-2$. As for $k = 1$, obviously, our formula Eq. (10) in Theorem 2 converges to Eq. (12) and, for example, $\mathcal{K}_{\alpha} > 2^{\alpha} - 1$ with $k = 1/2$ and $\alpha = 2$, equation (10) shows a tighter monogamy relation with larger lower bounds. A more vivid example to show the capability of Theorem 2 is as followed.

Example 1 Let us consider the three-qubit state $|\psi\rangle_{ABC}$ in the generalized Schmidt decomposition form

$$\begin{aligned} |\psi\rangle_{ABC} = \ & \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle \\ & + \lambda_3|110\rangle + \lambda_4|111\rangle, \end{aligned} \tag{13}$$

where $\lambda_i \geq 0$, $i = 0,1,2,3,4$, and $\sum_{i=0}^{4}\lambda_i^2 = 1$. From the definition of UE in Eq. (3), when $q = 2$ and s tends to 1, we have $T_2(|\psi\rangle_{A|BC}) = 1 - \mathrm{Tr}\rho_A^2$, $T_2(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle_{AB}\}}\sum_i p_i T_2(|\psi_i\rangle_{AB})$, where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi\rangle_{AB}\langle\psi|$ with $\sum_i p_i = 1$ and $\rho_A = \mathrm{Tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, $\rho_{AB} = \mathrm{Tr}_C|\psi\rangle_{ABC}\langle\psi|$. Then we get $T_2(|\psi\rangle_{A|BC}) = 2\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2)$, $T_2(\rho_{AB}) = 2\lambda_0^2\lambda_2^2$, and $T_2(\rho_{AC}) = 2\lambda_0^2\lambda_3^2$. Taking $\lambda_2 = \sqrt{2}/2$, $\lambda_0 = \lambda_1 = \lambda_3 = \lambda_4 = 1/2\sqrt{2}$, one can get $T_2(|\psi\rangle_{A|BC})^{\alpha} = (3/16)^{\alpha}$, $T_2(\rho_{AB})^{\alpha} = (1/8)^{\alpha}$, and $T_2(\rho_{AC})^{\alpha} = (1/32)^{\alpha}$. Thus, let $k = 1/4$, then we have $T_2(\rho_{AB})^{\alpha} + (2^{\alpha}-1)T_2(\rho_{AC})^{\alpha} = (1/8)^{\alpha} + (2^{\alpha}-1)(1/32)^{\alpha}$, and $T_2(\rho_{AB})^{\alpha} + \mathcal{K}_{\alpha}T_2(\rho_{AC})^{\alpha} = (1/8)^{\alpha} + (5^{\alpha}-4^{\alpha})(1/32)^{\alpha}$. Obviously, the new result is better than Eq. (12) for $\alpha \geq 1$, as shown in Fig. 1.

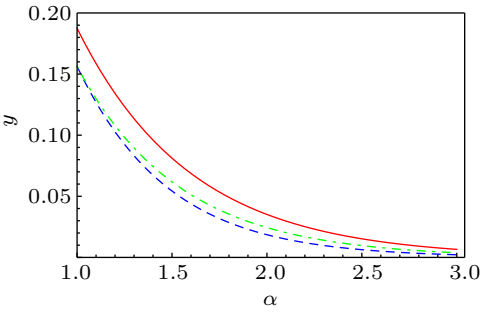


Fig. 1. The lower bounds of UE of $|\psi\rangle_{ABC}$ which varies with the parameter α is represented at y axis. The red line represents the values of UE of $|\psi\rangle_{ABC}$ in Example 1, the green line and blue line show the lower bound in our calculation and given in Ref. [24], respectively.

Now, we give out a universal monogamy inequality avoiding the conditions for Eqs. (10) and (12) are not always satisfied.

