Quantum dynamical resource theory under resource non-increasing framework

Si-Ren Yang(杨思忍) and Chang-Shui Yu(于长水)[†]

School of Physics, Dalian University of Technology, Dalian 116024, *China*

(Received 28 July 2022; revised manuscript received 26 September 2022; accepted manuscript online 17 November 2022)

We define the resource non-increasing (RNI) framework to study the dynamical resource theory. With this definition, we propose several potential quantification candidates under various free operation sets. For explicit demonstrations, we quantify the quantum dynamical coherence in the scenarios with and without post-selective measurements. Correspondingly, we show that the maximally incoherent operations (MIO) and the incoherent operations (IO) in the static coherence resource theory are free in the sense of dynamical coherence. We also provide operational meanings for the measures by the quantum discrimination tasks. Moreover, for the dynamical total coherence, we also present convenient measures and give the analytic calculation for the amplitude damping channel.

Keywords: quantum resource theory, quantum channels, quantum coherence

PACS: 03.67.–a, 03.67.Mn, 03.67.Hk, 34.80.Pa **DOI:** 10.1088/1674-1056/aca398

1. Introduction

In recent years, quantum features, such as quantum entanglement[1–4] and quantum discord[5–9] are taken regarded as quantum resources and quantitatively investigated under rigorous mathematical methods, i.e., quantum resource theories (QRTs), which have been systematically developed since quantum coherence was formally and quantitatively studied in Ref. [10]. QRTs are powerful approaches to investigating quantum characteristics and have deep effects on our recognization of quantum science. Up to now, QRTs have been widely applied to other quantum features: nonlocality,[11] contextuality,[12] non-Gaussianity[13] and asymmetry,[14] and so on. QRTs are usually defined by two fundamental ingredients: free states and free quantum operations. Free states are those states without any resources and free quantum operations are referred to as those that can not generate quantum resources if operated on free states. In such a framework, QRTs bring new insight into studying quantum features on the static level: measures of quantum features not only evaluate the weight of a system but also have operational meanings corresponding to specific quantum processes. Such a framework for static resources is quite rigorous, and can be used to develop connections with different areas. For example, in the QRTs of quantum coherence, researchers proposed multiple measures[15–21] and operational interpretations.[22–26] It has shown that quantum coherence closely correlated with other quantum features like quantum entanglement,[2–4] quantum discord,[5–9] quantum asymmetric,[27–29] *et al.*

Quantum states might get evolved under dynamic processes or quantum operations. In fact, quantum states can even be viewed as special quantum channels in some specific circumstances. Most quantum processes can be characterized by quantum channels which are mathematically the set of completely positive trace preserving (CPTP) maps. Corresponding to static states, quantum channels are dynamical quantum resources in quantum science and carry much more information than static systems. Therefore, one natural idea would be whether to investigate dynamical channels by QRTs. In this sense, one possible way is to follow and upgrade the two elements in static QRTs to dynamical levels, and find out what can be gotten in dynamical resources through the QRTs. With such motivations, researchers are working on this area and have made progresses.[30–44] Some studied the dynamical coherence via different free channels, for example, the Choi isomorphism of classical channels was considered.[30] In Ref. [32], the resource theory proposed free positive-operator-valued measures (POVMs) and free detection/creation incoherence in the sense of quantum computational setting.[33] In addition, the dynamical entanglement has been considered in Ref. [35], and a quantitative relationship between the dynamical coherence and the dynamical entanglement is introduced in Ref. [36].

In dynamical QRTs, the two ingredients are free operations and free superoperations (completely CP and TP preserving maps). The free operations are those without expected quantum features, and free superoperations are defined as those that could not map a free operation to a resourceful operation. However, one would find that the definitions of free superoperation sets depend on the physical considerations. Even for the same quantum feature, the free sets are not unique. Another tough problem is how to measure dynamical quantum features. Due to the diversity of channel representations, the analytical solutions rarely exist, thus measures are

[†]Corresponding author. E-mail: ycs@dlut.edu.cn

© 2023 Chinese Physical Society and IOP Publishing Ltd

usually given in numerical results. Moreover, realizations of superoperations are not unique either. To sum up, dynamical QRTs are developing in different directions, which leads to various QRTs.

In this paper, we direct a new path to define the free operation sets for quantum dynamical resources. We call it resource non-increasing (RNI) frameworks, in which the free channel will not increase the resourcefulness of the input states. Our postulates are inspired by the second law of thermodynamics which is widely accepted in scenarios of physical interest. The second law may be formulated by the observation that the entropy of isolated systems left to spontaneous evolution can not decrease. It can also provide necessary criteria for spontaneous processes. Thus, it is natural to identify a quantum channel is free if it can not generate or increase the quantum feature of the input. Inversely, we can characterize dynamical resourcefulness by the channel's ability to increase the input. It will show that our RNI mindset provides a straight comprehension for dynamical free and meanwhile guarantees RNI framework has no conflict with the well-defined RNG framework. We also refer to static resource theories and design appropriate superoperations for dynamical QRTs. The free dynamical sets in the RNI frameworks are fairly pellucid. We present several potential quantifications of dynamical resources under the mindset of distance measure or the magnitude of violation from dynamical free. To investigate the dynamical coherence, we demonstrate that maximally incoherent operations (MIO) and incoherent operations (IO) in the static coherence resource theory are free in the sense of dynamical coherence. In this sense, we give the corresponding measures of quantum dynamical coherence in cases with or without post-selective measurements. Semidefinite programming (SDP) is also applied to quantify dynamical coherence without post selective measurements. In addition, we also study the quantification of the dynamical total coherence, for which an analytic calculation is given for the amplitude damping channels.

We organize the remaining parts of this paper as follows. In Section 2, we review the dynamical QRTs, propose our RNI framework and present several alternative quantifications. In Section 3, we establish dynamical QRTs for quantum coherence in different scenarios. We illustrate the operational meanings of measures in quantum discrimination tasks. In Section 4, we investigate the dynamical total coherence and give an analytic calculation as an example. We summarize the paper in Section 5.

2. Resource theory of quantum channels

The RNI framework has a direct definition of the free dynamical set, i.e., the free operations could not increase any quantum static resource for arbitrary static input. Therefore, to demonstrate the RNI frameworks, we first need a short review of static quantum resources. As mentioned previously, for static resource theory, the free states are those without any resource, and the set of free states is denoted by $\mathbb{F}$. The free operations with its Kraus operators $\{K_i\}$ are defined by $K_i\delta K_i^\dagger \in \mathbb{F}$ for any free state δ. Thus a valid static resource measure can be given as follows.

Proposition 1 A static resource measure $\mathcal{M}$ for some certain quantum resource $\mathcal{R}$ should fulfill (i) faithfulness: $\mathcal{M} \geq 0$ and vanishes for free states; (iia) strong monotonicity: the average resource under selective free operations can not be increased; (iib) monotonicity: the resource of the state after free operations can not be increased; (iii) convexity: mixing states would not increase their resourcefulness. These constraints are widely applied in measuring entanglement, coherence, *et al.*[10]

It is known to all that the strong monotonicity combined with the convexity would lead to the general monotonicity. The measures that fulfill the strong monotonicity imply that such quantum channels are post-selective measurements allowed. These selective measurements exist in a scenario where the results are accessible with post-selective operations. With the static resource measure, we can present our free operations in the RNI framework (RNI-free operations).

