# Lie symmetry analysis and invariant solutions for the $(3+1)$-dimensional Virasoro integrable model

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Lie symmetry analysis is applied to a $(3+1)$-dimensional Virasoro integrable model and the corresponding similarity reduction equations are obtained with the different infinitesimal generators. Invariant solutions with arbitrary functions for the $(3+1)$-dimensional Virasoro integrable model, including the interaction solution between a kink and a soliton, the lump-type solution and periodic solutions, have been studied analytically and graphically.

**Keywords:** $(3+1)$-dimensional Virasoro integrable model, Lie symmetry, invariant solutions

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## 1. Introduction

It is well known that many phenomena in nature can be modeled by nonlinear partial differential equations. To fining more exact solutions of nonlinear partial differential equations is one of the most important subjects in nonlinear science. With the development of soliton theory, many effective methods have been proposed to study the nonlinear systems, such as Darboux transformation, the Painlevé analysis method, classical Lie group method, nonclassical Lie group method, the function expansion method, and so on.[1–17] On the one hand, many higher dimensional nonlinear systems can be regarded as the extension of the lower dimensional nonlinear systems, for instance, the Korteweg–de Vries (KdV) equation can be extended to Kadomtsev–Petviashvili (KP) equation. It is lucky that one can find some higher dimensional models which have infinite dimensional centerless Virasoro type symmetry algebra by the concrete realizations of the Kac–Moody–Virasoro symmetry algebra.[18,19]

In this paper, we will focus on a $(3+1)$-dimensional Virasoro integrable model

$$u_{xt}u_{xxx}-u_{xxt}u_{xx}+\frac{3}{2}u_xu_{xxx}u_{yz}-2u_{yz}u_{xx}^2 +mu_z(u_{xx}u_{xxy}-u_{xy}u_{xxx})-mu_y(u_{xx}u_{xxz}-u_{xz}u_{xxx})=0, \quad (1)$$

where $m$ being an arbitrary constant and $u=u(x,y,z,t)$. Equation (1) is derived as a $(3+1)$-dimensional Virasoro integrable model from the realization of the Virasoro symmetry algebra

$$[\sigma(f_1),\sigma(f_2)]=\sigma(\dot{f}_2f_1-\dot{f}_1f_2),$$

where the functions $f_1$ and $f_2$ are arbitrary functions of a single independent variable and the dots over the functions $f_1$ and $f_2$ present the derivatives with respect to the independent variable.[20] The authors have proved that Eq. (1) is not only Virasoro integrable but also Painlevé integrable when the constant $m=0$ through the standard WTC method.

We aim to secure various new exact solutions of the $(3+1)$-dimensional Virasoro integrable Eq. (1) that have not been earlier obtained via Lie group theory technique. Without loss of generality, we will perform the Lie group analysis and its symmetry reduction for Eq. (1) by fixing $m=1$. Then Eq. (1) becomes the following form:

$$u_{xt}u_{xxx}-u_{xxt}u_{xx}+\frac{3}{2}u_xu_{xxx}u_{yz}-2u_{yz}u_{xx}^2 +u_z(u_{xx}u_{xxy}-u_{xy}u_{xxx})-u_y(u_{xx}u_{xxz}-u_{xz}u_{xxx})=0. \quad (2)$$

As we know, the Lie group analysis and the abundant exact solutions for Eq. (2) have not been reported in the literature. The organization of the paper is as follows. In Section 2, the infinitesimal generators of Eq. (2) are obtained by means of Lie group analysis method. In Section 3, many reduction equations and the corresponding similarity solutions of Eq. (1) or Eq. (2) are obtained by different selection of the vector fields, which contain arbitrary functions and constants. The structures of the similarity solutions of Eq. (2) are shown analytically and graphically in Section 4. Section 5 gives the summary and discussion.