Theorem 3 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, if $kE_{q,s}(\rho_{AB_i}) \geq \sum_{l=i+1}^{N-1} E_{q,s}(\rho_{AB_l})$ for $i = 1,2,\ldots,m$, and $E_{q,s}(\rho_{AB_j}) \leq k\sum_{l=j+1}^{N-1} E_{q,s}(\rho_{AB_l})$ for $j = m+1,\ldots,N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, $0 < k \leq 1$, we have

$$\begin{aligned} & E^{\alpha}_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}) \\ \geq\ & \sum_{i=1}^{m}\mathcal{K}_{\alpha}^{i-1}E^{\alpha}_{q,s}(\rho_{AB_i}) + \mathcal{K}_{\alpha}^{m+1}(E^{\alpha}_{q,s}(\rho_{AB_{m+1}}) \\ & + \cdots + E^{\alpha}_{q,s}(\rho_{AB_{N-2}})) + \mathcal{K}_{\alpha}^{m}E^{\alpha}_{q,s}(\rho_{AB_{N-1}}) \end{aligned} \tag{14}$$

for $\alpha \geq 1$, $q \geq 2$, $0 \leq s \leq 1$, $qs \leq 3$, and $\mathcal{K}_\alpha = [(1+k)^\alpha - 1]/k^\alpha$.

Proof From the inequality in Lemma 1, we have

$$\begin{aligned}
&E^\alpha_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}) \\
&\geq E^\alpha_{q,s}(\rho_{AB_1}) + \mathcal{K}_\alpha \left(\sum_{i=2}^{N-1} E_{q,s}(\rho_{AB_i})\right)^\alpha \\
&\geq E^\alpha_{q,s}(\rho_{AB_1}) + \mathcal{K}_\alpha E^\alpha_{q,s}(\rho_{AB_2}) + \mathcal{K}^2_\alpha \left(\sum_{i=3}^{N-1} E_{q,s}(\rho_{AB_i})\right)^\alpha \\
&\geq \cdots \\
&\geq E^\alpha_{q,s}(\rho_{AB_1}) + \cdots + \mathcal{K}^{m-1}_\alpha E^\alpha_{q,s}(\rho_{AB_m}) \\
&\quad + \mathcal{K}^m_\alpha \left(\sum_{i=m+1}^{N-1} E_{q,s}(\rho_{AB_i})\right)^\alpha . \qquad (15)
\end{aligned}$$

Similarly, as $E_{q,s}(\rho_{AB_j}) \leq k\sum_{l=j+1}^{N-1} E_{q,s}(\rho_{AB_l})$ for $j = m+1,\ldots,N-2$, we get

$$\begin{aligned}
&E^\alpha_{q,s}(\rho_{A|B_{m+1}\cdots B_{N-1}}) \\
&\geq \mathcal{K}_\alpha E^\alpha_{q,s}(\rho_{AB_{m+1}}) + \left(\sum_{i=m+2}^{N-1} E_{q,s}(\rho_{AB_i})\right)^\alpha \\
&\geq \mathcal{K}_\alpha (E^\alpha_{q,s}(\rho_{AB_{m+1}}) + \cdots + E^\alpha_{q,s}(\rho_{AB_{N-2}})) + E^\alpha_{q,s}(\rho_{AB_{N-1}}). \qquad (16)
\end{aligned}$$

Combining Eqs. (15) and (16), we then have Theorem 3.

Similar to the statement above, equation (14) reduces to Eq. (7) of Ref. [16] for $\alpha = 1$ and $k = 1$. What is more, $\alpha > 1$ and $0 < k < 1$, $\mathcal{K}_\alpha > 2^\alpha - 1$ and all $1 \leq m \leq N-3$, formula (14) in Theorem 3 leads to a tighter monogamy relation with larger lower bounds than Eq. (15) in Ref. [24]. The monogamy relation based on entanglement measure UE in Theorem 3 is different from Theorem 2. It is noted that the classes of states in Theorem 3 do not must satisfy the conditions in Theorem 2. In this circumstance, Theorem 3 is better than Theorem 2.

Example 2 Consider the 4-qubit generlized W state

$$|W\rangle_{AB_1B_2B_3} = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle). \qquad (17)$$

Suppose s tends to 1 and $q = 2$, we then have $T_2(|W\rangle_{A|B_1B_2B_3}) = 3/8$, $T_2(\rho_{AB_i}) = 1/8$, $i = 1,2,3$, where $T_2(\rho_{AB_i})$ has the same definition in Example 1. It is easy to see $T_2(\rho_{AB_i})$ does not satisfy the condition (10) of Theorem 2 and Eq. (12) from Ref. [24]. From inequality (14) of Theorem 3, we have $T_2{}^\alpha(|W\rangle_{A|B_1B_2B_3}) \geq (2\mathcal{K}_\alpha + 1)(1/8)^\alpha$.