Theorem 1 RNI-free operations in the dynamical resource theory are consistent with the free operations in the static resource theory.

Proof Based on the idea of the RNI framework, the RNI-free operations $\mathcal{E}(\cdot) = \sum_i K_i \cdot K_i^\dagger$ with respect to a valid static resource measure $\mathcal{M}$ is defined as $\sum_i \mathrm{Tr}((K_i\rho K_i^\dagger))\mathcal{M}(K_i\rho K_i^\dagger/\mathrm{Tr}((K_i\rho K_i^\dagger)) \leq \mathcal{M}(\rho)$ for any density matrix ρ. If $\rho \in \mathbb{F}$, due to the faithfulness of static measures, $\mathcal{M}(\rho) = 0$, so $\mathcal{M}(K_i\rho K_i^\dagger/\mathrm{Tr}((K_i\rho K_i^\dagger)) = 0$, which is consistent with the definition of free operations in the static resource theory. On the contrary, if $\mathcal{E}(\cdot) = \sum_i K_i \cdot K_i^\dagger$ is the static free operations, based on the strong monotonicity, the mentioned definition for the RNI-free operations is also satisfied.

Note that if a quantum channel is considered in a black-box scenario, which means the measurement is non-selective, one can only require monotonicity in a non-increasing framework, i.e., $\mathcal{M}(\mathcal{E}(\rho)) \leq \mathcal{M}(\rho)$. This corresponds to the RNI framework in the sense of monotonicity. Similarly, the free operations can also be naturally defined subject to monotonicity, which will be directly used later and will not be explicitly elucidated here.

Another ingredient in dynamical QRTs is the free superoperations. Similar to the static resource theory, the free superoperation has a primal constraint that maps a free operation to another free operation. Besides, we stress that the construction of superoperations should not contain certain resourceful

ingredients. Now we propose our free superoperations with the following structure.

Definition 1 A superoperation represented by Kraus operators $\{\mathfrak{F}_n\}$ is free if and only if $\mathfrak{F}_n[\mathcal{N}]$ can be written into the sequence as

$$\mathfrak{F}_n[\mathcal{N}] = \mathcal{E}_{i_n,\Phi_n^{j_n}}, \cdots, \mathcal{E}_{i_2,\Phi_2^{j_2}} \mathcal{E}_{i_1,\Phi_1^{j_1}}, \tag{1}$$

where $\mathcal{E}_{i_n,\Phi_n^{j_n}}$ denotes the j_nth Kraus element of the superoperation $\{\mathcal{E}_{i_n,\Phi_n^{j_n}}\}$ with the corresponding free operation $\{\Phi_n^{j_n}\}$ and

$$\begin{aligned}
&\mathcal{E}_{i_n=0,\Phi_n^{j_n}}[\mathcal{N}] = \mathrm{Tr}[\mathcal{N}],\\
&\mathcal{E}_{i_n=1,\Phi_n^{j_n}}[\mathcal{N}] = \Phi_n^{j_n} \circ \mathcal{N}, \quad \mathcal{E}_{i_n=2,\Phi_n^{j_n}}[\mathcal{N}] = \Phi_n^{j_n} \otimes \mathcal{N},\\
&\mathcal{E}_{i_n=3,\Phi_n^{j_n}}[\mathcal{N}] = \mathcal{N} \circ \Phi_n^{j_n}, \quad \mathcal{E}_{i_n=4,\Phi_n^{j_n}}[\mathcal{N}] = \mathcal{N} \otimes \Phi_n^{j_n}.
\end{aligned} \tag{2}$$

The above definition implies the tensor product structure is automatically satisfied if one replaces $\Phi_n^{j_n}$ by identity operator $\mathbb{1}$. Furthermore, it is not difficult to find that our superoperations would be free for every single Kraus operator. Combing the two characteristics we can conclude that our superoperations are separately (every Kraus operator free) and completely (tensor product structure) free. Hence, our free superoperations meet the requirement of our motivation. With the two ingredients defined, we can define the measure of RNI dynamical resource as follows.

Definition 2 A measure $T(\cdot)$ quantifying dynamical resourcefulness of arbitrary quantum channel $\mathcal{N}$ is a qualified measure if the following are satisfied.

(i) Faithfulness: $T(\mathcal{N}) \geq 0$, where equality holds if $\mathcal{N}$ is free.

(ii) Monotonicity or strong monotonicity: $T(\mathcal{N}) \geq T(\mathfrak{F}[\mathcal{N}])$ or $T(\mathcal{N}) \geq \sum_n p_n T(\mathcal{N}_n)$ for free superoperations $\mathfrak{F} = \sum_n p_n \mathfrak{F}_n$ with $\mathcal{N}_n = \mathfrak{F}_n[\mathcal{N}]$ and $\sum_n p_n = 1$.

(iii) Convexity: $T(\mathcal{N})$ is convex.

The strong monotonicity is an operational constraint where the measures obey the monotonicity under selective measurements. Our free superoperations allow us to read out information of the post states. However, if one can not have enough information about the resourceful superoperation (such as a black box in some practical scenarios mentioned previously), it is enough to consider the monotonicity, even though in a QRT strong monotonicity is required. Our free superoperations can also be explicitly given in the sense of quantum computational settings, which is similar to Ref. [45].

Up to now, we have well established the fundamental requirements for a dynamical resource measure in the RNI framework. Thus in the following, we present two natural quantification approaches, one is based on the distance from the free operation set, and the other is directly based on the violation of the definition of free operations.

Definition 3 Dynamical resource of the channel $\mathcal{N}(\cdot) = \sum_i K_i \cdot K_i^\dagger$ can be measured by the minimal distance from the free set $\mathcal{S}$ with appropriate distance functions $\|\cdot\|_*$ as

$$T(\mathcal{N}) = \min_{\mathcal{F}\in\mathcal{S}} \|\mathcal{N} - \mathcal{F}\|_*, \tag{3}$$

or by the magnitude of the violation of the free operation as

$$\tilde{T}(\mathcal{N}) = \max\{\Delta\mathcal{M}_*(\mathcal{N}), 0\}, \tag{4}$$

with $\Delta\mathcal{M}_*(\mathcal{N}) = \max_\rho \sum_i \mathrm{Tr}[K_i\rho K_i^\dagger]\mathcal{M}\left(K_i\rho K_i^\dagger/\mathrm{Tr}[K_i\rho K_i]^\dagger\right)$ where $*$ denotes the proper distance functions in Eq. (3) or static resource measure in Eq. (10) such that both $T(\mathcal{N})$ or $\tilde{T}(\mathcal{N})$ satisfies Definition 2.

Finally, we would like to emphasize that the dynamical resource in the RNI framework makes sense for the quantum channel with given Kraus operators since our definitions are based on the strong monotonicity. Of course, one can consider the channel without post-selection in the sense of monotonicity.

3. Dynamical quantum coherence in resource non-increasing framework

Now we will consider the dynamical resource theory of coherence in the RNI framework. Since we have shown that the RNI-free operations are the same as free operations in the static resource theory, the RNI-incoherent operations in the dynamical resource theory naturally correspond to the incoherent operations in the static coherence quantification. As mentioned in the previous section, the RNI dynamical resource theory can be considered in the sense of both strong monotonicity and monotonicity. One can find that the free operations subject to the dynamical coherence are the incoherent operations (IO) and the maximally incoherent operations (MIO), respectively. Thus, we will directly employ the proposed approaches to quantify the dynamical coherence based on Definition 3.

