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Citation: Chin. Phys. B, 2023, 32 (**7**): 074501. DOI: 10.1088/1674-1056/ac9cbe
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Fractional Noether theorem and fractional Lagrange equation of multi-scale mechano-electrophysiological coupling model of neuron membrane

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(Received 14 August 2022; revised manuscript received 29 September 2022; accepted manuscript online 21 October 2022)

Noether theorem is applied to a variable order fractional multiscale mechano-electrophysiological model of neuron membrane dynamics. The variable orders fractional Lagrange equation of a multiscale mechano-electrophysiological model of neuron membrane dynamics is given. The variable orders fractional Noether symmetry criterion and Noether conserved quantities are given. The forms of variable orders fractional Noether conserved quantities corresponding to Noether symmetry generators solutions of the model under different conditions are discussed in detail, and it is found that the expressions of variable orders fractional Noether conserved quantities are closely dependent on the external nonconservative forces and material parameters of the neuron.

Keywords: Hamilton's principle, Noether theorem, fractional derivative, multiscale electromechanical coupling, neuron membrane

PACS: 45.20.Jj, 11.30.–j, 45.10.Hj, 46.70.Hg **DOI:** 10.1088/1674-1056/ac9cbe

1. Introduction

The action potential (AP) has been considered to be only an electric signal, and the well known Hodgkin–Huxley (H–H) model[1] is the foundation of this viewpoint. However, a recent experiment[2] found that the AP propagation accompanies mechanical deformation like swelling or contraction. So the standpoint of AP being a mechano-electrophysiological coupling process was proposed.[3] Engelbrecht *et al.*[4] gave a continuum mechanics model to explain this mechano-electrical coupling phenomenon, Chen *et al.*[5] gave a finite element computation model and Drapaca[6] proposed an electromechanical model using Hamilton's principle, and gave its Euler–Lagrange equation. Later, she extended this model to a fractional derivative scenario.[7] However, due to the complexity and nonlinearity, it is hard to solve the mathematical equations of these models. Symmetry is an important tool in finding the first integral, differential equation reduction and classification,[8] so we can apply symmetry in electromechanical coupling equations to find their conserved quantities, further to reduce orders of the differential equations. However, to our knowledge, the Noether symmetry and conserved quantities of neuron dynamics have not been investigated.

Symmetry is a higher rule in physics. Since Noether revealed the relations between symmetry and conserved quantities in her eminent paper,[9] Noether theorem has had various extensions and has been applied in physics and mechanics.[10–22] Fractional calculus is increasingly important as it can more exactly describe complex phenomena in science and engineering. Fractional Noether theorem has been extended to the fractional Lagrange system, Hamilton system, generalized Hamilton system, Birkhoff system, and so on.[23–33] However, there is no research on fractional Noether symmetry in mechano-electrophysiological coupling equations of neuron dynamics. Considering the visco-elasticity of neuron membranes, we will adopt a fractional derivative of variable orders. In this paper we will generalize fractional Noether theorem to neuron dynamics.

The structure of this paper is as follows. In Section 2 we will introduce a variable order fractional mechano-electrophysiological model of a neuron membrane with Riemann–Liouville fractional derivative, deduce its variable order fractional Lagrange equation, and show that the variable order fractional H–H equation can be deduced from the variable order fractional Lagrange equation. In Section 3 variable order fractional Noether symmetry and conserved quantities of the neuron dynamics are studied, and the criterion of variable order fractional Noether symmetry and the form of variable order fractional conserved quantities are given. In Section 4 we will specifically discuss the deduced variable fractional conserved quantities in various conditions. The final section is the conclusion.

2. The variable order fractional Lagrange equation of neuron membrane dynamics

Conduction and processing of information are the characteristics of neurons. Neurons make use of electric signals to conduct information. As we know, action potential plays a key role in producing electric signals in neurons. However,

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experimental observations prove that action potential production is not only because of ion transmission across neuron membranes, but also accompanied with deformation of the neuron membrane, a multiscale mechanics and electrophysiology coupling process. Recently, Drapaca[7] proposed a new fractional multiscale mechano-electrophysiological coupling model using Hamilton's principle, however its numerical solutions are in fact uncoupled. In this paper we will revisit the model, but study the more general case using Noether symmetry analysis.

2.1. The model

The membrane of a neuron consists of a phospholipid bilayer embedded channel protein. The propagation of electric signals in the neuron system is achieved by producing action potential accompanied with an ion channel open or shut. The action potential can induce deformation of the neuron membrane, and inversely the deformation of the neuron membrane can also induce action potential. We can use the Hodgkin–Huxley model to describe the electric process and a linear visco-elastic Kelvin–Voigt model to describe the mechanical process, and Maxwell models to control the activations of the Na^+ and K^+ channels, and the inactivation of the Na^+ channel, respectively. The coupling process is connected through membrane capacitance and displacement (see Fig. 1). As we know, the neuron can be considered as an axis-symmetric cylinder with circular cross section, so we can study half the neuron by symmetry.