In Ref. [18], the authors have presented

$$E^\alpha_{A|B_1B_2\cdots B_{N-1}} \geq E^\alpha_{AB_1} + \frac{\alpha}{\sqrt{2}} E^\alpha_{AB_2} + \cdots + \left(\frac{\alpha}{\sqrt{2}}\right)^{N-2} E^\alpha_{AB_{N-1}}$$

for $\alpha \geq \sqrt{2}$ with the conditions $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$, $i = 1,2,\ldots,N-2$ for N-qubit states; in Ref. [17], the authors have obtained $E^\alpha_{A|B_1B_2\cdots B_{N-1}} \geq E^\alpha_{A|B_1} + E^\alpha_{A|B_2} + \cdots + E^\alpha_{AB_{N-1}}$ for $\alpha \geq \sqrt{2}$. Because EoF coincides with concurrence for any $2 \otimes d$-dimensional pure state, we get

$$\begin{aligned}
E^\alpha_{A|B_1B_2\cdots B_{N-1}} &\geq E^\alpha_{AB_1} + (2^{\alpha/\sqrt{2}} - 1)E^\alpha_{AB_2} + \cdots \\
&\quad + \left(2^{\alpha/\sqrt{2}} - 1\right)^{N-2} E^\alpha_{AB_{N-1}}
\end{aligned}$$

for $\alpha \geq \sqrt{2}$ and $k = 1$ with the same conditions as q tends to 1 in Theorem 2. As $2^{\alpha/\sqrt{2}} - 1 \geq \alpha/\sqrt{2} \geq 1$ for $\alpha \geq \sqrt{2}$, obviously, our result is better than that in Refs. [17,18].

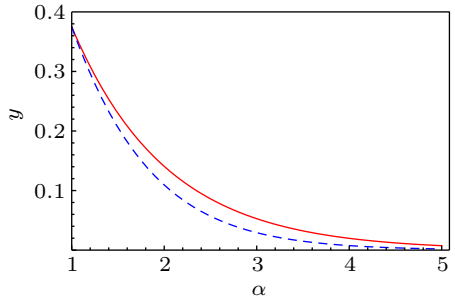


Fig. 2. The lower bounds of UE of $|W\rangle_{AB_1B_2B_3}$ which varies with the parameter α and let $k = 1$ is represented at y axis. The red line shows the UE of $|W\rangle_{AB_1B_2B_3}$ in Example 2 and blue line illustrates the lower bound of Theorem 3.

4. Tighter polygamy relations of multiqubit entanglement

Starting from this section we provide a class of polygamy relations in multiqubit systems in terms of UEoA, which are tighter than the existing ones. At the beginning, we introduce a lemma for later use.

Lemma 2 Suppose that k is a real number satisfying $0 < k \leq 1$. Then, for any real numbers x and t, $0 \leq t \leq k$, we have $(1+t)^x \leq 1 + \mathcal{K}_x t^x$ for $0 \leq x \leq 1$, where $\mathcal{K}_x = [(1+k)^x - 1]/k^x$.

Theorem 4 For any $2 \otimes 2 \otimes 2$ mixed state $\rho \in H_A \otimes H_B \otimes H_C$, if $kE^a_{q,s}(\rho_{AB}) \geq E^a_{q,s}(\rho_{AC})$, we have

$$(E^a_{q,s}(\rho_{A|BC}))^\beta \leq (E^a_{q,s}(\rho_{AB}))^\beta + \mathcal{K}_\beta (E^a_{q,s}(\rho_{AC}))^\beta, \qquad (18)$$

for $0 \leq \beta \leq 1$, $-q^2 + 4q - 3 \leq s \leq 1$, $1 \leq q \leq 2$, and $\mathcal{K}_\beta = [(1+k)^\beta - 1]/k^\beta$.

Proof It has been shown that $E^a_{q,s}(\rho_{A|BC}) \leq E^a_{q,s}(\rho_{AB}) + E^a_{q,s}(\rho_{AC})$ for arbitrary $2 \otimes 2 \otimes 2$ tripartite state ρ_{ABC}.[27] Then, if $kE^a_{q,s}(\rho_{AB}) \geq E^a_{q,s}(\rho_{AC})$, we have

$$\begin{aligned}
(E^a_{q,s}(\rho_{A|BC}))^\beta &\leq (E^a_{q,s}(\rho_{AB}) + E^a_{q,s}(\rho_{AC}))^\beta \\
&= (E^a_{q,s}(\rho_{AB}))^\beta \left(1 + \frac{E^a_{q,s}(\rho_{AC})}{E^a_{q,s}(\rho_{AB})}\right)^\beta \\
&\leq (E^a_{q,s}(\rho_{AB}))^\beta \left[1 + \mathcal{K}_\beta \left(\frac{E^a_{q,s}(\rho_{AC})}{E^a_{q,s}(\rho_{AB})}\right)^\beta\right] \\
&= (E^a_{q,s}(\rho_{AB}))^\beta + \mathcal{K}_\beta (E^a_{q,s}(\rho_{AC}))^\beta,
\end{aligned}$$

where the second inequality is due to Lemma 2.