Theorem 2 Let the free operation sets denoted by IO and MIO corresponding to the RNI coherence with and without post-selection scenarios, the dynamical coherence for a quantum channel $\mathcal{N}$ can be measured by

$$T_{1/\diamond}(\mathcal{N}) = \min_{\mathcal{F}\in\mathrm{IO/MIO}} \|\mathcal{N} - \mathcal{F}\|_{1/\diamond}, \tag{5}$$

where the induced trace norm and the diamond norm are defined as

$$\|\Phi\|_1 = \max\{\|\Phi(X)\|_1 : X \in \mathcal{L}(X), \|X\|_1 \leq 1\}, \tag{6}$$

$$\|\Phi^{\mathrm{A}\to\mathrm{B}}\|_\diamond = \|\Phi^{\mathrm{A}\to\mathrm{B}} \otimes \mathbb{1}^{\mathrm{C}}\|_1, \tag{7}$$

where $\mathrm{A}\to\mathrm{B}$ denotes the map between subsystems A and B, and C denotes the third subsystem.

Proof Firstly, $T_{1/\diamond}(\mathcal{N})$ vanishes for free operations since it is defined by the minimal distance. For the convexity, let us consider a channel $\mathcal{N}$ mixed by a set $\mathcal{N}_m$ with probabilities

q_m and denote the corresponding optimal free operation in set IO/MIO by $\mathcal{F}_m$, the average coherence is

$$\begin{aligned}&\sum_m q_m T_{1/\diamond}(\mathcal{N}_m) = \sum_m q_m \|\mathcal{N}_m - \mathcal{F}_m\|_{1/\diamond} \\ &\geq \|\sum_m q_m \mathcal{N}_m - \sum_m q_m \mathcal{F}_m\|_{1/\diamond} \geq \|\mathcal{N} - \mathcal{F}\|_{1/\diamond} \\ &= T_{1/\diamond}(\mathcal{N}). \end{aligned} \tag{8}$$

For the strong monotonicity, let us consider the free superchannel $\mathfrak{F} = \sum_m p_m \mathfrak{F}_m$ given in Eq. (1). One can denote $T_{1/\diamond}(\mathfrak{F}_m[\mathcal{N}]) = T_{1/\diamond}(\mathcal{N}_m)$. Since the induced trace norm is sub-multiplicative and sub-multiplicative with respect to tensor product, which indicates that $T_{1/\diamond}(\mathcal{N}_m) \leq T_{1/\diamond}(\mathcal{N})$ for every single Kraus operator in free superchannels. Thus, the strong monotinicity holds by the following inequality:

$$\sum_m p_m T_{1/\diamond}(\mathcal{N}_m) \leq \sum_m p_m T_{1/\diamond}(\mathcal{N}) = T_{1/\diamond}(\mathcal{N}). \tag{9}$$

Finally one can directly obtain the monotonicity based on the strong monotonicity and convexity.

Let $T_{1/\diamond,\mathrm{non}}$ denote $T_{1/\diamond}$ without post-selective measurements. It is shown that $T_{1/\diamond,\mathrm{non}}$ has a direct operational meaning in the quantum channels discrimination task and can be calculated by semidefinite programming (SDP). We will illustrate the details in the Appendix.

Besides the distance measures, one can also employ the maximal violation in Definition 3 to define the dynamical coherence as follows.

Theorem 3 Given a quantum channel $\mathcal{N}(\cdot) = \sum_n K_n \cdot K_n^\dagger$, the dynamical coherence can be well quantified by

$$\tilde{T}(\mathcal{N}) = \max\{\Delta\mathcal{M}_{np/p}(\mathcal{N}), 0\}, \tag{10}$$

where $\mathcal{D}$ denotes the set of all density matrices in the space, and in the case of without post-selective measurements

$$\Delta\mathcal{M}_{np}(\mathcal{N}) = \max_{\rho\in\mathcal{D}} \mathcal{C}(\mathcal{N}(\rho)) - \mathcal{C}(\rho), \tag{11}$$

in the case of with post-selective measurements

$$\Delta\mathcal{M}_{p}(\mathcal{N}) = \max_{\rho\in\mathcal{D}} \sum_n p_n \mathcal{C}(\rho_n) - \mathcal{C}(\rho), \tag{12}$$

where $p_n = \mathrm{Tr}((K_n \rho K_n^\dagger))$ and $\rho_n = K_n \rho K_n^\dagger / p_n$.

Proof At first, the definition of free operations in either scenario directly implies $\tilde{T}(\mathcal{N}) \geq 0$ which is saturated if and only if $\mathcal{N} \subset$ IO/MIO.

For the strong monotonicity, one will have to consider Definition 1 in detail, which shows free superoperations can be written as $\mathfrak{F}(\mathcal{N}) = \sum_m q_m \mathfrak{F}_m(\mathcal{N}) = \sum_m q_m \mathcal{N}_m$. From the following, one can find that every $\mathfrak{F}_m$ implies $\tilde{T}(\mathcal{N}_m) \leq \tilde{T}(\mathcal{N})$, which immediately leads to $\sum_m p_m \tilde{T}(\mathcal{N}_m) \leq \sum_m p_m \tilde{T}(\mathcal{N}) = \tilde{T}(\mathcal{N})$. (i) Discarding the system with $\mathcal{E}_{i_n=0}$ makes a free state. (ii) Attatching ancilla by $\mathcal{E}_{i_n=2,4}$ means $\mathcal{N}(\rho) = \mathrm{Tr}_\mathrm{A}[(\Theta_\mathrm{A} \otimes \mathcal{N})(\sigma_\mathrm{A} \otimes \rho)] = \mathrm{Tr}_\mathrm{A}[\Theta_\mathrm{A}(\sigma_\mathrm{A}) \otimes \mathcal{N}(\rho)] = 1 \cdot \mathcal{N}(\rho)$. (iii) Linking a free operation by $\mathcal{E}_{i_n=1}$ corresponds to $\mathcal{C}(\Theta \circ \mathcal{N}(\rho)) \leq \mathcal{C}(\mathcal{N}(\rho))$ for any state $\rho \in \mathcal{D}$ and any free channel Θ due to the monotonicity of static measures $\mathcal{C}$. (iv) Linking a free operation for $\mathcal{E}_{i_n=3}$ can be verified as

$$\begin{aligned}\tilde{T}(\mathcal{N}\circ\Theta) &= \max_\rho \sum_n p_n \mathcal{C}\left(\frac{K_n \rho_\Theta K_n^\dagger}{p_n}\right) - \mathcal{C}(\rho) \\ &\leq \max_\rho \sum_n p_n \mathcal{C}\left(\frac{K_n \rho_\Theta K_n^\dagger}{p_n}\right) - \mathcal{C}(\rho_\Theta) \\ &\leq \max_\rho \sum_n p_n \mathcal{C}\left(\frac{K_n \rho K_n^\dagger}{p_n}\right) - \mathcal{C}(\rho) \\ &= \tilde{T}(\mathcal{N}), \end{aligned} \tag{13}$$

where Θ represents any free channel and $\rho_\Theta = \Theta(\rho)$ is the post state operated by the free channel. The first inequality comes from the monotonicity of static measures, i.e., $\mathcal{C}(\rho_\Theta) \leq \mathcal{C}(\rho)$ and the second holds since the maximum overall the density matrix spaces is definitely not less than the maximum overall the subspace. Equation (13) mainly focuses on $\Delta\mathcal{M}_p(\mathcal{N})$. A similar proof can be easily obtained for $\Delta\mathcal{M}_{np}(\mathcal{N})$ (not given here). So (i)–(iv) prove the strong monotonicity.