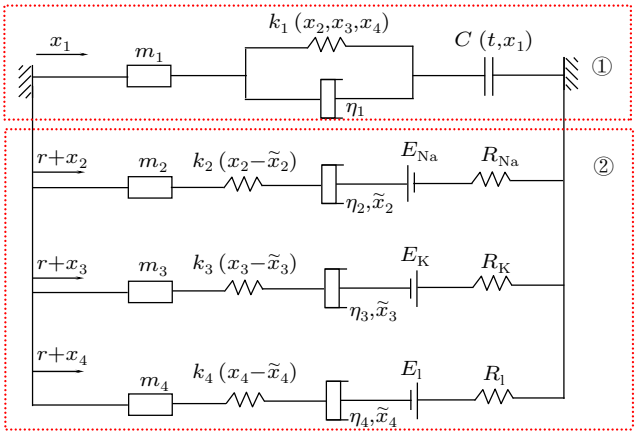


Fig. 1. Schematic of the mechano-electrophysiological coupling model of an axon membrane. The axon is considered as an axis-symmetric homogeneous circular cylinder with intracellular space filled with axoplasm, and the outer layer is the membrane. Using the symmetry and homogeneity of the column we only need to study half of the axon. At cellular scale we model the intracellular space as a viscoelastic material using the Kelvin model connected with axon capacitor (the dotted box ①), where (x_2,x_3,x_3) denote the deformation motion of the cytoskeleton with different ionic channels, which correspond to the m,n,h gate. x_1 is the displacement of the membrane. Mechanical motion or electrical stimuli can trigger the circuit. At subcellular scale ionic exchange obeys fractional Hodgkin–Huxley equations (the dotted box ②) and is controlled by Maxwell models.

We can express the macro-mechanical kinetic energy $T = \frac{1}{2}m_1({}_0D_t^{\alpha(t)}x_1)^2$, where m_1 denotes the half constant mass of the neuron, A is the cross-sectional area, and x_1 is the macroscopic (cell level) displacement that depends on time. In order to use the nonconservative Hamilton principle proposed in Ref. [34], we use the left and right Riemann–Liouville fractional derivative of variable orders with $0 \le \alpha(t) < 1$

$$ {}_0D_t^{\alpha(t)}f(t) = \frac{f(0)}{\Gamma(1-\alpha(t))t^{\alpha(t)}} + \int_0^t \frac{\mathrm{d}f(\tau)/\mathrm{d}\tau}{\Gamma(1-\alpha(t-\tau))(t-\tau)^{\alpha(t-\tau)}}\mathrm{d}\tau, \quad (t>0), \tag{1} $$

$$ {}_tD_a^{\alpha(t)}f(t) = \frac{f(a)}{\Gamma(1-\alpha(a-t))(a-t)^{\alpha(a-t)}} - \int_t^a \frac{\mathrm{d}f(\tau)/\mathrm{d}\tau}{(\Gamma(1-\alpha(\tau-t)))(\tau-t)^{\alpha(\tau-t)}}\mathrm{d}\tau, \quad (t>0). \tag{2} $$

However, definitions (1) and (2) will introduce additional memory effects for piecewise uniform accelerated motions.[35] Fortunately, when $0 \le \alpha(t) < 1$ the approximations

$$ {}_0D_t^{\alpha(t)}f(t) \approx {}_0\tilde{D}_t^{\alpha(t)}f(t), \quad {}_tD_a^{\alpha(t)}f(t) \approx {}_t\tilde{D}_a^{\alpha(t)}f(t) \tag{3} $$

hold[7] by power series expansion, and are identically satisfied for $\alpha(t) = \text{const} \in [0,1)$, whose approximate relation could satisfy the nonlocality criteria in Ref. [35], where ${}_0\tilde{D}_t^{\alpha(t)}$, ${}_t\tilde{D}_a^{\alpha(t)}$ are the Marchaud fractional derivatives of variable order with definition[36]

$$ {}_0\tilde{D}_t^{\alpha(t)}f(t) = \frac{f(0)}{\Gamma(1-\alpha(t))t^{\alpha(t)}} + \frac{1}{\Gamma(1-\alpha(t))}\int_0^t \frac{\mathrm{d}f(\tau)/\mathrm{d}\tau}{(t-\tau)^{\alpha(t)}}\mathrm{d}\tau, \quad (t>0), \tag{4} $$

$$ {}_t\tilde{D}_a^{\alpha(t)}f(t) = \frac{f(a)}{\Gamma(1-\alpha(t))(a-t)^{\alpha(t)}} - \frac{1}{\Gamma(1-\alpha(t))}\int_t^a \frac{\mathrm{d}f(\tau)/\mathrm{d}\tau}{(\tau-t)^{\alpha(t)}}\mathrm{d}\tau, \quad (t>0). \tag{5} $$