Here, if $E^a_{q,s}(\rho_{AB})=0$, the $E^a_{q,s}(\rho_{AB})$ and $E^a_{q,s}(\rho_{AC})$ are also equal to 0. Moreover, the assumption $kE^a_{q,s}(\rho_{AB})\geq E^a_{q,s}(\rho_{AC})$ is convenient for proving the theorem and without loss of generality.

With preceding lemmas, we show a Theorem for multiqubit systems below:

Theorem 5 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$ conditioned with $kE^a_{q,s}(\rho_{AB_i})\geq\sum_{j=i+1}^{N-1}E^a_{q,s}(\rho_{AB_j})$ for $i=1,\ldots,N-2$, we have

$$(E^a_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta\leq\sum_{j=1}^{N-1}\mathcal{K}_\beta^{j-1}(E^a_{q,s}(\rho_{AB_j}))^\beta \tag{19}$$

for $0\leq\beta\leq1$, $-q^2+4q-3\leq s\leq1$ and $1\leq q\leq2$, and $\mathcal{K}_\beta=[(1+k)^\beta-1]/k^\beta$.

Proof From the inequality in Lemma 2,

$$\begin{aligned}
&(E^a_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta\\
&\leq(E^a_{q,s}(\rho_{AB_1}))^\beta+\mathcal{K}_\beta\left(\sum_{j=2}^{N-1}E^a_{q,s}(\rho_{AB_j})\right)^\beta\\
&\leq(E^a_{q,s}(\rho_{AB_1}))^\beta+\mathcal{K}_\beta(E^a_{q,s}(\rho_{AB_2}))^\beta+\mathcal{K}_\beta^2\left(\sum_{j=3}^{N-1}E^a_{q,s}(\rho_{AB_j})\right)^\beta\\
&\leq\cdots\\
&\leq(E^a_{q,s}(\rho_{AB_1}))^\beta+\mathcal{K}_\beta(E^a_{q,s}(\rho_{AB_2}))^\beta+\cdots\\
&\quad+\mathcal{K}_\beta^{N-2}(E^a_{q,s}(\rho_{AB_{N-1}}))^\beta.
\end{aligned}$$

Recently, in Ref. [24] a class of multiqubit polygamy inequalities was found based on β-th power of concurrence. For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$ and $0\leq\beta\leq1$, $-q^2+4q-3\leq s\leq1$, $1\leq q\leq2$,

$$(E^a_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta\leq\sum_{j=1}^{N-1}(2^\beta-1)^{j-1}(E^a_{q,s}(\rho_{AB_j}))^\beta \tag{20}$$

with constraint $kE^a_{q,s}(\rho_{AB_i})\geq\sum_{j=i+1}^{N-1}E^a_{q,s}(\rho_{AB_j})$ for each $i=1,\ldots,N-2$. Through the following example, we can deeply realize that the Eq. (19) induce a tighter polygamy relation than Eq. (20) of Ref. [24] as $\mathcal{K}_\beta\leq2^\beta-1$ with $0\leq\beta\leq1$.