For the convexity, let us first prove the scenario with postseletive measurements. Consider the dynamical coherence of mixing a set of channels $\{\mathcal{N}_i\}$ with probabilities $\{q_i\}$. Let a state ρ undergoes channel $\mathcal{N}_i$ with its Kraus operator K_{in} , the post-measurement state is denoted by $\sigma_{i,n} = K_{in} \rho K_{in}^\dagger$, then we have

$$\begin{aligned}\tilde{T}\left(\sum_i q_i \mathcal{N}_i\right) &= \max_{\rho\in\mathcal{D}} \sum_{i,n} q_i \mathrm{Tr}(\sigma_{i,n}) \mathcal{C}\left(\frac{\sigma_{i,n}}{\mathrm{Tr}(\sigma_{i,n})}\right) - \mathcal{C}(\rho) \\ &= \sum_{i,n} q_i \mathrm{Tr}(\sigma_{i,n}^0) \mathcal{C}\left(\frac{\sigma_{i,n}^0}{\mathrm{Tr}(\sigma_{i,n}^0)}\right) - \mathcal{C}(\rho^0) \\ &\leq \sum_{i,n} q_i \mathrm{Tr}(\sigma_{i,n}^i) \mathcal{C}\left(\frac{\sigma_{i,n}^i}{\mathrm{Tr}(\sigma_{i,n}^i)}\right) - \mathcal{C}(\rho^i) \\ &= \sum_{i,n} q_i \max_{\rho^i\in\mathcal{D}} p_n^i \mathcal{C}(\rho_n^i) - \mathcal{C}(\rho^i) \\ &= \sum_i q_i \tilde{T}(\mathcal{N}_i), \end{aligned} \tag{14}$$

where the superscript on $\sigma_{i,n}^0$ denotes the optimal state of the maximum in the second equality and $\sigma_{i,n}^i$ is the optimal one for $\mathcal{N}_i$.

In the case without post-selection, for the mixed channel $\sum_i q_i \mathcal{N}_i$, we have

$$\begin{aligned}\tilde{T}(\sum_i q_i \mathcal{N}_i) &= \max_{\rho\in\mathcal{D}} \mathcal{C}(\sum_i q_i \mathcal{N}_i(\rho)) - \mathcal{C}(\rho) \\ &= \mathcal{C}(\sum_i q_i \mathcal{N}_i(\sigma)) - \mathcal{C}(\sigma) \\ &\leq \sum_i q_i \mathcal{C}(\mathcal{N}_i(\sigma)) - \mathcal{C}(\sigma) \end{aligned}$$

$$\leq \sum_i q_i \max_{\rho_i\in\mathcal{D}} \mathcal{C}(\mathcal{N}_i(\rho_i)) - \mathcal{C}(\rho_i)$$
$$= \sum_i q_i \tilde{T}(\mathcal{N}_i), \tag{15}$$

where σ denotes the optimal state for the mixed channel $\sum_i q_i\mathcal{N}_i$, and the second equality comes from the convexity of the static measures.

Up to now, we have proved the convexity, which will directly leads to the monotonicity associated with the strong monotonicity.

The dynamical coherence in Eq. (11) has a similar form to cohering power in Ref. [46], but our maximum is taken over all the density matrices rather than incoherent states. The MIO set is proposed as the maximal set of free operations in the static QRT, and we will show that the free set of RNI with the non-selective measurements is exactly the MIO. The RNI free set subjects to MIO can provide a new operational interpretation for the RNG framework. In this sense, one can find an alternative dynamical coherence measure assisted by the dephasing channel $\Delta(\cdot) = \sum_i \langle i|\cdot|i\rangle|i\rangle\langle i|$ as follows.

Theorem 4 Given a quantum channel $\mathcal{N}(\cdot) = \sum_n K_n \cdot K_n^\dagger$, the dynamical coherence without post measurements can be quantified by

$$T_{\mathrm{a,non}}(\mathcal{N}) = \min_{\mathcal{F}\in\mathrm{MIO},\delta\in\mathcal{I}} \|(\mathcal{N}-\mathcal{F})\delta\|_1$$
$$= \min_{\mathcal{F}\in\mathrm{MIO}} \|(\mathcal{N}-\mathcal{F})\Delta\|_1. \tag{16}$$

Proof (i) The distance functions guarantee $T_{\mathrm{a,non}}$ can faithfully detect dynamical coherence. (ii) The convexity holds because of the absolute homogeneity and the triangle inequality. Considering two quantum channels $\mathcal{N}$ and $\mathcal{M}$ such that $T_{\mathrm{a,non}}(\mathcal{N}) = \|\mathcal{N}-\mathcal{F}_1\|_1$ and $T_{\mathrm{a,non}}(\mathcal{M}) = \|\mathcal{M}-\mathcal{F}_2\|_1$, for any $0\leq t\leq 1$, one can find

$$\begin{aligned}
&T_{\mathrm{a}}(t\mathcal{N}+(1-t)\mathcal{M})\\
&= \min_{\mathcal{F}\in\mathrm{MIO}} \|(t\mathcal{N}+(1-t)\mathcal{M}-\mathcal{F})\Delta\|_1\\
&\leq \|(t\mathcal{N}+(1-t)\mathcal{M})\Delta - (t\mathcal{F}_1+(1-t)\mathcal{F}_2)\Delta\|_1\\
&= \|t(\mathcal{N}\Delta-\mathcal{F}_1\Delta)+(1-t)(\mathcal{M}\Delta-\mathcal{F}_2\Delta)\|_1\\
&\leq t\|(\mathcal{N}-\mathcal{F}_1)\Delta\|_1+(1-t)\|(\mathcal{M}-\mathcal{F}_2)\Delta\|_1\\
&= tT_{\mathrm{a}}(\mathcal{N})+(1-t)T_{\mathrm{a}}(\mathcal{M}),
\end{aligned} \tag{17}$$

which is the convexity. (iii) The strong monotonicity $T(\mathcal{N}) \geq \sum_i q_i T(\mathcal{N}_i)$ can be proved by

$$\begin{aligned}
\sum_i q_i T_{\mathrm{a}}(\mathcal{N}_i) &= \sum_i q_i \min_{\mathcal{F}_i\in\mathrm{MIO}} \|(\mathcal{N}_i-\mathcal{F}_i)\Delta\|_1\\
&= \sum_i q_i \|(\mathcal{N}_i-\mathcal{F}_i^*)\Delta\|_1\\
&\leq \sum_i q_i \min_{\mathcal{X}\in\mathrm{MIO}} \|(\mathcal{F}_i-\mathcal{F}_i(X))\Delta\|_1\\
&\leq \min_{\mathcal{X}\in\mathrm{MIO}} \|(\mathcal{N}-X)\Delta\|_1 = T_{\mathrm{a}}(\mathcal{N}),
\end{aligned} \tag{18}$$

where $\mathcal{N}_i = \mathfrak{F}_i(N)$ and $\mathfrak{F}_i$ are Kraus operators of superoperation $\mathfrak{F}$, $\mathcal{F}^*$ represents the channel achieving its minimum, the second last inequality holds for that $\mathcal{F}_i(X)$ can not be the optimal, and the last inequality is valid due to the submultiplicativity. (iv) The strong monotonicity combined with convexity leads to monotonicity.