In this paper we will use a Riemann–Liouville fractional derivative of variable orders with $0 \le \alpha(t) < 1$, $|\mathrm{d}\alpha/\mathrm{d}t| \ll 1$, $t \in (0,a)$.

The macro mechanical potential energy $V = -\frac{1}{2}k_1(x_2,x_3,x_4)x_1^2$, the micro mechanical potential energy $V^* = -\frac{1}{2}k_2(x_2-\tilde{x}_2)^2 - \frac{1}{2}k_3(x_3-\tilde{x}_3)^2 - \frac{1}{2}k_4(x_4-\tilde{x}_4)^2$, and the micro kinetic energy of the sub-cytoskeleton (subcellular scale) $T^* = \frac{1}{2}m_2({}_0D_t^{\beta(t)}x_2)^2 + \frac{1}{2}m_3({}_0D_t^{\beta(t)}x_3)^2 + \frac{1}{2}m_4({}_0D_t^{\beta(t)}x_4)^2$, where m_2,m_3,m_3 are subcytoskeleton masses in the Maxwell model, and x_2,x_3,x_4 are the displacements of masses (subcellular level). The electric energy $W_{\rm e} = \frac{1}{2}\frac{e_C^2}{C}$, where e_C is the sum of the charges in the capacitor, and C is the capacitance of the capacitor in the H–H electric circuit.

Introduce the mechanical dissipation function (work of viscous force) $\psi_{\rm m} = \eta_1 x_1\,{}_0D_t^{\alpha(t)}x_1 + \eta_2\tilde{x}_2\,{}_0D_t^{\beta(t)}\tilde{x}_2 + \eta_3\tilde{x}_3\,{}_0D_t^{\beta(t)}\tilde{x}_3 + \eta_4\tilde{x}_4\,{}_0D_t^{\beta(t)}\tilde{x}_4$, where $\eta_1,\eta_2,\eta_3,\eta_4$ are damping coefficients in the Kelvin and Maxwell elements. The electric dissipation function $\psi_{\rm e} = (1/2)R_{\rm Na}({}_0D_t^{\gamma(t)}e_{\rm Na})^2 +$

$(1/2)R_{\rm K}({}_0D_t^{\gamma(t)}e_{\rm K})^2+(1/2)R_{\rm l}({}_0D_t^{\gamma(t)}e_{\rm l})^2+E_{\rm Na}({}_0D_t^{\gamma(t)}e_{\rm Na})+E_{\rm K}({}_0D_t^{\gamma(t)}e_{\rm K})+E_{\rm l}({}_0D_t^{\gamma(t)}e_{\rm l})$, where $e_{\rm Na}$, $e_{\rm K}$, $e_{\rm l}$ denote the electric charges of the Na^+, K^+ and leakage channels, respectively, $R_{\rm Na}$, $R_{\rm K}$, $R_{\rm l}$ are the resistances, and $E_{\rm Na}$, $E_{\rm K}$, $E_{\rm l}$ are the reverse electric potentials in the H–H electric circuit, where fractional derivatives of electric charges denote the transmembrane current induced by membrane deformation or action potential.

2.2. The fractional Lagrange equation of variable orders

Based on the electric charge conservation law, we have a holonomic constraint to charge: $e_C-e_{\rm Na}-e_{\rm K}-e_{\rm l}=0$, so the freedom of the coupling system is 10. Introduce generalized coordinates to express universally the spatial variables and electrical variables, q_s $(s=1,\ldots,10)$, where $q_1=x_1$, $q_2=x_2$, $q_3=x_3$, $q_4=x_4$, $q_5=e_{\rm Na}$, $q_6=e_{\rm K}$, $q_7=e_{\rm l}$, $q_8=\tilde{x}_2$, $q_9=\tilde{x}_3$, $q_{10}=\tilde{x}_4$. The Lagrangian of the neuronal mechano-electrophysiological model is

$$\begin{aligned}L(t,q_s,{}_0D_t^{\alpha_i(t)}q_s) &= T(t,{}_0D_t^{\alpha_i(t)}q_s)+T^*(t,{}_0D_t^{\alpha_i(t)}q_s)\\ &\quad -V(q_s)-V^*(q_s)-W_{\rm e}(t,q_s).\end{aligned}\tag{6}$$