## 2. Lie symmetry analysis of Eq. (2)

In this section, the classical Lie group method is applied to Eq. (2) and the infinitesimal generators of Eq. (2) are obtained directly. First of all, we make a one-parameter Lie group transformation for Eq. (2)

$$x\to x+\varepsilon\xi,\ \ y\to y+\varepsilon\eta,\ \ z\to z+\varepsilon\chi,$$
$$t\to t+\varepsilon\tau,\ \ u\to u+\varepsilon\Phi, \quad (3)$$

where $\varepsilon$ is a small parameter. The infinitesimal generators $\{\xi,\eta,\chi,\tau,\Phi\}$ are undetermined functions depending on

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$x,y,z,t,u$. The corresponding vector field is

$$\boldsymbol{V}=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\chi\frac{\partial}{\partial z}+\tau\frac{\partial}{\partial t}+\Phi\frac{\partial}{\partial u}, \tag{4}$$

which must satisfy the Lie invariant condition

$$\mathrm{Pr}^{(3)}\boldsymbol{V}(\Delta)|_{\Delta=0}=0, \tag{5}$$

where

$$\begin{aligned}\Delta \;=\; & u_{xt}u_{xxx}-u_{xxt}u_{xx}+\frac{3}{2}u_xu_{xxx}u_{yz}-2u_{yz}u_{xx}^2\\ & +u_z(u_{xx}u_{xxy}-u_{xy}u_{xxx})-u_y(u_{xx}u_{xxz}-u_{xz}u_{xxx}),\end{aligned}$$

and $\mathrm{Pr}^{(3)}\boldsymbol{V}$ is the third prolongation of Eq. (4)

$$\begin{aligned}\mathrm{Pr}^{(3)}\boldsymbol{V}=\boldsymbol{V}&+\Phi^x\frac{\partial}{\partial x}+\Phi^y\frac{\partial}{\partial y}+\Phi^z\frac{\partial}{\partial z}+\Phi^{xx}\frac{\partial^2}{\partial x^2}+\Phi^{xy}\frac{\partial^2}{\partial x\partial y}\\ &+\Phi^{xz}\frac{\partial^2}{\partial x\partial z}+\Phi^{xt}\frac{\partial^2}{\partial x\partial t}+\Phi^{yz}\frac{\partial^2}{\partial y\partial z}+\Phi^{xxx}\frac{\partial^3}{\partial x^3}\\ &+\Phi^{xxy}\frac{\partial^3}{\partial x^2\partial y}+\Phi^{xxz}\frac{\partial^3}{\partial x^2\partial z}+\Phi^{xxt}\frac{\partial^3}{\partial x^2\partial t}.\end{aligned}$$

Then we can get Lie invariant condition

$$\begin{aligned}&\frac{3}{2}\Phi^xu_{xxx}u_{yz}+\Phi^z(u_{xx}u_{xxy}-u_{xy}u_{xxx})-\Phi^y(u_{xx}u_{xxz}+u_{xz}u_{xxx})\\ &+\Phi^{xxy}u_zu_{xx}+\Phi^{xt}u_{xxx}-\Phi^{xy}u_zu_{xxx}\\ &+\Phi^{xx}(u_zu_{xxy}-u_yu_{xxz}-u_{xxt}-4u_{yz})+\Phi^{yz}\left(\frac{3}{2}u_{xxx}u_x-2u_{xx}^2\right)\\ &-\Phi^{xz}u_yu_{xxx}+\Phi^{xxx}(u_{xt}+\frac{3}{2}u_xu_{yz}-u_zu_{xy}-u_yu_{xz})\\ &-\Phi^{xxz}u_yu_{xx}-\Phi^{xxt}u_{xx}=0,\end{aligned} \tag{6}$$

where $\Phi^z$, $\Phi^x$, $\Phi^y$, $\Phi^{xx}$, $\Phi^{xy}$, $\Phi^{xz}$, $\Phi^{xt}$, $\Phi^{yz}$, $\Phi^{xxx}$, $\Phi^{xxy}$, $\Phi^{xxt}$, and $\Phi^{xxz}$ are invariant coefficients. From the standard procedure of Lie group method, we can compute these coefficients and substitute them into the invariant Eq. (6) of the original Eq. (2). Then with the help of Eq. (2), we will have the determined equations for the functions $\{\xi,\eta,\chi,\tau,\Phi\}$