Example 3 Let us consider again the three-qubit state $|\psi\rangle_{ABC}$ of Eq. (13). From the definition of UEoA in Eq. (5), when $q=2$ and s tends to 1, we have $T_2(|\psi\rangle_{A|BC})=1-\mathrm{Tr}\rho_A^2$, $T_2(\rho_{AB})=\max_{\{p_i,|\psi_i\rangle_{AB}\}}\sum_i p_iT_2(|\psi_i\rangle_{AB})$, where the maximum is taken over all possible pure state decompositions of $\rho_{AB}=\sum_i p_i|\psi\rangle_{AB}\langle\psi|$ and $\sum_i p_i=1$ and $\rho_A=\mathrm{Tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, $\rho_{AB}=\mathrm{Tr}_C|\psi\rangle_{ABC}\langle\psi|$. Then we have $T_2^a(|\psi\rangle_{A|BC})=2\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2)$, $T_2^a(\rho_{AB})=2\lambda_0^2(\lambda_2^2+\lambda_4^2)$, and $T_2^a(\rho_{AC})=2\lambda_0^2(\lambda_3^2+\lambda_4^2)$. Let $\lambda_2=\sqrt{2}/2$, $\lambda_0=\lambda_1=\lambda_3=\lambda_4=1/2\sqrt{2}$, one can get $T_2^a(|\psi\rangle_{A|BC})^\beta=(3/16)^\beta$, $T_2^a(\rho_{AB})^\beta=(5/32)^\beta$, and $T_2^a(\rho_{AC})^\beta=(1/16)^\beta$. Thus, let $k=1/2$, then we have $T_2^a(\rho_{AB})^\beta+(2^\beta-1)T_2^a(\rho_{AC})^\beta=(5/32)^\beta+(2^\beta-1)(1/16)^\beta$, and $T_2^a(\rho_{AB})^\beta+\mathcal{K}_\beta T_2^a(\rho_{AC})^\beta=(5/32)^\beta+(3^\beta-2^\beta)(1/16)^\beta$. It is obvious that the result in this work is better than Eq. (20) of Ref. [24] while $0\leq\beta\leq1$, see Fig. 3.

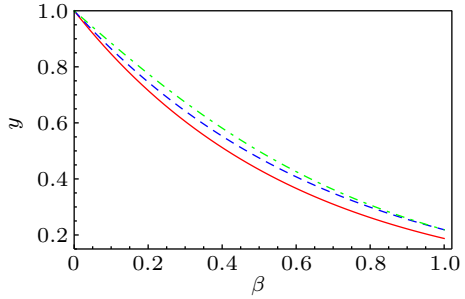


Fig. 3. The axis y denotes the upper bound of the UEoA of $|\psi\rangle_{ABC}$, which are functions of β. The red solid line represents the UEoA of $|\psi\rangle_{ABC}$ in Eq. (13), blue dashed line represents the upper bound of our result, green dot–dashed line represents the upper bound given in Ref. [24].

To obtain a widespread result getting rid of the fact that conditions of Eq. (19) in Theorem 5 and Eq. (20) in Ref. [24] are not generally satisfied, we show the following Theorem.

Theorem 6 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, if $kE^a_{q,s}(\rho_{AB_i})\geq\sum_{l=i+1}^{N-1}E^a_{q,s}(\rho_{AB_l})$ for $i=1,\ldots,m$, and $E^a_{q,s}(\rho_{AB_j})\leq k\sum_{l=j+1}^{N-1}E^a_{q,s}(\rho_{AB_l})$ for $j=m+1,\ldots,N-2$, $\forall1\leq m\leq N-3$, $N\geq4$, and $0<k\leq1$ we have

$$\begin{aligned}
&(E^a_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta\\
&\leq\sum_{i=1}^{m}\mathcal{K}_\beta^{i-1}(E^a_{q,s}(\rho_{AB_m}))^\beta\\
&\quad+\mathcal{K}_\beta^{m+1}((E^a_{q,s}(\rho_{AB_{m+1}}))^\beta+\cdots+(E^a_{q,s}(\rho_{AB_{N-2}}))^\beta)\\
&\quad+\mathcal{K}_\beta^m(E^a_{q,s}(\rho_{AB_{N-1}}))^\beta
\end{aligned}\tag{21}$$

for $0\leq\beta\leq1$, $-q^2+4q-3\leq s\leq1$, $1\leq q\leq2$, and $\mathcal{K}_\beta=[(1+k)^\beta-1]/k^\beta$.