The virtual work of nonconservative generalized forces is

$$\delta W(t,q_s,{}_0D_t^{\alpha_i(t)}q_s)=-\left(Q_s-\frac{\partial(\psi_{\rm m}+\psi_{\rm e})}{\partial{}_0D_t^{\alpha_i(t)}q_s}\right)\delta q_s.\tag{7}$$

The Hamilton principle of the nonconservative mechano-electrophysiological system of the axon membrane is

$$\int_0^a(\delta L+\delta W)\,{\rm d}t=0.\tag{8}$$

By expanding the above equation, and using the communication relation ${}_0D_t^{\alpha_i(t)}\delta=\delta{}_0D_t^{\alpha_i(t)}$ which holds for the holonomic constrained system, and the relation of the fractional integral by parts[23–25,34]

$$\begin{aligned}&\int_0^a\left(\frac{\partial L}{\partial{}_0D_t^{\alpha_i(t)}q_s}{}_0D_t^{\alpha_i(t)}\delta q_s\right){\rm d}t\\ &=\int_0^a\left({}_tD_a^{\alpha_i(t)}\frac{\partial L}{\partial{}_0D_t^{\alpha_i(t)}q_s}\delta q_s\right){\rm d}t\end{aligned}\tag{9}$$

and the end points relations $\delta q(0)=0$, $\delta q(a)=0$, we can get the fractional multi-scale mechano-electrophysiological coupling Lagrange equation of neuronal membrane dynamics

$$\begin{aligned}&{}_tD_a^{\alpha_i(t)}\frac{\partial L}{\partial{}_0D_t^{\alpha_i(t)}q_s}+\frac{\partial L}{\partial q_s}+\frac{\partial\psi}{\partial{}_0D_t^{\alpha_i(t)}q_s}=Q_s,\\ &(s=1,\ldots,10,\ \ \alpha^{i=s=1}(t)=\alpha(t),\\ &\alpha^{i=s=2-4,8-10}(t)=\beta(t),\ \ \alpha^{i=s=5-7}(t)=\gamma(t)),\end{aligned}\tag{10}$$

where $\psi=\psi_{\rm m}+\psi_{\rm e}$. The coupling equation of motion describes the changes of ions between the outer layer and intracellular space and deformation of the neuron membrane.

When $\psi=Q_s=0$, we can get the fractional Lagrange equation of a conservative system

$${}_tD_a^{\alpha_i(t)}\frac{\partial L}{\partial{}_0D_t^{\alpha_i(t)}q_s}+\frac{\partial L}{\partial q_s}=0.\tag{11}$$

Putting the expression of Lagrangian L into Eq. (10), we can get the Euler–Lagrange differential equations

$$\begin{aligned}&m_1{}_tD_a^{\alpha_1(t)}{}_0D_t^{\alpha_1(t)}q_1-k(q_8,q_9,q_{10})q_1\\ &-\eta_1{}_0D_t^{\alpha_1(t)}q_1+\frac{1}{2}\frac{\partial C}{\partial u}\left(\frac{q_C}{C}\right)^2,\end{aligned}\tag{12}$$

$$\begin{aligned}&-\frac{1}{2}\left(\frac{\partial k_2}{\partial q_1}(q_2-q_8)^2+\frac{\partial k_3}{\partial q_1}(q_3-q_9)^2\right.\\ &\left.+\frac{\partial k_4}{\partial q_1}(q_4-q_{10})^2\right)=Q_1,\end{aligned}\tag{13}$$

$$m_2{}_tD_a^{\alpha_2(t)}{}_0D_t^{\alpha_2(t)}q_2-k_2(q_2-q_8)=Q_2,\tag{14}$$

$$m_3{}_tD_a^{\alpha_3(t)}{}_0D_t^{\alpha_3(t)}q_3-k_3(q_3-q_9)=Q_3,\tag{15}$$

$$m_4{}_tD_a^{\alpha_4(t)}{}_0D_t^{\alpha_4(t)}q_4-k_4(q_4-q_{10})=Q_4,\tag{16}$$

$$R_{\rm Na}{}_0D_t^{\alpha_5(t)}q_5+E_{\rm Na}=V,\tag{17}$$

$$R_{\rm K}{}_0D_t^{\alpha_6(t)}q_6+E_{\rm K}=V,\tag{18}$$

$$R_{\rm l}{}_0D_t^{\alpha_7(t)}q_7+E_{\rm l}=V,\tag{19}$$

$$-\eta_2{}_0D_t^{\alpha_8(t)}q_8+k_2(q_2-q_8)-\frac{1}{2}\frac{\partial k_1}{\partial q_8}q_1^2=Q_8,\tag{20}$$