The $T_{\mathrm{a,non}}$ is a success probability in channel discrimination tasks if the participant allows the specific free dephasing operation or the incoherent states. This numerical result is very close to the one studied in Ref. [45], where the authors analyzed detection-incoherent settings. Since the MIO is the maximal free set in static QRTs of coherence, constraints (such as applying a dephasing channel) would definitely shrink the free set (to dephasing incoherent channels).

4. Dynamical total coherence in the RNI framework

Quantum total coherence is one type of basis-independent coherence of which the static QRT was studied in Ref. [47]. Similarly, it can also be investigated in the sense of dynamical resource theory, i.e., the dynamical total coherence of a channel. In Ref. [47], it is explicitly given that the free operations with post-selective measurements are the mixed unitary channels defined as $\mathcal{U}(\cdot) = \sum_x q_x U_x(\cdot)U_x^\dagger$ with unitary U and $\sum_x q_x = 1$, and the free operations without post-selective measurements are the unital channels given by $\mathcal{A}(\cdot) = \sum_x A_x(\cdot)A_x^\dagger$ with $\sum_x A_x(\cdot)A_x^\dagger = I$. Following the above section, for a given quantum channel $\mathcal{N}$ one can straightforwardly get the corresponding measures of the dynamical total coherence by replacing the set IO/MIO in Theorem 2 and replacing the static coherence measure $\mathcal{M}$ by a proper total coherence measure. One can also easily show that these measures satisfy the necessary conditions for a valid dynamic resource theory.

Considering the computability, we would like to mention that the static total coherence measure based on the l_2 norm is a good measure. In this sense, an explicit measure of the dynamical total coherence can be raised similar to Theorem 7 as follows.

Theorem 5 For a quantum channel $\mathcal{N}$ with Kraus operators $\{K_n\}$, the dynamical total coherence in the RNI framework with post-selective measurements can be quantified as

$$\tilde{T}_{l_2}(\{K_n\}) = \max\{\max_\rho \sum_n p_n \mathrm{Tr}[\rho_n^2] - \mathrm{Tr}[\rho^2], 0\}, \tag{19}$$

and without post-selective measurements can be quantified as

$$\tilde{T}_{l_2}(\mathcal{N}) = \max\{\max_\rho \mathrm{Tr}\left[\sum_n p_n\rho_n\right]^2 - \mathrm{Tr}[\rho^2], 0\}, \tag{20}$$

where $\rho_n = K_n\rho K_n^\dagger/p_n$ and $p_n = \mathrm{Tr}(K_n\rho K_n^\dagger)$.

Proof Since the static total coherence based on l_2 norm is a qualified measure (i.e., satisfies all constraints for measures

including the monotonicity and the strong monotonicity),[48] this theorem can entirely be understood as an explicit example of Theorem 3.

As demonstrations, we consider the dynamical coherence of amplitude damping channels which characterize the energy dissipation in the quantum process. With a given dissipation rate η, the Kraus operators of this channel can be given as[49]

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta} \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & \sqrt{\eta} \\ 0 & 0 \end{bmatrix}. \tag{21}$$

Given a density matrix in Bloch representaion

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-\mathrm{i}y \\ x+\mathrm{i}y & 1-z \end{bmatrix}, \tag{22}$$

where the parameters x, y, and z are satisfied $x^2+y^2+z^2 \leq 1$, the dynamical coherence measured by $\tilde{T}_{l_2}(\{K_n\})$ with Eq. (19) reads

$$\begin{aligned} &\tilde{T}_{l_2}(\{K_n\}) \\ &= \max\left\{ \max_{x,y,z} \frac{(1+z)^2+(1-\eta)^2(1-z)^2+2(1-\eta)(x^2+y^2)}{2(2-\eta+\eta z)} \right. \\ &\left. +\frac{\eta(1-z)}{2} - \frac{1+x^2+y^2+z^2}{2}, 0 \right\}. \end{aligned} \tag{23}$$

To maximize $\tilde{T}_{l_2}(\{K_n\})$, we would like to apply the method of residual multipliers

$$\mathcal{L}(x,y,z,\lambda) = f(x,y,z) + \lambda(x^2+y^2+z^2-1), \tag{24}$$

where $f(x,y,z)$ is the main part (ignore the maximize) of Eq. (23) and the residual part is the qubit constraint. The maximum of $f(x,y,z)$ can be obtained when the partial derivatives for all parameters equal zero and the inequality constraint holds for the equality. Thus, we conclude the following equations:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = \dfrac{2(1-\eta)x}{2-\eta+\eta z} - x + 2\lambda x = 0, \\ \dfrac{\partial \mathcal{L}}{\partial y} = \dfrac{2(1-\eta)y}{2-\eta+\eta z} - y + 2\lambda y = 0, \\ \dfrac{\partial \mathcal{L}}{\partial z} = -2\eta z^3 + (2\eta-6)z^2 + (2\eta=4)z - 2\eta + 2 = 0, \\ \lambda(x^2+y^2+z^2-1) = 0. \end{cases} \tag{25}$$

It should be noted that for any $x^2+y^2+z^2=1$, $f(x,y,z)=0$, which implies that the maximum lies within the feasible region. Thus, we have the optimal solution of Eq. (25) as

$$\begin{cases} \lambda = x = y = 0, \\ z = \dfrac{2\eta+\sqrt{9-8\eta}-3}{2\eta}. \end{cases} \tag{26}$$

Thus the analytic total coherence measure for amplitude damping channels is given by

$$\tilde{T}_{l_2}(\{K_n\}) = \frac{9\left(\sqrt{9-8\eta}-3\right)-4\eta\left(2\eta+2\sqrt{9-8\eta}-9\right)}{4\eta^2}. \tag{27}$$

For the situation without post-selective measurements, the dynamical total coherence of Eq. (20) is

$$\tilde{T}_2(\mathcal{N}) = \max\left\{ \sum_{i=1}^{3} \frac{\xi_i^2 \tilde{a}_i^2}{2(1-\xi_i^2)} + \frac{a_i^2}{2}, 0 \right\}, \tag{28}$$

where $a_i = (1/2)\mathrm{Tr}\mathcal{N}(\sigma_i)$, $M_{ij} = (1/2)\mathrm{Tr}\sigma_i\mathcal{N}(\sigma_j)$, and ξ_i is the singular value of the matrix M, $|\tilde{a}\rangle = U^{\mathrm{T}}|a\rangle$ with U determined by the singular value decomposition $M = U\Lambda V^{\mathrm{T}}$.[50] In Fig. 1 we compare the dynamical total coherence of amplitude damping channels with or without post-selective measurement. It is clear that the dynamical coherence with post-selective measurements is larger than that post-selective measurements.

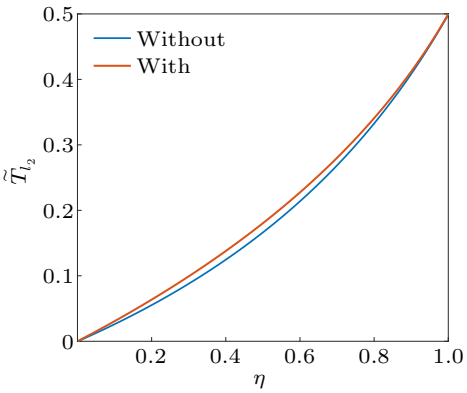


Fig. 1. The dynamical total coherence by $\tilde{T}_2$ versus η.