$$\begin{aligned}&\tau_u=0,\;\;\tau_x=0,\;\;\tau_z=0,\;\;\tau_y=0,\;\;\Phi_x=0,\;\;\Phi_{xx}=0,\\ &\Phi_{uy}=0,\;\;\Phi_{uuu}=0,\;\;\Phi_{uuy}=0,\;\;\Phi_{uut}=0,\;\;\Phi_{uuz}=0,\\ &\xi_{xxt}=0,\;\;\xi_{xxx}=0,\;\;\xi_u=0,\;\;\xi_y=0,\;\;\xi_z=0,\;\;\xi_{xx}=0,\\ &\eta_z=0,\;\;\eta_u=0,\;\;\eta_x=0,\;\;\chi_y=0,\;\;\chi_u=0,\;\;\chi_x=0,\\ &\Phi_y-\chi_t=0,\;\;\Phi_z+\eta_t=0,\;\;\Phi_y-\chi_t=0,\\ &2\Phi_{ut}-2\xi_{xt}+3\Phi_{yz}=0,\;\;2\xi_{xt}-2\Phi_{yz}-\Phi_{ut}=0,\\ &\chi_z+\eta_y-\tau_t-\Phi_u=0.\end{aligned} \tag{7}$$

Solving the determined Eq. (7), we can get the corresponding infinitesimal generators

$$\begin{aligned}&\xi=\left(\frac{f_1'}{2}+c_1\right)x+f_2,\;\;\eta=(-f_1'+c_2)y+f_3,\\ &\chi=f_1'z+f_4,\;\;\tau=f_1+c_4t+c_3,\\ &\Phi=(c_2-c_4-f_1')u+f_1''yz+f_4'y-f_3'z+f_5,\end{aligned} \tag{8}$$

where $f_1$, $f_2$, $f_3$, $f_4$, and $f_5$ are arbitrary functions about $t$ and $c_1$, $c_2$, $c_3$, $c_4$ are arbitrary constants. Then the vector field is given as

$$\begin{aligned}\boldsymbol{V} \;=\; & c_1\boldsymbol{V}_1+c_2\boldsymbol{V}_2+c_3\boldsymbol{V}_3+c_4\boldsymbol{V}_4+\boldsymbol{V}_5(f_1)\\ & +\boldsymbol{V}_6(f_2)+\boldsymbol{V}_7(f_3)+\boldsymbol{V}_8(f_4)+\boldsymbol{V}_9(f_5),\end{aligned}$$

where

$$\begin{aligned}&\boldsymbol{V}_1=x\frac{\partial}{\partial x},\;\;\boldsymbol{V}_2=y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u},\;\;\boldsymbol{V}_3=\frac{\partial}{\partial t},\;\;\boldsymbol{V}_4=t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u},\\ &\boldsymbol{V}_5(f_1)=\frac{1}{2}xf_1'\frac{\partial}{\partial x}-yf_1'\frac{\partial}{\partial y}+zf_1'\frac{\partial}{\partial z}+f_1\frac{\partial}{\partial t}+(yzf_1''-uf_1')\frac{\partial}{\partial u},\\ &\boldsymbol{V}_6(f_2)=f_2\frac{\partial}{\partial x},\;\;\boldsymbol{V}_7(f_3)=f_3\frac{\partial}{\partial y}-zf_3'\frac{\partial}{\partial u},\\ &\boldsymbol{V}_8(f_4)=f_4\frac{\partial}{\partial z}+yf_4'\frac{\partial}{\partial u},\;\;\boldsymbol{V}_9(f_5)=f_5\frac{\partial}{\partial u}.\end{aligned} \tag{9}$$

That is to say, the $(3+1)$-dimensional Virasoro integrable Eq. (2) admits the Lie algebra spanned by the vectors $\boldsymbol{V}_j$ $(j=1,2,\ldots,9)$.

## 3. Symmetry reduction and invariant solutions of Eq. (2)

In this section, we will consider the symmetry reduction and invariant solutions of Eq. (2) by selecting different vector fields. In order to find the exact similarity solutions of Eq. (2), we fix the arbitrary functions in the infinitesimal generators to derive the abundant solutions for the Virasoro integrable Eq. (2). We perform the symmetry reduction equations with the aid of the symmetries gained in Eq. (9) and their corresponding invariant solutions of Eq. (2) are illustrated in the following cases.

**Case 1** $\boldsymbol{V}_1=x\frac{\partial}{\partial x}$.

Solving the characteristic equation

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{0}, \tag{10}$$

we can obtain similarity solution and similarity variables, respectively

$$u=F(Y,Z,T),\;\;Y=y,\;\;Z=z,\;\;T=t,$$

where $F(Y,Z,T)$ is an arbitrary function. It is evident that the arbitrary function $F(y,z,t)$ satisfies Eq. (2).