$$-\eta_3{}_0D_t^{\alpha_9(t)}q_9+k_3(q_3-q_9)-\frac{1}{2}\frac{\partial k_1}{\partial q_9}q_1^2=Q_9,\tag{21}$$

$$-\eta_4{}_0D_t^{\alpha_{10}(t)}q_{10}+k_4(q_4-q_{10})-\frac{1}{2}\frac{\partial k_1}{\partial q_{10}}q_1^2=Q_{10},\tag{22}$$

where $V=U_i=q_C/C$ is the potential of the capacitor.

Here the Euler–Lagrange equations are different from the equations in Ref. [7], because we suppose the elastic parameters of the cytoskeleton depend on the macro-deformation of the membrane. Kirchhoff's current law demands ${}_0D_t^{\gamma(t)}(q_C+q_{\rm Na}+q_{\rm K}+q_{\rm l})=I$,[7] where I is a known external electric current applied on the membrane.

Putting Eqs. (17)–(19) into Kirchhoff's current law, the fractional Hodgkin–Huxley equation of the membrane potential can be found

$$\begin{aligned}{}_0D_t^{\gamma(t)}(CV) &= I-\frac{1}{R_{\rm Na}}(V-E_{\rm Na})\\ &\quad -\frac{1}{R_{\rm K}}(V-E_{\rm K})-\frac{1}{R_{\rm l}}(V-E_{\rm l}).\end{aligned}\tag{23}$$

Because of lack of information on the mechanotransduction process of axons, the expressions of m_2, m_3, m_4, k_2, k_3, k_4, η_2, η_3, η_4, Q_s are difficult to know, so in Ref. [7], the author made a simplification to neglect them by a scales comparison. However, these factors should affect the characteristics of axon action. So, in the following study we will treat the

general cases by Noether symmetry analysis to study these neglected parameters and how they affect conserved quantities and the expression of these parameters, which may be useful for numerical solutions of the mechano-electrophysiological coupling equations of axons.

3. Variable order fractional Noether theorem of neuronal membrane dynamics

We introduce a one-parameter infinitesimal Lie transformation group in space $(t, q_s, \dot{q}_s)$

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^* &= q_s + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad s = 1, \ldots, 10, \end{aligned} \tag{24}$$

where ε is an infinitesimal parameter, and $\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are infinitesimal transformation generators. The infinitesimal generator vector

$$X^{(0)} = \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\frac{\partial}{\partial t} + \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\frac{\partial}{\partial q_s} \tag{25}$$

is the operator for the infinitesimal generator of the one-parameter Lie group of transformations (24) in space $(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$. The first prolongation of the infinitesimal generator vector[34] is

$$\begin{aligned} X^{(1)} = X^{(0)} + \Bigg(& {}_0D_t^{\alpha_i(t)}(\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - \dot{q}_s\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})) \\ & + \frac{\mathrm{d}}{\mathrm{d}t}\,{}_0D_t^{\alpha_i(t)}q_s\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\Bigg)\frac{\partial}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}, \end{aligned} \tag{26}$$

which defines a first extended one-parameter Lie group of transformation in space $(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ by partial derivatives, $(\dot{})$ means first derivative to t.

The Hamilton action is

$$S(\gamma) = \int_0^a L(t, q_s, {}_0D_t^{\alpha_i(t)}q_s)\,\mathrm{d}t. \tag{27}$$

Under the infinitesimal transformation, the curve γ is transformed to curve γ^*. The corresponding Hamilton action is transformed to

$$S(\gamma^*) = \int_{0^*}^{a^*} L(t^*, q_s^*, {}_{0^*}D_{t^*}^{\alpha_i(t)}q_s^*)\,\mathrm{d}t^*. \tag{28}$$

The variation ΔS of Hamilton action S is the main linear part of the difference $S(\gamma^*) - S(\gamma)$ to infinitesimal parameter ε, and we have

$$\Delta S = \int_0^a [\Delta L + L(\Delta t)^{\cdot}]\,\mathrm{d}t, \tag{29}$$

where Δ denotes anisochronous variation, and δ denotes isochronous variation. Expanding the above equation, we have

$$\begin{aligned} \Delta S = \int_0^a \Bigg\{ & L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t + \frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial q_s}\Delta q_s \\ & + \frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}\Delta\,{}_0D_t^{\alpha_i(t)}q_s\Bigg\}\mathrm{d}t \end{aligned} \tag{30}$$

