Performance optimization on finite-time quantum Carnot engines and refrigerators based on spin-1/2 systems driven by a squeezed reservoir

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We investigate the finite-time performance of a quantum endoreversible Carnot engine cycle and its inverse operation — Carnot refrigeration cycle, employing a spin-1/2 system as the working substance. The thermal machine is alternatively driven by a hot boson bath of inverse temperature $\beta_{\rm h}$ and a cold boson bath at inverse temperature $\beta_{\rm c}$ ($>\beta_{\rm h}$). While for the engine model the hot bath is constructed to be squeezed, in the refrigeration cycle the cold bath is established to be squeezed, with squeezing parameter r. We obtain the analytical expressions for both efficiency and power in heat engines and for coefficient of performance and cooling rate in refrigerators. We find that, in the high-temperature limit, the efficiency at maximum power is bounded by the analytical value $\eta_+ = 1-\sqrt{\mathrm{sech}(2r)(1-\eta_{\rm C})}$, and the coefficient of performance at the maximum figure of merit is limited by $\varepsilon_+ = \frac{\sqrt{\mathrm{sech}(2r)(1+\varepsilon_{\rm C})}}{\sqrt{\mathrm{sech}(2r)(1+\varepsilon_{\rm C})}-\varepsilon_{\rm C}} - 1$, where $\eta_{\rm C} = 1-\beta_{\rm h}/\beta_{\rm c}$ and $\varepsilon_{\rm C} = \beta_{\rm h}/(\beta_{\rm c}-\beta_{\rm h})$ are the respective Carnot values of the engines and refrigerators. These analytical results are identical to those obtained from the Carnot engines based on harmonic systems, indicating that the efficiency at maximum power and coefficient at maximum figure of merit are independent of the working substance.

Keywords: performance optimization, squeezed bath, quantum Carnot engine, quantum Carnot refrigerator

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1. Introduction

Heat engines convert thermal energy into mechanical work and refrigerators operate in reverse operation. These thermal machines based on quantum systems[1–11] provide excellent platforms for understanding the non-equilibrium thermodynamics of open systems, in addition to their potential application to utilizing energy. For the standard heat engines, the efficiency is bounded from above by the Carnot efficiency, $\eta_{\rm C} = 1-\beta_{\rm h}/\beta_{\rm c}$. In contrast, the upper bound imposed on the efficiency may surpass the Carnot value due to quantum effects, such as squeezing, quantum coherence,[12–15] and quantum correlations.[12,16–20] These effects may also lead to the similar behavior that the quantum refrigerators can go beyond the Carnot limit.[21–29]

The thermal machines are of little practical use when they achieve the maximal efficiency (maximal coefficient of performance) at vanishing power (cooling rate). It is indicated that these machines should operate in finite time and they may be considered within framework of finite time thermodynamics which began with the seminar work[30] where the famous Curzon–Ahlborn efficiency $\eta_{\rm CA} = 1-\sqrt{1-\eta_{\rm C}}$ was obtained. The heat engines were always optimized under the maximal power,[8,31–40] and the objective function for optimizing the refrigerators was usually the figure of merit χ, where $\chi := \varepsilon Q_{\rm c}^{\rm R}/\tau_{\rm cyc}$ (with cooling load $Q_{\rm c}^{\rm R}$ and total cycle period $\tau_{\rm cyc}$). It has been shown that, adopting χ as an objective function, there is so-called Curzon–Ahlborn coefficient, $\varepsilon_{\rm CA} = \sqrt{\varepsilon_{\rm C}+1}-1$, with $\varepsilon_{\rm C} = \beta_{\rm h}/(\beta_{\rm c}-\beta_{\rm h})$ being the Carnot coefficient of performance. Both quantum engines and quantum refrigerators in finite time have been shown to outperform their classical counterparts. For example, either the efficiency at maximum power or the coefficient of performance can be dramatically enhanced by reservoir squeezing, and it may be larger than the Carnot value with no violation of the second law of thermodynamics.[12,31,41–52]

Besides the quantum Otto cycle, the quantum Carnot model in finite time has attracted much interest and produced meaningful results.[41,49,53–62] Among them, the special emphasis has been put on the possible universal behavior of efficiency at maximum power in standard quantum heat engines, with the universality the same as that of the Curzon–Ahlborn efficiency. Another interesting finding is that the efficiency for a squeezed-bath-driven Carnot engine working with harmonic systems (obeying the Bose–Einstein statistics) takes the same form as that from a quantum Otto engine based on harmonic systems with reservoir squeezing. Here a natural question arising is: how is the finite-time performance of Carnot machines (including heat engines and refrigerators) under reservoir squeezing when the working systems obey the Fermi–

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Dirac statistics? Is there the so-called generalized Curzon–Ahlborn efficiency (coefficient of performance) in the heat engines in the presence of reservoir squeezing?