**Case 2** $\boldsymbol{V}_2=y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u}$.

From the characteristic equation

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{u}, \tag{11}$$

the similarity solution and their variables are

$$u=yF(X,Z,T),\;\;X=x,\;\;Z=z,\;\;T=t. \tag{12}$$

Substituting Eq. (12) into Eq. (2), we obtain the first nontrivial reduction as

$$F_{XXX}F_{XT}-F_{XX}^2F_Z-F_{XXT}F_{XX}+\frac{1}{2}F_{XXX}F_XF_Z$$
$$-FF_{XXZ}F_{XX}+FF_{XXX}F_{XZ}=0. \tag{13}$$

Some special solutions of Eq. (13) and the corresponding solutions of Eq. (2) are given as follows:

$$F_1=X+G(Z,T),\ \ u_1=xy+yG_1(z,t),$$
$$F_2=XG(Z,T),\ \ u_2=xyG_2(z,t),$$

where $G_1(z,t)$ and $G_2(z,t)$ are two arbitrary functions.

**Case 3** $\boldsymbol{V}_3=\frac{\partial}{\partial t}$.

It is easy to solve the equation

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}u}{0},$$

and we have the similarity solution

$$u=F(X,Y,Z), \tag{14}$$

with $X=x$, $Y=y$, and $Z=z$. Substituting the reduction Eq. (14) into Eq. (2), we obtain the second reduction as

$$\frac{3}{2}F_{XXX}F_{YZ}F_X-2F_{XX}^2F_{YZ}+F_{XXY}F_{XX}F_Z$$
$$-F_{XXX}F_{XY}F_Z-F_{XXZ}F_{XX}F_Y+F_{XXX}F_{XZ}F_Y=0. \tag{15}$$

Then the exact solutions of Eq. (2) are

$$u_3=x+G_3(y,z),\ \ u_4=xG_4(y,z),$$

which can be obtained directly from the reduction Eq. (15) with two arbitrary functions $G_3(y,z)$ and $G_4(y,z)$.

**Case 4** $\boldsymbol{V}_4=t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}$.

The corresponding characteristic equation is formed as

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{t}=\frac{\mathrm{d}u}{-u}, \tag{16}$$

and the similarity reductions are

$$u=\frac{F(X,Y,Z)}{t},\ \ X=x,\ \ Y=y,\ \ Z=z. \tag{17}$$

Making use of the Eq. (17) in Eq. (2), we have the third reduced equation

$$F_{XX}^2-2F_{XX}^2F_{YZ}-F_{XXX}F_X+\frac{3}{2}F_{XXX}F_{YZ}F_X+F_{XXY}F_{XX}F_Z$$
$$-F_{XXX}F_{XY}F_Z-F_{XXZ}F_{XX}F_Y+F_{XXX}F_{XZ}F_Y=0. \tag{18}$$

Now a special solution of Eq. (18) is

$$F=X+G_5(Y,Z).$$

Subsequently, the exact solution for Eq. (2) becomes

$$u_5=\frac{x+G_5(y,z)}{t},$$

where $G_5(y,z)$ is an arbitrary function.

**Case 5** $\boldsymbol{V}_7(f_3)=f_3\frac{\partial}{\partial y}-zf_3'\frac{\partial}{\partial u}$.

The characteristic equation associated to the vector field $\boldsymbol{V}_7$ can be written as

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{f_3}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{-zf_3'}. \tag{19}$$

It is easy to find the similarity variables are $X=x$, $Z=z$, $T=t$ and the similarity solution

$$u=-\frac{yzf_3'}{f_3}+F(X,Z,T), \tag{20}$$

which can be solved by Eq. (19). Substituting Eq. (20) into Eq. (2), we can achieve the fourth reduction equation

$$f_3(F_{XXX}F_{XT}-F_{XXT}F_{XX})$$
$$+f_3'(2F_{XX}^2-\frac{3}{2}F_{XXX}F_X+F_{XXZ}F_{XX}Z-ZF_{XXX}F_{XZ})=0. \tag{21}$$

The special solution to this reduction Eq. (21) is

$$F(X,Z,T)=H_1(Z,T)+H_2(Z,T)X, \tag{22}$$

and the explicit solution of Eq. (2) is derived as

$$u=-\frac{yzf_3'}{f_3}+H_1(z,t)+H_2(z,t)x, \tag{23}$$

where $H_1(z,t)$ and $H_2(z,t)$ are two arbitrary functions.