and using the relation

$$\Delta\,{}_0D_t^{\alpha_i(t)}f = {}_0D_t^{\alpha_i(t)}\delta f + \frac{\mathrm{d}}{\mathrm{d}t}\,{}_0D_t^{\alpha_i(t)}f\Delta, \tag{31}$$

$$\delta\,{}_0D_t^{\alpha_i(t)}f = {}_0D_t^{\alpha_i(t)}\delta f, \tag{32}$$

$$\delta f = \Delta f - \dot{f}\Delta t, \tag{33}$$

where f is an arbitrary function of t, we can get

$$\begin{aligned} \Delta S = \int_0^a \Bigg\{ & L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t + \frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial q_s}\Delta q_s + \frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s} \\ & \times({}_0D_t^{\alpha_i(t)}(\Delta q_s - \dot{q}_s\Delta t) + \frac{\mathrm{d}}{\mathrm{d}t}\,{}_0D_t^{\alpha_i(t)}q_s\Delta t)\Bigg\}\mathrm{d}t. \end{aligned} \tag{34}$$

Put the infinitesimal transformation Eq. (24) into Eq. (30), and the following expression can be obtained:

$$\begin{aligned} \Delta S = \int_0^a \Bigg\{ & L\frac{\mathrm{d}}{\mathrm{d}t}\xi_0 + \frac{\partial L}{\partial t}\xi_0 + \frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s} \\ & \times({}_0D_t^{\alpha_i(t)}(\xi_s - \dot{q}_s\xi_0) + \frac{\mathrm{d}}{\mathrm{d}t}\,{}_0D_t^{\alpha_i(t)}q_s\xi_0)\Bigg\}\mathrm{d}t. \end{aligned} \tag{35}$$

Definition 1 If the variation of the Hamilton action satisfies

$$\Delta S = 0, \tag{36}$$

the infinitesimal transformation (24) is a Noether symmetrical transformation.

Based on definition 1, we can get the Noether symmetry criterion.

Criterion 1 If the infinitesimal generators $\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfy

$$L(t, q_s, {}_0D_t^{\alpha_i(t)}q_s)\dot{\xi}_0 + X^1\left(L(t, q_s, {}_0D_t^{\alpha_i(t)}q_s)\right) = 0, \tag{37}$$

the transformation invariance is named Noether symmetry, which is also called variational symmetry. For Noether symmetry we can deduce the invariant.

Theorem 1 For the Lagrange system (11), if the generators $\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of the infinitesimal transformations have Noether symmetry (37), there exist fractional conserved quantities as

$$\begin{aligned} I_{\mathrm{N}} = {} & L(t, q_s, {}_0D_t^{\alpha_i(t)}q_s)\xi_0 \\ & + \int_0^a \Bigg\{ \frac{\partial L(t, q_s, {}_0D_t^{\alpha_i(t)}q_s)}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}\,{}_0D_t^{\alpha_i(t)}(\xi_s - \dot{q}_s\xi_0) \\ & - {}_tD_a^{\alpha_i(t)}\frac{\partial L(t, q_s, {}_0D_t^{\alpha_i(t)}q_s)}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}(\xi_s - \dot{q}_s\xi_0)\Bigg\}\mathrm{d}t \\ = {} & \text{const.}, \quad (a \in [0, 1]), \end{aligned} \tag{38}$$

which are called variable order fractional Noether conserved quantities. We can verify this theorem by defining $\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{N}} = 0$, which denotes that I_{N} is a conserved quantity.[26]

Proof

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{N}} =& L\frac{\mathrm{d}}{\mathrm{d}t}\xi_0+\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_s}\dot{q}_s\xi_0+\frac{\partial L}{\partial {}_0D_t^{\alpha_i(t)}q_s}\frac{\mathrm{d}}{\mathrm{d}t}{}_0D_t^{\alpha_i(t)}q_s\xi_0\\&+\frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}{}_0D_t^{\alpha_i(t)}(\xi_s-\dot{q}_s\xi_0)\\&-{}_tD_a^{\alpha_i(t)}\frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}(\xi_s-\dot{q}_s\xi_0)\\=&-\left(\frac{\partial L}{\partial q_s}+{}_tD_a^{\alpha_i(t)}\frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}\right)(\xi_s-\dot{q}_s\xi_0)=0.\end{aligned}\tag{39}$$

In the proving process of this theorem we have used condition (37) and variable order fractional Lagrange equation (11). In fact we can generalize the Noether symmetry to non-conservative dynamical systems.