5. Conclusion and discussion

Quantum channels are dynamical resources that contain more information than static states. Investigating dynamical resources in QRTs has a pre-request that identifies the free sets in different physical backgrounds. It should be understood that the range of the RNI framework does not exceed the well-defined resource non-generating (RNG) framework, but it gives a new sight to determine the free set under different scenarios. We design free superoperations with fundamental ingredients and give some measures to quantify dynamical resources. To quantify the dynamical resources, we design two ways: one is based on the distance measures; the other is the maximal violation compared to free operations. Both methods are natural and have been proved under metric constraints. As a demonstration, we quantify dynamical (total) coherence in our frameworks. Under consideration of physical scenarios, MIO and IO are free (incoherent) operations corresponding to the case without and with post selective measurements, in dynamical coherence theories, one would find the RNI-free set is IO when post-selective measurements are allowed. Of course, it is suitable for other types of operations such as SIO, DIO, etc., respectively. The operational meanings in quantum tasks are also given for the dynamical coherence. In particular, the analytical calculation is given for the dynamical coherence

of the amplitude damping channel, and the semidefinite programming is provided for dynamical coherence without post selective measurements. In the end, as an open question, one may dig into more connotations with RNI applying to other resources.

Appendix A: Operational meaning of $T_{\diamond,\mathrm{non}}$

We first demonstrate the operational meaning of $T_{\diamond,\mathrm{non}}$ in channel discrimination tasks.[51–54] In such a task, Bob wants to distinguish two channels with allowed operations. Another participant Alice prepares a probabilistic state in classic register Z. The classic register will be state 0 with probability λ and state 1 with probability $1-\lambda$. Alice read out the state 0 or 1 in register Z, then sent an initial state (which is prepared in arbitrary register X) through quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, and the final state (ρ_1 and ρ_2) is sent to Bob in register Y. Bob has to determine the final state had experienced which quantum channel. And Bob tries best to improve his successful probability by maximize $(1/2)\|\lambda\rho_1-(1-\lambda)\rho_2\|$.

We set $\lambda=1/2$ for a better illustration. Meanwhile, with practical consideration, Bob is allowed to hold an auxiliary register R and manipulates some operations Ψ^{AR}. In a non-selective scenario, Bob can only rely on the information located in register Y. Now, Bob, has a successful probability[45,55] as

$$\begin{aligned}&\mathrm{Prob}(\mathcal{N}_1,\mathcal{N}_2)\\&=\frac{1}{2}+\frac{1}{4}\max_{\Psi^{\mathrm{AR}},\sigma^{\mathrm{AR}}}\left\|\Psi^{\mathrm{AR}}(\mathcal{N}_1-\mathcal{N}_2)^{\mathrm{A}}\otimes\mathbb{1}^{\mathrm{R}}\sigma^{\mathrm{AR}}\right\|_1,\end{aligned}\tag{A1}$$

where σ^{AR} denotes the bipartite final state in registers Y and R. And the allowed manipulations Ψ^{AR} can be determined by Bob if they are applied or not. If Bob holds the free set MIO and we minimize the diamond norm between resource channel $\mathcal{N}_1$ and $\mathcal{N}_2\in\mathrm{MIO}$, then we will find that

$$\begin{aligned}&T_\diamond(\mathcal{N}_1)\\&=\min_{\mathcal{N}_2^{\mathrm{A}}\in\mathrm{MIO}}\|\mathcal{N}_1^{\mathrm{A}}-\mathcal{N}_2^{\mathrm{A}}\|_\diamond\\&\geq\min_{\mathcal{N}_2^{\mathrm{A}}\in\mathrm{MIO}}\max_{\mathcal{F}^{\mathrm{AR}}\in\mathrm{MIO}}\|\mathcal{F}^{\mathrm{AR}}(\mathcal{N}_1-\mathcal{N}_2)^{\mathrm{A}}\|_\diamond\\&=\min_{\mathcal{N}_2^{\mathrm{A}}\in\mathrm{MIO}}\max_{\mathcal{F}^{\mathrm{AR}}\in\mathrm{MIO}}\|\mathcal{F}^{\mathrm{AR}}(\mathcal{N}_1-\mathcal{N}_2)^{\mathrm{A}}\otimes\mathbb{1}^{\mathrm{R}}\|_1\\&=\min_{\mathcal{N}_2^{\mathrm{A}}\in\mathrm{MIO}}\max_{\mathcal{F}^{\mathrm{AR}}\in\mathrm{MIO},P}\max_{\sigma^{\mathrm{AR}}}\mathrm{Tr}\left[P\mathcal{F}^{\mathrm{AR}}(\mathcal{N}_1-\mathcal{N}_2)^{\mathrm{A}}\otimes\mathbb{1}^{\mathrm{AR}}\sigma^{\mathrm{AR}}\right]\\&=\min_{\mathcal{N}_2^{\mathrm{A}}\in\mathrm{MIO}}\max_{\mathcal{F}^{\mathrm{AR}}\in\mathrm{MIO},\sigma^{\mathrm{AR}}}\|\mathcal{F}(\mathcal{N}_1-\mathcal{N}_2)^{\mathrm{A}}\otimes\mathbb{1}^{\mathrm{AR}}\sigma^{\mathrm{AR}}\|_1,\end{aligned}\tag{A2}$$

where the inequality originates from the monotonicity of dynamical measures. The second last equality is an alternative definition for the trace norm $\|\cdot\|=\max_P\mathrm{Tr}[P(\cdot)]$, where P denotes projective operators. Since any projective operators are in the set of MIO, thus we have the last equality.

Comparing Eq. (A2) with Eq. (A1), one can easily see that in such a scenario the probability which is exactly given by

$$\mathrm{Prob}(\mathcal{N}_1,\mathcal{N}_2)=\frac{1}{2}+\frac{1}{4}T_{\diamond,\mathrm{non}}(\mathcal{N}_1),\tag{A3}$$

since the maximum can be reached at least when Bob does an identity operation (which belongs to MIO, i.e., a free operation). Hence, the boundary gives the operational meaning to $T_{\diamond,non}$.

Furthermore, if Bob does not hold the ancilla reference R, one will directly derive an operational meaning for $T_{1,\mathrm{non}}$ based on Eq. (A3).