The purpose of this paper is to address the problems by investigating the efficiency at maximum power and coefficient of performance at maximum figure of merit for the thermal machines working with spin$-1/2$ systems, which are quite different from the harmonic systems and obey the Fermi–Dirac statistics. In the present paper, we investigate the finite-time performance of Carnot engine and refrigerator, each of which is alternatively fueled by a squeezed and a thermal bath. In contrast to the working substance, these two baths for the engine or refrigerator are composed of harmonic oscillators and thus they obey Bose–Einstein statistics. We obtain general formulae for heat and work per cycle, and for the time spent on the isothermal processes. We find that, in the high-temperature limit, the efficiency at maximum power is identical to that obtained from quantum Otto and Carnot engines working with harmonic systems, and the coefficient of performance at maximum figure of merit is also the same as that derived from the quantum Carnot engines based on harmonic systems.

2. Machine models

We consider a quantum system with a magnetic moment $\boldsymbol{M}$ placed in a magnetic field $\boldsymbol{B}$. The direction of the magnetic field $\boldsymbol{B}$ is chosen constant and along the positive z axis. The magnitude of the magnetic field can change over time, but is not allowed to reach zero. The Hamiltonian of the interaction between the magnetic moment $\boldsymbol{M}$ in the quantum system and the magnetic field $\boldsymbol{B}$ is given as $\hat{H}(t) = -\boldsymbol{M}\cdot\boldsymbol{B} = 2\mu_{\rm B}\boldsymbol{S}\cdot\boldsymbol{B} = 2\mu_{\rm B}B_Z(t)\hat{S}_Z$, where $\mu_{\rm B}$ is the Bohr magnetron, $\boldsymbol{S}$ is a spin angular momentum. Let us define the time-dependent frequency $\omega(t) =: 2\mu_{\rm B}B_Z(t)$, where ω is positive since the spin angular momentum and magnetic moment are in opposite directions.[63] The quantum Carnot engine and refrigeration is composed of a spin-$1/2$ system. For this spin system, the Hamiltonian can be given by ($\hbar \equiv 1$)[64]

$$\hat{H}(t) = \omega(t)\hat{S}_z, \tag{1}$$

where $\hat{S}_z$ is the matrix by casting $1/2$ into the Pauli matrix. The system energy with inverse temperature β is given by

$$\langle\hat{H}\rangle = \omega(t)\langle\hat{S}_z\rangle = \omega S, \tag{2}$$

where $S = \langle\hat{S}_z\rangle = -\frac{1}{2}\tanh(\beta\omega/2)$ is the average polarization, which is equivalent to the expectation of $\hat{S}_z$.

Figures 1(a) and 1(b) are schematic diagrams of quantum Carnot engine and refrigerator in the (ω, S) plane, respectively. While for the heat engine the hot bath is squeezed, in the refrigerator the cold bath is set to be squeezed. Along each thermal-contact process, the spin-$1/2$ system is weakly coupled to a boson bath composed of harmonic oscillators. The heat bath excites and deexcites the spin system in a specific way where the detailed balance is satisfied, such that the system can reach the steady state in infinite long time. In such a steady state, the excitation number of the boson reservoir under squeezing can be expressed in terms of squeezing parameter r:[31,65] $n_{\rm sq} = n_{\rm th} + (2n_{\rm th}+1)\sinh^2(r)$, where $n_{\rm th} = 1/(\mathrm{e}^{\beta\omega}-1)$ denotes the thermal occupation number. Here, the squeezing parameter r quantifies the deviation from the thermal equilibrium. By introducing the effective inverse temperature $\beta^{\rm eff}$ of the squeezed bath, the excitation number can be written as $n_{\rm sq} = 1/(\mathrm{e}^{\omega\beta^{\rm eff}}-1)$, leading to

$$\beta^{\rm eff} = \omega^{-1}\ln\frac{n_{\rm th}+(2n_{\rm th}+1)\sinh^2(r)+1}{n_{\rm th}+(2n_{\rm th}+1)\sinh^2(r)}. \tag{3}$$

As in the classical endoreversible engine,[30] the inverse temperatures $\beta_{\rm c}'$ and $\beta_{\rm h}'$ ($<\beta_{\rm c}'$) of the working substance, which are different from those of the heat baths owing to finite heat flux between the system and heat bath, are assumed to be respective constants along these isothermal processes. These specific processes of the engine and refrigeration cycles are described as follows.