Definition 2 If the Hamilton action is generalized quasi-invariant under the infinitesimal transformation group, that is, the variation satisfies

$$\Delta S=-\int_0^a\left\{\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G_{\mathrm{N}})+\left(Q_s-\frac{\partial\psi}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}\right)\delta q_s\right\}\mathrm{d}t,\tag{40}$$

the infinitesimal transformation (24) is a generalized quasi-symmetrical transformation, where $G_{\mathrm{N}}(t,\boldsymbol{q},{}_0D_t^{\alpha_i(t)}\boldsymbol{q})$ is a gauge function, and $(Q_s-\frac{\partial\psi}{\partial\,{}_0D_t^{\alpha_i(t)}q_s})\delta q_s$ is the sum of the virtual work of the generalized non-conservative force.

Based on definition 2, we can get the generalized Noether symmetry criterion.

Criterion 2 If there exists a gauge function $G_{\mathrm{N}}(t,\boldsymbol{q},{}_0D_t^{\alpha_i(t)}\boldsymbol{q})$ that makes the infinitesimal generators $\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, $\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfy

$$\begin{aligned}&L(t,q_s,{}_0D_t^{\alpha_i(t)}q_s)\dot{\xi}_0+X^1\left(L(t,q_s,{}_0D_t^{\alpha_i(t)}q_s)\right)\\&+\left(Q_s-\frac{\partial\psi}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}\right)(\xi_s-\dot{q}_s\xi_0)\\&+\dot{G}_{\mathrm{N}}(t,q_s,{}_0D_t^{\alpha_i(t)}q_s)=0,\end{aligned}\tag{41}$$

the infinitesimal transformation is named variable order fractional quasi-Noether symmetry. The Noether symmetry can always lead to conserved quantities.

Theorem 2 For the Lagrange equation Eq. (10) of neuronal membrane dynamics, if the infinitesimal generators $\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, $\xi_s(s,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfy equation (41) (criterion 2), the system has the following first integrals:

$$\begin{aligned}I_{\mathrm{N}} =& L(t,q_s,{}_0D_t^{\alpha_i(t)})\xi_0+\int_0^a\left\{\frac{\partial L(t,q_s,{}_0D_t^{\alpha_i(t)})}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}{}_0D_t^{\alpha_i(t)}\right.\\&\left.\times(\xi_s-\dot{q}_s\xi_0)-{}_tD_a^{\alpha_i(t)}\frac{\partial L(t,q_s,{}_0D_t^{\alpha_i(t)})}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}(\xi_s-\dot{q}_s\xi_0)\right\}\mathrm{d}t\\&+G_{\mathrm{N}}(t,q_s,{}_0D_t^{\alpha_i(t)}q_s)=\text{const.},\quad(a\in[0,1]),\end{aligned}\tag{42}$$

which are variable order fractional Noether conserved quantities.

Proof

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{N}} =& L\frac{\mathrm{d}}{\mathrm{d}t}\xi_0+\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_s}\dot{q}_s\xi_0+\frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}\frac{\mathrm{d}}{\mathrm{d}t}{}_0D_t^{\alpha_i(t)}q_s\xi_0\\&+\frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}{}_0D_t^{\alpha_i(t)}(\xi_s-\dot{q}_s\xi_0)\\&-{}_tD_a^{\alpha_i(t)}\frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}(\xi_s-\dot{q}_s\xi_0)+\frac{\mathrm{d}}{\mathrm{d}t}G\\=&-\left(\frac{\partial L}{\partial q_s}+{}_tD_a^{\alpha_i(t)}\frac{\partial L}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}-Q_s+\frac{\partial\psi}{\partial\,{}_0D_t^{\alpha_i(t)}q_s}\right)\\&\times(\xi_s-\dot{q}_s\xi_0)=0.\end{aligned}\tag{43}$$

In the proving process of this proposition we have used condition (41) and variable order fractional Lagrange equation (10).

4. Solutions of Noether symmetry generators and conserved quantities

Putting the exact form of Lagrangian L (6) and dissipative function ψ into the Noether identity Eq. (41), we have