Appendix B: Semidefinite programming for $T_{\diamond,\mathrm{non}}$

A quantum channel $\mathcal{N}$ has its Choi representation (sometimes called Choi–Jamiolkowski isomorphism)[56,57] as

$$J(\mathcal{N})=\mathbb{1}\otimes\mathcal{N}(\phi_+),\tag{B1}$$

where ϕ_+ is unnormalized maximally entangled state $\phi_+=\sum_{ij}|i\rangle\langle j|\otimes|i\rangle\langle j|$. Its dynamical coherence can be measured by

$$T_\diamond(\mathcal{N})=\min_{\mathcal{F}\in\mathrm{FREE}}\|\mathcal{N}-\mathcal{F}\|_\diamond.\tag{B2}$$

Under the RNI and non-selective background, the free set is MIO. Thus, the dynamical coherence can be measured by

$$T_{\diamond,\mathrm{non}}(\mathcal{N})=\min_{\mathcal{F}\in\mathrm{MIO}}\|\mathcal{N}-\mathcal{F}\|_\diamond,\tag{B3}$$

and evaluated by semidefinite programming (with polynomial algorithms[58]). According to its definition, diamond norm of operator $\|\mathcal{N}-\mathcal{F}\|_\diamond$ has its primal problem

$$\begin{aligned}&\text{Primal}\\&\text{minimize}: \quad 2\|\mathrm{Tr}_{\mathrm{B}}(Z)\|_\infty\\&\text{s.t.} \quad Z\geq J(\mathcal{N}-\mathcal{F}),\ \ Z\geq 0.\end{aligned}\tag{B4}$$

Then $T_{\diamond,\mathrm{non}}(\mathcal{N})$ is the optimal value of

$$\begin{aligned}&\text{minimize}: \quad 2\|\mathrm{Tr}_{\mathrm{B}}(Z)\|_\infty\\&\text{s.t.} \quad Z\geq J(\mathcal{N}-\mathcal{F}),\ \ Z\geq 0,\\&\qquad \mathcal{F}\in\mathrm{MIO}.\end{aligned}\tag{B5}$$

Applying constraints on $\mathcal{F}$ and Choi representation properties for MIO, the primal problem becomes

$$\begin{aligned}&\text{minimize}: \quad 2\|\mathrm{Tr}(Z)\|_\infty\\&\text{s.t.} \quad Z\geq J(\mathcal{N})-M,\ \ Z\geq 0\\&\qquad M\geq 0,\ \ \mathrm{Tr}_{\mathrm{B}}(M)=\mathbb{1}_{\mathrm{A}},\\&\qquad \mathrm{Tr}_{\mathrm{A}}(M)-\Delta\mathrm{Tr}_{\mathrm{A}}(M)=0,\end{aligned}\tag{B6}$$

which equals to

$$\begin{aligned} \text{minimize}: \quad & a \\ \text{s.t.} \quad & a \geq 0, \\ & \mathbb{1}_{\mathrm{A}} \cdot a - 2\mathrm{Tr}_{\mathrm{B}}(Z) \geq 0, \\ & Z \geq J(\mathcal{N}) - M, \ \ Z \geq 0, \\ & M \geq 0, \ \ \mathrm{Tr}_{\mathrm{B}}(\mathrm{M}) = \mathbb{1}_{\mathrm{A}}, \\ & \mathrm{Tr}_{\mathrm{A}}(\mathrm{M}) - \Delta \mathrm{Tr}_{\mathrm{A}}(\mathrm{M}) = 0. \end{aligned} \tag{B7}$$

The Lagrangian of the primal problem is given by

$$\begin{aligned} & \mathcal{L}(a, Z, M, \widetilde{X}, X, Y_1, Y_2) \\ = \ & a + \mathrm{Tr}[(2\mathrm{Tr}_{\mathrm{B}}(Z) - a\mathbb{1}_{\mathrm{A}})\widetilde{X}] + \mathrm{Tr}[(J(\mathcal{N}) - M - Z)X] \\ & + \mathrm{Tr}[(\mathrm{Tr}_{\mathrm{A}} M - \Delta \mathrm{Tr}_{\mathrm{A}}\mathrm{M})Y_1] + \mathrm{Tr}[(\mathrm{Tr}_{\mathrm{B}} M - \mathbb{1}_{\mathrm{A}})Y_2], \end{aligned} \tag{B8}$$

and its dual function (see more details for prime and dual problems in Ref. [59]) is

$$\begin{aligned} & q(\widetilde{X}, X, Y_1, Y_2) \\ = \ & \inf_{a,Z,M} \mathcal{L}(a, Z, M, \widetilde{X}, X, Y_1, Y_2) \\ = \ & \inf_{a,Z,M} \mathrm{Tr}[J(\mathcal{N})] - \mathrm{Tr}[Y_2] + a[1 - \mathrm{Tr}(\widetilde{X})] \\ & + \mathrm{Tr}[(2 \cdot \widetilde{X} \otimes \mathbb{1}_{\mathrm{B}} - X)Z] \\ & + \mathrm{Tr}[(\mathbb{1}_{\mathrm{A}} \otimes Y_1 - \mathbb{1}_{\mathrm{A}} \otimes \Delta Y_1 + Y_2 \otimes \mathbb{1}_{\mathrm{B}} - X)M]. \end{aligned} \tag{B9}$$

The dual function has value $\mathrm{Tr}[J(\mathcal{N})X] - \mathrm{Tr}[Y_2]$ if $\widetilde{X} \leq 1 \wedge 2 \cdot \widetilde{X} \otimes \mathbb{1}_{\mathrm{B}} - X \geq 0 \wedge \mathbb{1}_{\mathrm{A}} \otimes Y_1 - \mathbb{1} \otimes \Delta Y_1 + Y_2 \otimes \mathbb{1}_{\mathrm{B}} - X \geq 0$ and $-\infty$ in other cases.

Thus, the dual problem is to maximize $\mathrm{Tr}[J(\mathcal{N})X] - \mathrm{Tr}[Y_2]$ with constraints for $\widetilde{X}, X, Y_1, Y_2$. To make the constraint clear and easy reading, we simplify the four constraints: (i) and (ii) $\widetilde{X}$ is nonnegative and trace less than one; (iii) $X \geq 0$; (iv) $2 \cdot \widetilde{X} \otimes \mathbb{1}_{\mathrm{B}} - X$ to one constraint as $X \leq 2 \cdot \rho \otimes \mathbb{1}_{\mathrm{B}}$ which ρ is a density matrix . Then we can construct a trace one $\widetilde{X}' := (1/\mathrm{Tr}\widetilde{X})\widetilde{X}$, positive semidefinite will keep $2 \cdot \widetilde{X}' \otimes \mathbb{1}_{\mathrm{B}} - X$ positive semidefinite for all X, which satisfies $2\widetilde{X} \otimes \mathbb{1}_{\mathrm{B}} - X \geq 0$. Hence, the dual problem is

$$\begin{aligned} & \text{Dual} \\ \text{maximize} \quad & \mathrm{Tr}[J(\mathcal{N})X] - \mathrm{Tr}[Y_2] \\ \text{s.t.} \quad & X \leq 2 \cdot \rho \otimes \mathbb{1}_{\mathrm{B}} : \rho \text{ is density matrix}, \\ & \mathbb{1}_{\mathrm{A}} \otimes Y_1 - \mathbb{1} \otimes \Delta Y_1 + Y_2 \otimes \mathbb{1}_{\mathrm{B}} - X \geq 0, \\ & X \geq 0, \ \ Y_1 = Y_1^{\dagger}, \ \ Y_2 = Y_2^{\dagger}. \end{aligned} \tag{B10}$$

In the end, we have to show that the primal and the dual problem reach the same optimal value, which is the strong duality holds in this programming. The strong duality obtains if the Slater condition[60] holds: there exists some Z^* and W^* satisfy all the equality constraints in primal problem and all the inequality constraints strictly hold. It is easy to find that the Slater condition holds when

$$Z^* = \mathbb{1}_{\mathrm{A}} \otimes \mathbb{1}_{\mathrm{B}} + J(\mathcal{N}), \tag{B11}$$

$$M^* = \frac{1}{|B|} \mathbb{1}_{\mathrm{A}} \otimes \mathbb{1}_{\mathrm{B}}, \tag{B12}$$

and the strong duality would promise the optimal value can be reached.