Proof From Eq. (8) and Lemma 2, we have

$$\begin{aligned}
&(E^a_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta\\
&\leq(E^a_{q,s}(\rho_{AB_1}))^\beta+\mathcal{K}_\beta\left(\sum_{j=2}^{N-1}E^a_{q,s}(\rho_{AB_j})\right)^\beta\\
&\leq(E^a_{q,s}(\rho_{AB_1}))^\beta+\mathcal{K}_\beta(E^a_{q,s}(\rho_{AB_2}))^\beta+\mathcal{K}_\beta^2\left(\sum_{j=3}^{N-1}E^a_{q,s}(\rho_{AB_j})\right)^\beta\\
&\leq\cdots\\
&\leq(E^a_{q,s}(\rho_{AB_1}))^\beta+\cdots+\mathcal{K}_\beta^{m-1}(E^a_{q,s}(\rho_{AB_m}))^\beta\\
&\quad+\mathcal{K}_\beta^m\left(\sum_{i=m+1}^{N-1}E^a_{q,s}(\rho_{AB_i})\right)^\beta.
\end{aligned}\tag{22}$$

Similarly, as $E^a_{q,s}(\rho_{AB_j})\leq k\sum_{l=j+1}^{N-1}E^a_{q,s}(\rho_{AB_l})$ for $j=m+1,\ldots,N-2$, we can get

$$\begin{aligned}
&\left(\sum_{i=m+1}^{N-1}E^a_{q,s}(\rho_{AB_i})\right)^\beta\\
&\leq\mathcal{K}_\beta(E^a_{q,s}(\rho_{AB_{m+1}}))^\beta+\left(\sum_{i=m+2}^{N-1}E^a_{q,s}(\rho_{AB_i})\right)^\beta
\end{aligned}$$

$$\leq \mathcal{K}_\beta((E^a_{q,s}(\rho_{AB_{m+1}}))^\beta + \cdots + (E^a_{q,s}(\rho_{AB_{N-2}}))^\beta) + (E^a_{q,s}(\rho_{AB_{N-1}}))^\beta. \quad (23)$$

Combining Eqs. (22) and (23), we then obtain Theorem 6.

A polygamy relation based on UEoA is shown in Theorem 6 and comparing Eq. (19) of Theorem 5 with Eq. (21) of Theorem 6, the inequality of Eq. (21) still holds when those classes of states not satisfying the conditions of Eq. (19) in Theorem 5 and Eq. (20) in Ref. [24].

Example 4 Consider the 4-qubit generlized W state

$$|W\rangle_{AB_1B_2B_3} = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle). \quad (24)$$

When s tends to 1 and $q = 2$, we have $T^a_2(|W\rangle_{A|B_1B_2B_3}) = 3/8$, $T^a_2(\rho_{AB_i}) = 1/8$ for $i = 1,2,3$, where $T^a_2(\rho_{AB_i})$ has the same definition in Example 3. It is easy to see $T^a_2(\rho_{AB_i})$ does not satisfy the conditions (19) in Theorem 5 and Eq. (20) in Ref. [24]. However, from inequality (21) of Theorem 6, we have $T^{a\beta}_2(|W\rangle_{A|B_1B_2B_3}) \leq (2\mathcal{K}_\beta + 1)(1/8)^\beta$, as shown in Fig. 4.

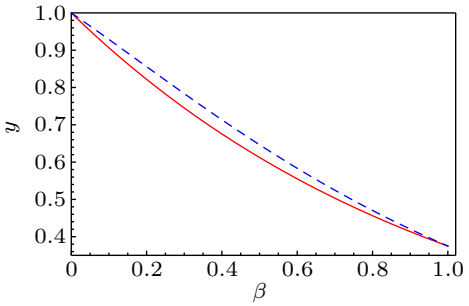


Fig. 4. The y denotes the upper bound of UEoA of $|W\rangle_{AB_1B_2B_3}$ and let $k = 1$. Solid (red) line is exactly the UEoA of $|W\rangle_{AB_1B_2B_3}$, dashed (blue) line represents the upper bound of $T^a_2(|W\rangle_{A|B_1B_2B_3})$ in Eq. (21) for $0 \leq \beta \leq 1$.

5. Conclusion

Entanglement monogamy is a fundamental property of multipartite entangled states. In this paper, a class of tighter monogamy inequalities of multiqubit entanglement in terms of the α-th power of unified-(q,s) entanglement for $\alpha \geq 1$ are established. Also, we established a class of polygamy inequalities of multiqubit entanglement based on β-th power of unified-(q,s) entanglement of assistance. Since RE and TE are special cases for UE, our results present a general class of the monogamy and polygamy relations for bipartite entanglement measures based on unified-(q,s) entropy and one obtains tighter monogamy or polygamy relations than the existing ones for some classes of quantum states directly. The method used in this paper is valuable for getting tighter monogamy relations and polygamy relations to characterize entanglement distribution in multiqubit system.

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