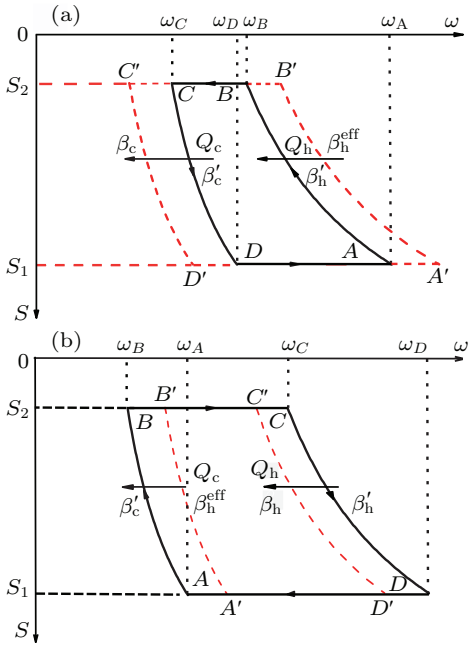


Fig. 1. The cycle of operation for the endoreversbile Carnot (a) heat engine and (b) refrigerator. The actual cyclic operation $A\to B\to C\to D\to A$ is different from the reversible cycle $A'\to B'\to C'\to D'\to A'$. $\beta_{\rm c}>\beta_{\rm c}'>\beta_{\rm h}'>\beta_{\rm h}$ for the heat engine (a) and $\beta_{\rm c}'>\beta_{\rm c}^{\rm eff}>\beta_{\rm h}>\beta_{\rm h}'$ for the refrigerator (b). $\beta_{\rm c,h}'$ denote the inverse temperatures of the system during the hot and cold isotherms, and $\beta_{\rm c,h}$ ($\beta_{\rm c,h}^{\rm eff}$) indicate the temperatures (effective temperatures) of the cold and hot baths.

We first briefly describe the Carnot engine cycle [cf. Fig. 1(a)] consisting of two isothermal and two adiabatic processes. Hot isothermal stroke $A\to B$: the system with constant inverse temperature $\beta_{\rm h}'$ is weakly coupled to a squeezed reservoir at effective inverse temperature $\beta_{\rm h}^{\rm eff}$[31,42,44] in time duration $\tau_{\rm h}$. Adiabatic expansion $B\to C$: the system with varying

frequency ω undergoes unitary evolution over time duration $\tau_{\rm hc}$. The von Neumann entropy of the system is constant and thus the average population S is kept constant along this stroke. Cold isothermal stroke $C \to D$: the system releases heat by weakly coupling to a thermal reservoir at inverse temperature $\beta_{\rm c}$. The time duration for this system–bath interaction is denoted by $\tau_{\rm c}$. Adiabatic compression $D \to A$: the system returns to the initial state A while keeping constant entropy during a time period $\tau_{\rm ch}$.

The four consecutive strokes for the refrigerator sketched in Fig. 1(b) are now described. Isothermal cooling stroke $A \to B$: the system is weakly coupled to the cold squeezed bath of constant effective inverse temperature $\beta_{\rm c}^{\rm eff}$ [31,42,44] in time period $\tau_{\rm c}^{\rm R}$. Adiabatic compression $B \to C$: the working system, which is isolated from the two baths, undergoes a unitary evolution over time duration $\tau_{\rm ch}^{\rm R}$. The third stroke $C \to D$ is similar to the first, but the hot bath of constant inverse temperature $\beta_{\rm h}^{\rm R}$ is thermal. During the time duration $\tau_{\rm h}^{\rm R}$, the frequency changes from ω_C to ω_D. The fourth adiabatic expansion branch $D \to A$: the working system takes a unitary evolution in time duration $\tau_{\rm hc}^{\rm R}$.

3. Optimization on the performance of the machines

The differential elements of work extracted and heat injection can be identified respectively (see Appendix A) as

$$\mathrm{d}W = S\mathrm{d}\omega, \quad \mathrm{d}Q = \omega \mathrm{d}S. \tag{4}$$

In physical terms, the work is produced by changing the frequency and heat occurs through the transition between energy levels.

3.1. Efficiency at maximum power for Carnot engines

We first consider the performance of quantum Carnot engine in finite time. From Eq. (4), the heat absorbed by the system along the hot and cold isothermal strokes, can be determined according to $Q_{\rm h} = \int_{S_1}^{S_2} \omega \mathrm{d}S$, and $Q_{\rm c} = \int_{S_2}^{S_1} \omega \mathrm{d}S$, leading to

$$\begin{aligned} Q_{\rm h} =& \frac{1}{2}\omega_A \tanh\left(\frac{\beta_{\rm h}'\omega_A}{2}\right) \\ &- \frac{1}{2}\omega_B \tanh\left(\frac{\beta_{\rm h}'\omega_B}{2}\right) - \frac{1}{\beta_{\rm h}'}\ln\frac{\cosh\left(\frac{\beta_{\rm h}'\omega_A}{2}\right)}{\cosh\left(\frac{\beta_{\rm h}'\omega_B}{2}\right)}, \end{aligned} \tag{5}$$

$$\begin{aligned} Q_{\rm c} =& \frac{1}{2}\frac{\beta_{\rm h}'}{\beta_{\rm c}'}\omega_B \tanh\left(\frac{\beta_{\rm h}'\omega_B}{2}\right) - \frac{1}{2}\frac{\beta_{\rm h}'}{\beta_{\rm c}'}\omega_A \tanh\left(\frac{\beta_{\rm h}'\omega_A}{2}\right) \\ &- \frac{1}{\beta_{\rm c}'}\ln\frac{\cosh\left(\frac{\beta_{\rm h}'\omega_B}{2}\right)}{\cosh\left(\frac{\beta_{\rm h}'\omega_A}{2}\right)}. \end{aligned} \tag{6}$$