**Case 6** $\boldsymbol{V}_8(f_4)=f_4\frac{\partial}{\partial z}+yf_4'\frac{\partial}{\partial u}$.

We solve the characteristic equation

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{f_4}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{yf_4'},$$

and obtain the similarity solution and similarity variables

$$u=\frac{yzf_4'}{f_4}+F(X,Y,T),\ \ X=x,\ \ Y=y,\ \ T=t. \tag{24}$$

Substituting Eq. (24) into Eq. (2), we can obtain the fifth reduction equation as

$$F_{XXX}(2f_4F_{XT}-2f_4'YF_{XY}+3f_4'F_X)-4F_{XX}^2f_4'$$
$$-2f_4F_{XXT}F_{XX}+2f_4'YF_{XXY}F_{XX}=0. \tag{25}$$

**Case 7** $\boldsymbol{V}_1+\boldsymbol{V}_2=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u}$.

With the characteristic equation

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{u}, \tag{26}$$

one can obtain the similarity solution and similarity variables

$$u=xF(X,Z,T),\ \ X=\frac{y}{x},\ \ Z=z,\ \ T=t. \tag{27}$$

Substituting Eq. (27) into Eq. (2), we can obtain the sixth reduced equation as

$$5X^2F_{XXX}F_{XZ}F_X-3XF_{XXX}F_{XZ}F-2XF_{XXX}F_XF_Z$$

$$+15XF_{XX}F_{XZ}F_X-9FF_{XX}F_{XZ}-4X^2F_{XZ}F_{XX}^2-6F_{XX}F_XF_Z$$
$$-2X^2F_{XX}F_{XXZ}F_X-2XF_{XX}^2F_Z+2X^2F_{XXX}F_{XT}-2XF_{XXX}F_T$$
$$-2X^2F_{XXT}F_{XX}+6XF_{XX}F_{XT}-6F_{XX}F_T=0. \tag{28}$$

The special solutions of the reduction equation Eq. (28) can be written as

$$F(X,Z,T)=X+G_6(Z,T), \tag{29}$$

where $G_6(Z,T)$ is an arbitrary function.

**Case 8** $\boldsymbol{V}_1+\boldsymbol{V}_9=x\frac{\partial}{\partial x}+f_5\frac{\partial}{\partial u}$.

Using the characteristic equation

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{f_5},$$

one can obtain similarity solution and new variables

$$u=f_5\ln x+F(Y,Z,T),\ \ Y=y,\ \ Z=z,\ \ T=t. \tag{30}$$

Substituting Eq. (30) into Eq. (2), one can obtain the seventh reduced equation

$$f_5F_{YZ}+f_5'=0. \tag{31}$$

Hence, the exact explicit solution of Eq. (2) is

$$u=f_5\ln x+G_7(y,t)+G_8(z,t)-\frac{f_5'}{f_5}yz, \tag{32}$$

where $G_7(y,t)$ and $G_8(z,t)$ are two arbitrary functions of the indicated variables.

**Case 9** $\boldsymbol{V}_2+\boldsymbol{V}_6=f_2\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u}$.

Using the characteristic equation

$$\frac{\mathrm{d}x}{f_2}=\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{u}, \tag{33}$$

we obtain similarity solution and new variables

$$u=yF(X,Z,T),\ \ X=x-f_2\ln y,\ \ Z=z,\ \ T=t. \tag{34}$$

Substituting Eq. (34) into Eq. (2), we can obtain the eighth reduction equation

$$4f_2F_{XX}^2F_{XZ}-2F_{XX}^2F_Z+2F_{XXX}F_{XT}-2F_{XXT}F_{XX}$$
$$-5f_2F_{XXX}F_{XZ}F_X+F_{XXX}F_XF_Z+2f_2F_{XXZ}F_{XX}F_X$$
$$-2FF_{XXZ}F_{XX}+2FF_{XXX}F_{XZ}=0. \tag{35}$$

The exact solutions of Eq. (35) are given with two arbitrary functions

$$F_1(X,Z,T)=X+G_9(Z,T),\ \ F_2=XG_{10}(Z,T),$$

and the corresponding explicit solutions of Eq. (2) are

$$u=y(x-f_2\ln y+G_9(z,t)),\ \ u=y(x-f_2\ln y)G_{10}(z,t). \tag{36}$$

**Case 10** $\boldsymbol{V}_4+\boldsymbol{V}_8=t\frac{\partial}{\partial t}+f_4\frac{\partial}{\partial z}+(yf_{4t}-u)\frac{\partial}{\partial u}$.