Example The quantum channel $\mathcal{K}$ has two Kraus operators

$$K_0 = \begin{bmatrix} 0.2096 & -0.3956 \\ -0.2564 & -0.3719 \end{bmatrix}, \tag{B13}$$

$$K_1 = \begin{bmatrix} -0.6197 & 0.6418 \\ -0.7116 & -0.5415\,p \end{bmatrix}. \tag{B14}$$

By solving SDP in software CVX,[61] this quantum channel has total coherence of $+0.186758$.

Acknowledgment

Project supported by the National Natural Science Foundation of China (Grant Nos. 12175029, 11775040, and 12011530014).

References

[1] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865
[2] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 *Phys. Rev. Lett.* **78** 2275
[3] Girolami D and Yadin B 2017 *Entropy* **19** 124
[4] Yu C S and Song H S 2009 *Phys. Rev. A* **80** 022324
[5] Ollivier H and Zurek W H 2001 *Phys. Rev. Lett.* **88** 017901
[6] Datta A, Shaji A and Caves C M 2008 *Phys. Rev. Lett.* **100** 050502
[7] Cavalcanti D, Aolita L, Boixo S, Modi K, Piani M and Winter A 2010 *Phys. Rev. A* **83** 483
[8] Luo S 2008 *Phys. Rev. A* **77** 042303
[9] Giorda P and Paris M G A 2010 *Phys. Rev. Lett.* **105** 020503
[10] Baumgratz T, Cramer M and Plenio M B 2014 *Phys. Rev. Lett.* **113** 140401
[11] Brunner N, Cavalcanti D, Pironio S, Scarani V and Wehner S 2014 *Rev. Mod. Phys.* **86** 419
[12] Kirchmair G, Zähringer F, Gerritsma R, Kleinmann M, Gühne O, Cabello A, Blatt R and Roos C F 2009 *Nature* **460** 494
[13] Strobel H, Muessel W, Linnemann D, Zibold T, Hume D B, Pezzè L, Smerzi A and Oberthaler M K 2014 *Science* **345** 424
[14] Marvian I and Spekkens R 2014 *Nat. Commun.* **5** 3821
[15] Yu X D, Zhang D J, Xu G F and Tong D M 2016 *Phys. Rev. A* **94** 060302
[16] Rana S, Parashar P and Lewenstein M 2016 *Phys. Rev. A* **93** 012110
[17] Yao Y, Xiao X, Ge L and Sun C P 2015 *Phys. Rev. A* **92** 022112
[18] Zhao H Q and Yu C S 2018 *Sci. Rep.* **8** 299
[19] Yu C S 2017 *Phys. Rev. A* **95** 042337
[20] Bu K, Singh U, Fei S M, Pati A K and Wu J 2017 *Phys. Rev. Lett.* **119** 150405
[21] Wu Z, Zhang L, Fei S M and Li-Jost X 2020 *J. Phys. A: Math. Theor.* **54** 015302
[22] Winter A and Yang D 2016 *Phys. Rev. Lett.* **116** 120404
[23] Napoli C, Bromley T R, Cianciaruso M, Piani M, Johnston N and Adesso G 2016 *Phys. Rev. Lett.* **116** 150502
[24] Rana S, Parashar P, Winter A and Lewenstein M 2017 *Phys. Rev. A* **96** 052336
[25] Zhu H, Hayashi M and Chen L 2018 *Phys. Rev. A* **97** 022342

[26] Patel D, Patro S, Vanarasa C, Chakrabarty I and Pati A K 2021 *Phys. Rev. A* **103** 022422

[27] Marvian I, Spekkens R W and Zanardi P 2016 *Phys. Rev. A* **93** 052331

[28] Piani M, Cianciaruso M, Bromley T R, Napoli C, Johnston N and Adesso G 2016 *Phys. Rev. A* **93** 042107

[29] Ioffe L and Mézard M 2007 *Phys. Rev. A* **75** 032345

[30] Korzekwa K, Czachórski S, Puchała Z and Życzkowski K 2018 *New J. Phys.* **20** 043028

[31] Wang X, Wilde M M and Su Y 2019 *New J. Phys.* **21** 103002

[32] Bischof F, Kampermann H and Bruß D 2019 *Phys. Rev. Lett.* **123** 110402

[33] Saxena G, Chitambar E and Gour G 2020 *Phys. Rev. Res.* **2** 023298

[34] Xu J 2021 *Phys. Lett. A* **387** 127028

[35] Gour G and Scandolo C M 2020 *Phys. Rev. Lett.* **125** 180505

[36] Theurer T, Satyajit S and Plenio M B 2020 *Phys. Rev. Lett.* **125** 130401

[37] Chitambar E and Gour G 2019 *Rev. Mod. Phys.* **91** 025001

[38] Liu Y and Yuan X 2020 *Phys. Rev. Res.* **2** 012035

[39] Kuroiwa K and Yamasaki H 2020 *Quantum* **4** 355

[40] Díaz M G, Desef B, Rosati M, Egloff D, Calsamiglia J, Smirne A, Skotiniotis M and Huelga S F 2020 *Quantum* **4** 249

[41] Designolle S, Uola R, Luoma K and Brunner N 2021 *Phys. Rev. Lett.* **126** 220404

[42] Gour G and Winter A 2019 *Phys. Rev. Lett.* **123** 150401

[43] Masini M, Theurer T and Plenio M B 2021 *Phys. Rev. A* **103** 042426

[44] Hsieh C Y 2021 *PRX Quantum* **2** 020318

[45] Theurer T, Egloff D, Zhang L and Plenio M B 2019 *Phys. Rev. Lett.* **122** 190405

[46] Mani A and Karimipour V 2015 *Phys. Rev. A* **92** 032331

[47] Yang S R and Yu C S 2018 *Ann. Phys.* **388** 305

[48] Yu C S, Yang S R and Guo B Q 2016 *Quantum Inf. Proc.* **15** 3773

[49] Nielsen M A and Chuang I 2000 *Quantum computation and quantum information* (Cambridge: Cambridge University Press)

[50] Yang S R and Yu C S 2021 arXiv: 2110.14267

[51] Chiribella G, D'Ariano G M and Perinotti P 2008 *Phys. Rev. Lett.* **101** 180501

[52] Cooney T, Mosonyi M and Wilde M M 2016 *Commun. Math. Phys.* **344** 797

[53] Hayashi M 2009 *IEEE Trans. Inf. Theor.* **55** 3807

[54] Duan R, Feng Y and Ying M 2009 *Phys. Rev. Lett.* **103** 210501

[55] Matthews W, Wehner S and Winter A 2009 *Commun. Math. Phys.* **291** 813

[56] Choi M D 1975 *Linear Algebra Appl.* **10** 285

[57] Jamio lkowski A 1972 *Rep. Math. Phys.* **3** 275

[58] Khachiyan L 1980 *USSR Comput. Math. Phys.* **20** 53

[59] Watrous J 2018 *The Theory of Quantum Information* (Cambridge: Cambridge university press)

[60] Giorgi G and Kjeldsen T H 2014 *Traces and Emergence of Nonlinear Programming* (New York: Springer)

[61] Grant M and Boyd S *Cvx: Matlab software for disciplined convex programming*