In deriving Eqs. (5) and (6), we have used the relations $\beta_{\rm h}'\omega_B = \beta_{\rm c}'\omega_C, \beta_{\rm h}'\omega_A = \beta_{\rm c}'\omega_D$. which are obtained from the two quantum adiabatic strokes. Since no heat is exchanged during the two adiabatic strokes, these two equations (5) and (6) set the total work extracted per cycle, $W = Q_{\rm h} + Q_{\rm c}$, and thus determine the power output $P = W/\tau_{\rm cyc}$, where $\tau_{\rm cyc} = \tau_{\rm h} + \tau_{\rm c} + \tau_{\rm hc} + \tau_{\rm ch}$ is the total cycle period. Note that, if the quantum Carnot cycle is reversible, the system temperature approaches the bath temperature, leading to $\beta_{\rm h}' \to \beta_{\rm h}^{\rm eff}$. Thus, the heats exchanged along the two isochores [see Eqs. (5) and (6)] as a function of $\beta_{\rm h}'$ are r-dependent.

We show in Appendix A that, the times required for completing the hot and cold isothermal strokes, $\tau_{\rm h}$ and $\tau_{\rm c}$, can be given by

$$\tau_{\rm h} = -\frac{1}{\alpha}\int_{\omega_A}^{\omega_B} \frac{(\mathrm{d}S/\mathrm{d}\omega)\mathrm{d}\omega}{\mathrm{e}^{q\beta_{\rm h}^{\rm eff}\omega}[2(1+\mathrm{e}^{\beta_{\rm h}^{\rm eff}\omega})S + \mathrm{e}^{\beta_{\rm h}^{\rm eff}\omega} - 1]}, \tag{7}$$

$$\tau_{\rm c} = -\frac{1}{\alpha}\int_{\omega_C}^{\omega_D} \frac{(\mathrm{d}S/\mathrm{d}\omega)\mathrm{d}\omega}{\mathrm{e}^{q\beta_{\rm c}\omega}[2(1+\mathrm{e}^{\beta_{\rm c}\omega})S + \mathrm{e}^{\beta_{\rm c}\omega} - 1]}, \tag{8}$$

where α and q, with $\alpha > 0$ and $-1 < q < 0$, are parameters depending on the system and bath. For the times spent on the two adiabatic strokes, we can use, $\tau_{\rm hc} = (2\mu)^{-1}\ln(\omega_B/\omega_C)$ and $\tau_{\rm ch} = (2\mu)^{-1}\ln(\omega_A/\omega_D)$,[66,67] where μ is called a nonadiabatic parameter dependent on the working substance. It follows that the total time required for completing the two adiabatic strokes takes the form of

$$\tau_{\rm adi} := \tau_{\rm ch} + \tau_{\rm hc} = \mu^{-1}\ln\left(\frac{\beta_{\rm c}'}{\beta_{\rm h}'}\right). \tag{9}$$

In the high-temperatures limit where $\beta\omega \ll 1$ and $\tanh(\beta\omega) \approx \beta\omega$, the heats (5) and (6) simplify to

$$Q_{\rm h} = -\beta_{\rm h}'(\omega_B^2 - \omega_A^2)/8 = \frac{2}{\beta_{\rm h}'}(S_1^2 - S_2^2), \tag{10}$$

$$Q_{\rm c} = -\frac{\beta_{\rm h}'^2}{\beta_{\rm c}'}(\omega_A^2 - \omega_B^2)/8 = -\frac{2}{\beta_{\rm c}'}(S_1^2 - S_2^2), \tag{11}$$

which give rise to

$$W = 2\left(-\frac{1}{\beta_{\rm c}'} + \frac{1}{\beta_{\rm h}'}\right)(S_1^2 - S_2^2). \tag{12}$$

The machine efficiency, $\eta = W/Q_{\rm h}$, then follows

$$\eta = 1 - \frac{\beta_{\rm h}'}{\beta_{\rm c}'}. \tag{13}$$

At this limit where the effective inverse temperature of the hot squeezed bath $\beta_{\rm h}^{\rm eff}$, given by Eq. (3), simplifies to $\beta_{\rm h}\,\mathrm{sech}(2r)$, the times spent along the two isothermal strokes, Eqs. (7) and (8), can be obtained after some simple algebra

$$\tau_{\rm h} = \frac{\ln\left(\frac{S_2}{S_1}\right)}{4\alpha\left[\frac{\beta_{\rm h}\,\mathrm{sech}(2r)}{\beta_{\rm h}'} - 1\right]}, \quad \tau_{\rm c} = \frac{\ln\left(\frac{S_1}{S_2}\right)}{4\alpha\left(\frac{\beta_{\rm c}}{\beta_{\rm c}'} - 1\right)}. \tag{14}$$