By solving the characteristic equation

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{f_4}=\frac{\mathrm{d}t}{t}=\frac{\mathrm{d}u}{yf_4'-u},$$

the following invariant functions are generated:

$$u=\frac{yf_4+F(X,Y,Z)}{t},\ \ X=x,\ \ Y=y,\ \ Z=z-\int\frac{f_4}{t}\mathrm{d}t. \tag{37}$$

Substituting Eq. (37) into Eq. (2), the ninth similarity equation is obtained as

$$2F_{XX}^2-4F_{XX}^2F_{YZ}-2F_{XXX}F_X+3F_{XXX}F_{YZ}F_X+2F_{XX}F_{XXY}F_Z$$
$$-2F_{XXX}F_{XY}F_Z-2F_YF_{XXZ}F_{XX}+2F_{XXX}F_{XZ}F_Y=0.$$

Consequently, some exact solutions of Eq. (2) are given as follows:

$$u=\frac{1}{t}\left(yf_4+x+G_{11}(y,z-\int\frac{f_4}{t}\mathrm{d}t)\right),$$
$$u=\frac{1}{t}\left(yf_4+xG_{12}(y,z-\int\frac{f_4}{t}\mathrm{d}t)\right), \tag{38}$$

where $G_{11}$ and $G_{12}$ are arbitrary functions of the indicated variables.

**Case 11** $\boldsymbol{V}_3+\boldsymbol{V}_6(f_2)=\frac{\partial}{\partial t}+f_2\frac{\partial}{\partial x}$.

From the characteristic equation

$$\frac{\mathrm{d}x}{f_2}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}u}{0},$$

we can obtain the similarity solution and new variables

$$u=F(X,Y,Z),\ \ X=x-\int f_2\mathrm{d}t,\ \ Y=y,\ \ Z=z. \tag{39}$$

Utilizing the invariants (39), we can obtain the tenth reduction equation

$$F_{XXX}(3F_XF_{YZ}-2F_{XY}F_Z+2F_{XZ}F_Y)$$
$$+2F_{XX}(F_{XXY}F_Z-2F_{YZ}F_{XX}-F_{XXZ}F_Y)=0. \tag{40}$$

A particular solution to Eq. (40) is

$$F(X,Y,Z)=X+G_{13}(Y,Z),$$

and the corresponding explicit solution of Eq. (2) is

$$u=x-\int f_2\mathrm{d}t+G_{13}(y,z),$$

where $G_{13}(y,z)$ is an arbitrary function. Due to the arbitrariness of the function $G_{13}(y,z)$, one can gain different similarity solutions of the original Eq. (2), such as the solitary solution, rational solution and dromions in $yz$-plane.

**Case 12** $\boldsymbol{V}_6(f_2)+\boldsymbol{V}_7(f_3)+\boldsymbol{V}_9(f_5)=f_2\frac{\partial}{\partial x}+f_3\frac{\partial}{\partial y}+(f_5-zf_3')\frac{\partial}{\partial u}$.

Solving the characteristic equation

$$\frac{\mathrm{d}x}{f_2}=\frac{\mathrm{d}y}{f_3}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{f_5-zf_3'},$$

one can find the similarity solution and new variables

$$u=\frac{xf_5-zxf_3'}{f_2}+F(Z,T,Y),$$
$$Y=yf_2-xf_3,\ \ Z=z,\ \ T=t. \tag{41}$$

Substituting Eq. (41) into Eq. (2), we can obtain the eleventh reduced equation

$$\begin{aligned}&\left(5f_3f_2^3F_{YZ}F_Y+3f_2^2Zf_3'F_{YZ}+2f_3f_2^2F_{YT}+4f_2^2f_3'F_Y\right.\\&-3f_2^2f_5F_{YZ}+2Zf_2f_3'')F_{YYY}-2\left(Zf_3'f_2'+f_2f_5'-f_5f_2'\right)F_{YYY}\\&-2f_3f_2^2(f_2F_{YYZ}F_Y+F_{YYT})F_{YY}-4f_2^2(f_3f_2F_{YZ}+f_3')F_{YY}^2=0.\end{aligned} \tag{42}$$

It is difficult to obtain the exact solutions of Eq. (42) because the existence of three arbitrary functions. If fixing three arbitrary functions, one can construct abundant exact solutions of Eq. (42).