With consideration of Eqs. (9), (12) and (14), we obtain the power output as

$$P = \frac{W}{\tau_{\rm cyc}} = \mathcal{A}\frac{\beta_{\rm c}' - \beta_{\rm h}'}{\beta_{\rm h}'\beta_{\rm c}'(\mathcal{B} + \mathcal{C})}, \tag{15}$$

where

$$\mathcal{A}=\frac{8\alpha(S_1^2-S_2^2)}{\ln\frac{S_1}{S_2}},\quad \mathcal{B}=\frac{4\alpha}{\mu\ln\frac{S_1}{S_2}}\ln\frac{\beta_{\rm c}'}{\beta_{\rm h}'},$$
$$\mathcal{C}=\frac{\beta_{\rm h}'\beta_{\rm c}-\beta_{\rm c}'\beta_{\rm h}\,{\rm sech}(2r)}{[\beta_{\rm h}'-\beta_{\rm h}\,{\rm sech}(2r)](\beta_{\rm c}-\beta_{\rm c}')}.$$

To proceed, we discuss the efficiency at maximum power by assuming the maximal and minimal values of spin polarization, S_1 and S_2 in Fig. 1(a), to be fixed. For given ratio α/μ, the efficiency at maximum power can be determined by numerically solving the set of equations, $\{\partial P/\partial\beta_{\rm h}'=0,\ \partial P/\partial\beta_{\rm c}'=0\}$. Our numerical calculations, which are not plotted in figure, show that the efficiency at maximum power increases as α/μ decreases, and it reaches its upper bound in the sudden limit where the time taken for the two adiabatic strokes is negligible, i.e., $\alpha/\mu\to 0$. In this sudden limit, setting $\partial P/\partial\beta_{\rm h}'=0$ and $\partial P/\partial\beta_{\rm c}'=0$, togher with Eq. (15), gives rise to the optimal temperatures of the working substance:

$$\beta_{\rm h}'=\frac{2\sqrt{\beta_{\rm c}}\beta_{\rm h}\,{\rm sech}(2r)}{\sqrt{\beta_{\rm c}}+\sqrt{\beta_{\rm h}\,{\rm sech}(2r)}},\quad \beta_{\rm c}'=\frac{2\beta_{\rm c}\sqrt{\beta_{\rm h}\,{\rm sech}(2r)}}{\sqrt{\beta_{\rm c}}+\sqrt{\beta_{\rm h}\,{\rm sech}(2r)}}.$$

It follows, inserting these optimal values of $\beta_{\rm h}'$ and $\beta_{\rm c}'$ into Eq. (13), the upper bound of efficiency at maximum power reads

$$\eta_+=1-\sqrt{{\rm sech}(2r)}\sqrt{1-\eta_{\rm C}}. \tag{16}$$

This agrees well with the results obtained from the Carnot engines and Otto engines working with harmonic oscillators. Under no squeezing ($r=0$), the so-called Curzon–Ahlborn efficiency $\eta_{\rm CA}=1-\sqrt{1-\eta_{\rm C}}$ is recovered, as expected.[30]

With consideration of Eq. (3), in the high-temperature limit the effective inverse temperature of the hot reservoir can be obtained as $\beta_{\rm h}^{\rm eff}={\rm sech}(2r)\beta_{\rm h}$. The generalized version of the Carnot efficiency, $\eta_{\rm C}^{\rm gen}:=1-\beta_{\rm h}^{\rm eff}/\beta_{\rm c}$, turns out to be $\eta_{\rm C}^{\rm gen}=1-{\rm sech}(2r)(1-\eta_{\rm C})$. Equation (16) shows that the machine efficiency is significantly enhanced by reservoir squeezing. We should note from Eq. (16) that the efficiency at maximum power may be larger than the standard Carnot efficiency $\eta_{\rm C}$ if r satisfies the relation of $r>\frac{1}{2}\ln\frac{1+\sqrt{(2-\eta_{\rm C})\eta_{\rm C}}}{1-\eta_{\rm C}}$, but it is always smaller than the generalized Carnot value $\eta_{\rm C}^{\rm gen}$, thereby implying that the second law of thermodynamics is not contradicted as it should.

3.2. Coefficient of performance for the refrigerator under maximal figure of merit

In this section, we investigate the coefficient of performance for the Carnot refrigerator [sketched in Fig. 1(b)] under maximal figure of merit. The heats absorbed by the working substance in the refrigeration cycle are determined according to $Q_{\rm c}^{\rm R}=\int_{S_1}^{S_2}\omega\,{\rm d}S$ and $Q_{\rm h}^{\rm R}=\int_{S_2}^{S_1}\omega\,{\rm d}S$ to arrive at

$$Q_{\rm h}^{\rm R}=\frac{1}{2}\frac{\beta_{\rm c}'}{\beta_{\rm h}'}\omega_B\tanh\left(\frac{\beta_{\rm c}'\omega_B}{2}\right)-\frac{1}{2}\frac{\beta_{\rm c}'}{\beta_{\rm h}'}\omega_A\tanh\left(\frac{\beta_{\rm c}'\omega_A}{2}\right)$$
$$-\frac{1}{\beta_{\rm h}'}\ln\frac{\cosh\left(\frac{\beta_{\rm c}'\omega_B}{2}\right)}{\cosh\left(\frac{\beta_{\rm c}'\omega_A}{2}\right)}, \tag{17}$$

$$Q_{\rm c}^{\rm R}=\frac{1}{2}\omega_A\tanh\left(\frac{\beta_{\rm c}'\omega_A}{2}\right)-\frac{1}{2}\omega_B\tanh\left(\frac{\beta_{\rm c}'\omega_B}{2}\right)$$
$$-\frac{1}{\beta_{\rm c}'}\ln\frac{\cosh\left(\frac{\beta_{\rm c}'\omega_A}{2}\right)}{\cosh\left(\frac{\beta_{\rm c}'\omega_B}{2}\right)}. \tag{18}$$

In deriving these, we have used the relations $\beta_{\rm h}'\omega_C=\beta_{\rm c}'\omega_B$ and $\beta_{\rm h}'\omega_D=\beta_{\rm c}'\omega_A$. Similar to the cycle of heat engine, the reservoir squeezing affects $\beta_{\rm c}'$ but not $\beta_{\rm h}'$, and thus it changes $Q_{\rm c}^{\rm R}$ and $Q_{\rm h}^{\rm R}$.

In the high-temperature limit, Eqs. (17) and (18) simplify to

$$Q_{\rm h}^{\rm R}=-\frac{\beta_{\rm c}'^2}{8\beta_{\rm h}'}(\omega_A^2-\omega_B^2)=-\frac{2}{\beta_{\rm h}'}(S_1^2-S_2^2), \tag{19}$$

$$Q_{\rm c}^{\rm R}=-\frac{\beta_{\rm c}'}{8}(\omega_B^2-\omega_A^2)=\frac{2}{\beta_{\rm c}'}(S_1^2-S_2^2). \tag{20}$$

It follows that, the coefficient of performance for the refrigerator, $\varepsilon=-Q_{\rm c}^{\rm R}/(Q_{\rm h}^{\rm R}+Q_{\rm c}^{\rm R})$, is given by

$$\varepsilon=\frac{\beta_{\rm h}'}{\beta_{\rm c}'-\beta_{\rm h}'}. \tag{21}$$

For the refrigeration cycle, the time required for completing the cold and hot isothermal strokes can be determined by using the same approach as adopted in the engine cycle (see Appendix A). In the high temperature limit in which $\beta_{\rm c}^{\rm eff}=\beta_{\rm c}\,{\rm sech}(2r)$, these thermal-contact times $\tau_{\rm h}^{\rm R}$ and $\tau_{\rm c}^{\rm R}$ turn out to be

$$\tau_{\rm h}^{\rm R}=\frac{\ln(\frac{S_1}{S_2})}{4\alpha\left(\frac{\beta_{\rm h}}{\beta_{\rm h}'}-1\right)},\quad \tau_{\rm c}^{\rm R}=\frac{\ln(\frac{S_2}{S_1})}{4\alpha\left[\frac{\beta_{\rm c}\,{\rm sech}(2r)}{\beta_{\rm c}'}-1\right]}. \tag{22}$$

Note that the time spent on the two adiabatic strokes, $\tau_{\rm adi}^{\rm R}$, is still determined according to Eq. (9). The figure of merit, $\chi=\varepsilon Q_{\rm c}^{\rm R}/\tau_{\rm cyc}$, with $\tau_{\rm cyc}^{\rm R}=\tau_{\rm h}^{\rm R}+\tau_{\rm c}^{\rm R}+\tau_{\rm adi}$, can then be obtained by using Eqs. (9) (18), (21) and (22), leading to

$$\chi=\mathcal{A}\frac{\beta_{\rm h}'}{\beta_{\rm c}'(\beta_{\rm c}'-\beta_{\rm h}')(\mathcal{B}+\mathcal{D})}, \tag{23}$$

where $\mathcal{A}$ and $\mathcal{B}$ were defined in Eq. (15), and

$$\mathcal{D}:=\frac{\beta_{\rm h}'\beta_{\rm c}\,{\rm sech}(2r)-\beta_{\rm c}'\beta_{\rm h}}{(\beta_{\rm h}'-\beta_{\rm h})[\beta_{\rm c}\,{\rm sech}(2r)-\beta_{\rm c}']}.$$