## 4. Invariant solutions of Eq. (2)

In this section, we will give several concrete invariant solutions of Eq. (2) by selecting the proper arbitrary functions in the solutions obtained in the last section. The first invariant solutions can be obtained by fixing the arbitrary function

$$F_1(y,z,t)=-\frac{16(2t^2+3tz-3ty+z^2-4zy-5y^2-12)}{[(2t+z-5y)^2+(t+z+y)^2+24]^2},$$
$$F_2(y,z,t)=\frac{8y-t-9z}{t^2+(2z-3y)t+9z^2-4zy+y^2+1},$$

in Case 1, then we will have the exact solutions of Eq. (2)

$$u_1=-\frac{16(2t^2+3tz-3ty+z^2-4zy-5y^2-12)}{[(2t+z-5y)^2+(t+z+y)^2+24]^2}, \tag{43}$$
$$u_2=\frac{8y-t-9z}{t^2+(2z-3y)t+9z^2-4zy+y^2+1}, \tag{44}$$

and the corresponding plots are given in Figs. 1(a) and 1(b), respectively.

The second invariant solution of Eq. (2) in Case 2 can be got by selecting the arbitrary functions

$$G_{21}(z,t)=\mathrm{e}^{\cos t}+\arctan\mathrm{e}^{2z-t},\ \ G_{22}=\mathrm{sech}^2(z+2t),$$

and the invariant solutions are obtained as follows:

$$u_3=xy(\mathrm{e}^{\cos t}+\arctan\mathrm{e}^{2z-t}),\ \ u_4=xy\,\mathrm{sech}^2(z+2t). \tag{45}$$

If fixing the arbitrary function in Case 7

$$G_6(z,t)=\tanh(z+t)+\frac{1}{\cosh^2(t-z)}, \tag{46}$$

then the third similarity solution of Eq. (2) is given as

$$u_5=y+x\left[\tanh(z+t)+\frac{1}{\cosh^2(t-z)}\right], \tag{47}$$

and the corresponding plot is shown in Fig. 3.