To describe the finite-time performance of the refrigerator, we analyze the optimized coefficient of performance (ε^*) by maximizing χ in Eq. (23) with respect to $\beta_{\rm h}'$ and $\beta_{\rm c}'$. In the sudden limit when the total time spent on the two adiabatic strokes $\tau_{\rm adi}$ can be neglected and thus $\alpha/\mu\to 0$, we find that the coefficient of performance is analytically obtained as

$$\varepsilon_+=\frac{\sqrt{{\rm sech}(2r)(1+\varepsilon_{\rm C})}}{\sqrt{{\rm sech}(2r)(1+\varepsilon_{\rm C})-\varepsilon_{\rm C}}}-1. \tag{24}$$

If the squeezing parameter $r = 0$, it reduces to the so-called Curzon–Ahlborn coefficient of performance, the lower bound of our machine model, $\varepsilon_- = \varepsilon_{\rm CA} = \sqrt{1+\varepsilon_{\rm C}} - 1$.

The analytical expression (24) for the sudden case with $\alpha/\mu \to 0$ is larger than the numerical calculations of ε^* with $\alpha/\mu \neq 0$, and thus this analytical result is the upper bound of the coefficient of performance at maximum figure of merit. We note that, as expected from the second law of thermodynamics, the optimized coefficient of performance must be smaller than the generalized Carnot value, $\varepsilon_{\rm C}^{\rm gen} := \frac{\varepsilon_{\rm C}}{(1+\varepsilon_{\rm C})\,{\rm sech}(2r)-\varepsilon_{\rm C}}$, though it may be larger than the standard Carnot value $\varepsilon_{\rm C}$ due to the presence of reservoir squeezing. Quite interestingly, the coefficient of performance at maximum figure of merit for our model is identical to that obtained from the Carnot refrigerator working with harmonic system, thereby indicating that the coefficient of performance for quantum Carnot refrigerators under maximum figure merit is independent of the nature of working substance.

4. Conclusions

We have studied a quantum Carnot heat engine driven by a hot squeezed bath and a quantum Carnot refrigerator fueled by a cold squeezed bath, employing spin-1/2 systems as the working substance. We showed that, in the high-temperature limit, the efficiency at maximum power (16) is identical to the one from both quantum Otto and Carnot engines working with harmonic systems, and the coefficient of performance (24) is also the same as that derived from quantum Carnot engines using harmonic systems as the working substance. Throughout the paper, we only considered the optimization and analysis of the performance of the quantum Otto heat engine and refrigerator in the high-temperature limit. In the finite-temperature domain, the expression of power output P is explicitly dependent on the frequency ω. For example, if we expand the heats (5) and (6) up to the second order (instead of the first order), the heats exchanged along the two thermal-contact processes are still given by Eqs. (5) and (6) due to $\tanh(\beta\omega) \approx \beta\omega + O[(\beta\omega)^3]$, but the time durations (7) and (8) spent on the hot and cold thermal-contact processes respectively become

$$\tau_{\rm h} = \left[\ln\left(\frac{S_2}{S_1}\right) + \ln\frac{1+\beta_{\rm h}^{\rm eff}\omega_A}{1+\beta_{\rm h}^{\rm eff}\omega_B}\right] \Big/ \left[4\alpha\left(\frac{\beta_{\rm h}^{\rm eff}}{\beta_{\rm h}'} - 1\right)\right],$$
$$\tau_{\rm c} = \left[\ln\left(\frac{S_1}{S_2}\right) + \ln\frac{1+\beta_{\rm c}\beta_{\rm h}'\omega_B/\beta_{\rm c}'}{1+\beta_{\rm c}\beta_{\rm h}'\omega_A/\beta_{\rm c}'}\right] \Big/ \left[4\alpha\left(\frac{\beta_{\rm c}}{\beta_{\rm c}'} - 1\right)\right].$$

Accordingly, the power output, $P = W/\tau_{\rm cyc}$, is explicitly dependent on the frequency ω, and the efficiency at maximum power is thus dependent on the frequency ω. This holds similarly in the refrigerator. The finite-time machine performance in finite-temperature domain is therefore expected to be different from that obtained in the high temperature limit. Extending our discussion to finite temperatures and even to low temperatures would be of great interest, since in those regimes the machine performance is dependent on the external field associated with the frequency, and the non-negligible quantum effects would significantly affect the machine performance. Moreover, in contrast to the macroscopic heat engines where heat and work are deterministic, the heat and work are stochastic variables and the fluctuations of work and heat are considerable. The topic of fluctuations in these quantum Carnot machines is thus of great interest and deserved to be studied in future.

Appendix A: Derivation of the time spent on an isothermal process

Based on the semigroup formalism,[63,68] the motion of an operator for an open system weakly coupled a heat bath of inverse temperature β can be described by $\mathrm{d}\hat{X}/\mathrm{d}t = \mathrm{i}[\hat{H},\hat{X}] + \frac{\partial \hat{X}}{\partial t} + L_{\rm D}(\hat{X})$,[68–71] where $L_{\rm D}(\hat{X}) = \sum_a \gamma_\alpha(\hat{V}_\alpha^+[\hat{X},\hat{V}_\alpha] + [\hat{V}_\alpha^+,\hat{X}]\hat{V}_\alpha)$ is a dissipation term originating from the system–thermal interaction. Here $\hat{V}_\alpha$ and $\hat{V}_\alpha^\dagger$ are Kraus operators, and γ_α is phenomenological positive coefficients. Setting $\hat{X} = \hat{H}$, with $\hat{H}$ given by Eq. (1) in the main text, the rate of the change in the system energy is obtained as

$$\left\langle \frac{\mathrm{d}\hat{H}}{\mathrm{d}t} \right\rangle = \left\langle \frac{\partial \hat{H}}{\partial t} \right\rangle + \langle L_{\rm D}(\hat{H})\rangle = \frac{\mathrm{d}\omega}{\mathrm{d}t} S + \omega\frac{\mathrm{d}S}{\mathrm{d}t}, \tag{A1}$$

where S was defined by the main-text Eq. (2). The instantaneous power and heat flow are identified as $\mathrm{d}W/\mathrm{d}t = S\mathrm{d}\omega/\mathrm{d}t$ and $\mathrm{d}Q/\mathrm{d}t = \omega\mathrm{d}S/\mathrm{d}t$, respectively, leading to Eq. (4) in the main text. For the spin system under consideration, $\hat{V}_\alpha$ and $\hat{V}_\alpha^+$ are chosen to be the spin creation and annihilation operators, $\hat{S}_+ = \hat{S}_x + \mathrm{i}\hat{S}_y$ and $\hat{S}_- = \hat{S}_x - \mathrm{i}\hat{S}_y$, yielding

$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}t} = \mathrm{i}\omega[\hat{S}_z,\hat{X}] + \frac{\partial \hat{X}}{\partial t} + \gamma_+(\hat{S}_-[\hat{X},\hat{S}_+] + [\hat{S}_-,\hat{X}]\hat{S}_+) + \gamma_-(\hat{S}_+[\hat{X},\hat{S}_-] + [\hat{S}_+,\hat{X}]\hat{S}_-). \tag{A2}$$

Substituting $\hat{X} = \hat{S}_z$ into Eq. (A2) and taking the expectation value, we then have

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -2(\gamma_+ + \gamma_-)S - (\gamma_- - \gamma_+), \tag{A3}$$

where γ_- and γ_+ represent transition rates from the excited to the ground level and vice versa, respectively. For an isothermal process where ω is explicitly time dependent, the instantaneous detailed balance relation $\gamma_-/\gamma_+ = \mathrm{e}^{\beta\omega(t)} = (1-2S)/(1+2S)$ is assumed to be satisfied, such that the system can asymptotically evolve to the thermal equilibrium in a specific way when the external field is frozen at $\omega(t) = \omega$. Hence, γ_+ and γ_- can be parameterized as

$$\gamma_+ = \alpha\,\mathrm{e}^{q\beta\omega}, \quad \gamma_- = \alpha\,\mathrm{e}^{(1+q)\beta\omega}, \tag{A4}$$

where q and α are constant parameters dependent on the system and heat bath, with $-1 < q < 0$ and $\alpha > 0$. Substituting Eq. (A4) into Eq. (A3), we obtain

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\alpha \mathrm{e}^{q\beta\omega}[2(1+\mathrm{e}^{\beta\omega})S + (\mathrm{e}^{\beta\omega}-1)]. \tag{A5}$$

This allows us to obtain the time taken for the isothermal process in which the external field varies from ω_i to ω_f, which reads

$$\tau = -\frac{1}{\alpha}\int_{\omega_i}^{\omega_f} \frac{(\mathrm{d}S/\mathrm{d}\omega)\mathrm{d}\omega}{\mathrm{e}^{q\beta\omega}[2(1+\mathrm{e}^{\beta\omega})S + \mathrm{e}^{\beta\omega}-1]}, \tag{A6}$$

which produces Eqs. (7) and (8) in the main part. The instantaneous average polarization in Eq. (A6) can be expressed as $S = -\frac{1}{2}\tanh(\beta'\omega/2)$ by introducing system temperature β'.

In the high-temperature limit when $\beta\omega \ll 1$ and $S \approx -\beta'\omega/4$, an approximation for Eq. (A6) can be obtained after some algebraic manipulation:

$$\tau = \frac{1}{2\alpha}\int_{\omega_i}^{\omega_f} \frac{\mathrm{d}\omega}{2(\beta/\beta'-1)\omega} = \frac{1}{4\alpha(\beta/\beta'-1)} \ln\left(\frac{S_f}{S_i}\right), \tag{A7}$$

where S_i and S_f are the average polarizations at the initial and the final instants during the isothermal process.

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