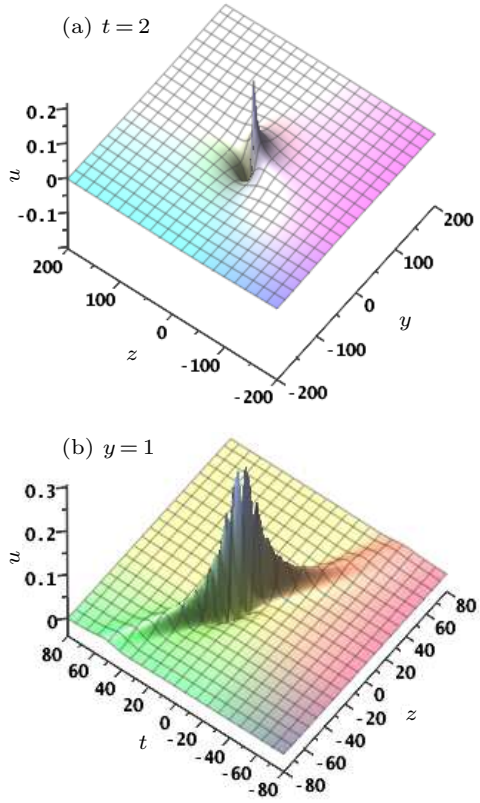



**Fig. 1.** The first invariant solutions (43) and (44) of Eq. (2).

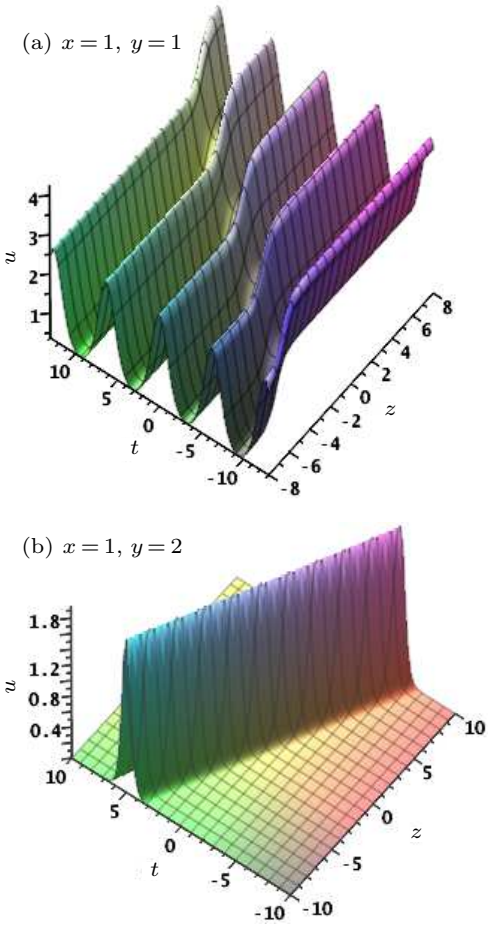



**Fig. 2.** The second invariant solution (45) of Eq. (2), which can be regarded as the periodic solution and solitary solution respectively.

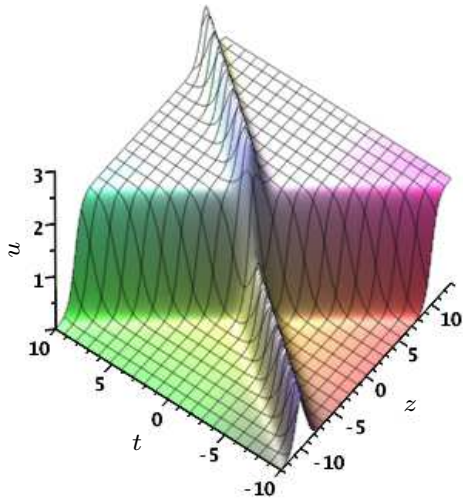



**Fig. 3.** The third invariant solution (47) of Eq. (2) with $x = 1$, $y = 1$, which can be regarded as the interaction between a kink and a soliton.

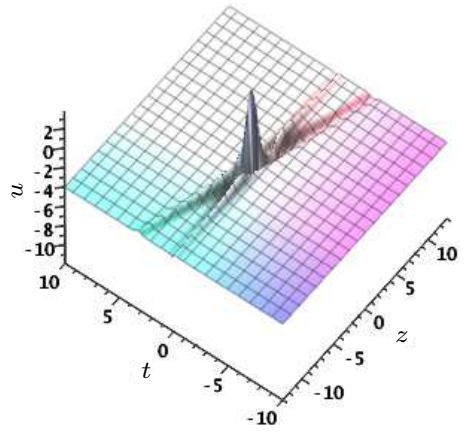



**Fig. 4.** The fourth invariant solution (48) of Eq. (2) with $x = 0$, $y = 1$, which can be regarded as a lump-type solution.

Fixing the arbitrary function in Case 10 as

$$f_4 = -4t,\ \ G_{11}(y, z+4t) = \frac{4[y^2-(z+4t)^2+3]}{[y^2+(z+4t)^2+3]^2},$$

so we obtain the fourth invariant solution

$$u_6 = \frac{1}{t}\left(x - 4yt + \frac{4[y^2-(z+4t)^2+3]}{[y^2+(z+4t)^2+3]^2}\right). \tag{48}$$

The structure of the solution (48) is shown explicitly in Fig. 4.

## 5. Summary

A $(3+1)$-dimensional nonlinear system, which is integrable in the sense of possessing the Virasoro symmetry algebra, is studied by means of symmetry reduction method. The infinitesimal generators with five arbitrary functions and four arbitrary constants are solved directly from the usual Lie symmetry procedure. The corresponding similarity reduction equations have been proposed with new similarity variables and different types of similarity solutions are obtained analytically by selecting proper arbitrary functions. These invariant solutions include the interaction solution between a kink and a soliton, the lump-type solution, periodic solutions, etc. Because of the existence of the arbitrary functions and constants in the solution expression of the similarity reduction equations, we get abundant interesting exact solutions as possible. Other integrable properties about this Virasoro integrable system, such as Lax pair and bilinear form, will be studied in future.

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