$$\begin{aligned}&\xi_1\left(k_1q_1+\frac{1}{2}V^2\frac{\partial C(q_1)}{\partial q_1}\right)+m_1\,{}_0D_t^{\alpha(t)}q_1\left({}_0D_t^{\alpha(t)}(\xi_1-\dot{q}_1\dot{\xi}_0)+\frac{\mathrm{d}}{\mathrm{d}t}{}_0D_t^{\alpha(t)}q_1\xi_0\right)+\left(Q_1-\eta_1\,{}_0D_t^{\alpha(t)}q_1\right)(\xi_1-\dot{q}_1\xi_0)\\&-\xi_2\left(\frac{1}{2}q_1^2\frac{\partial k_1}{\partial q_2}+k_2(q_2-q_8)\right)+m_2\,{}_0D_t^{\beta(t)}q_2\left({}_0D_t^{\beta(t)}(\xi_2-\dot{q}_2\dot{\xi}_0)+\frac{\mathrm{d}}{\mathrm{d}t}{}_0D_t^{\alpha(t)}q_2\xi_0\right)+Q_2(\xi_2-\dot{q}_2\xi_0)\\&-\xi_3\left(\frac{1}{2}q_1^2\frac{\partial k_1}{\partial q_3}+k_3(q_3-q_9)\right)+m_3\,{}_0D_t^{\beta(t)}q_3\left({}_0D_t^{\beta(t)}(\xi_3-\dot{q}_3\dot{\xi}_0)+\frac{\mathrm{d}}{\mathrm{d}t}{}_0D_t^{\alpha(t)}q_3\xi_0\right)+Q_3(\xi_3-\dot{q}_3\xi_0)\\&-\xi_4\left(\frac{1}{2}q_1^2\frac{\partial k_1}{\partial q_4}+k_4(q_4-q_{10})\right)+m_4\,{}_0D_t^{\beta(t)}q_4\left({}_0D_t^{\beta(t)}(\xi_4-\dot{q}_4\dot{\xi}_0)+\frac{\mathrm{d}}{\mathrm{d}t}{}_0D_t^{\alpha(t)}q_4\xi_0\right)+Q_4(\xi_4-\dot{q}_4\xi_0)\\&+k_2(q_2-q_8)+k_3(q_3-q_9)+k_4(q_4-q_{10})+\left(Q_5-R_{\mathrm{Na}}\,{}_0D_t^{\gamma(t)}q_5-E_{\mathrm{Na}}\right)(\xi_5-\dot{q}_5\xi_0)+\left(Q_6-R_{\mathrm{K}}\,{}_0D_t^{\gamma(t)}q_6-E_{\mathrm{K}}\right)(\xi_6-\dot{q}_6\xi_0)\\&+\left(Q_7-R_1\,{}_0D_t^{\gamma(t)}q_7-E_1\right)(\xi_7-\dot{q}_7\xi_0)+\left(Q_8-\eta_2\,{}_0D_t^{\beta(t)}q_8\right)(\xi_8-\dot{q}_8\xi_0)+\left(Q_9-\eta_3\,{}_0D_t^{\beta(t)}q_9\right)(\xi_9-\dot{q}_9\xi_0)\\&+\left(Q_{10}-\eta_4\,{}_0D_t^{\beta(t)}q_{10}\right)(\xi_{10}-\dot{q}_{10}\xi_0)+L\dot{\xi}_0+\dot{G}_{\mathrm{N}}=0.\end{aligned}\tag{44}$$

Next let us discuss the structures of Noether conserved quantities when external nonpotential forces $Q_s \neq 0$ ($s = 1,2,3,4,8,9,10$). We will show that the forms of generalized potential forces impact the integrability of the system. In the following we use the approximate relation (3) to simplify the calculus, and for convenience we still use the symbols ${}_0D_t^{\alpha_i(t)}, {}_tD_a^{\alpha_i(t)}$.

When $Q_1 = \eta_1\,{}_0D_t^{\alpha(t)}q_1$, $Q_8 = \eta_2\,{}_0D_t^{\beta(t)}q_8$, $Q_9 = \eta_3\,{}_0D_t^{\beta(t)}q_9$, $Q_{10} = \eta_4\,{}_0D_t^{\beta(t)}q_{10}$, the external non-potential force is synchronized with the viscoelastic deformation of the axon membrane material. Let us study conserved quantities for this case. One generator solution of Noether identity is

$$\begin{aligned}&\xi_0 = 0,\ \ \xi_1 = 0,\ \ \xi_2 = \xi_3 = \xi_4 = \pm 1,\ \ \xi_5 = \dot{q}_5,\\ &\xi_6 = \dot{q}_6,\ \ \xi_7 = \dot{q}_7,\ \ \xi_8 = \xi_9 = \xi_{10} = \pm 1,\end{aligned} \tag{45}$$

and we can work out the gauge function

$$G_{\rm N} = \pm\sum_{s=2}^{4}\frac{m_s t^{-\beta(t)}}{\Gamma(1-\beta(t))}\int_0^a {}_0D_t^{\beta(t)}q_s\,{\rm d}t \pm \int_0^a \tilde{Q}_{s=2,3,4}\,{\rm d}t. \tag{46}$$

The corresponding Noether conserved quantities are

$$I_{\rm N} = \mp\sum_{s=2}^{4}\int_0^a {}_tD_T^{\beta(t)}(m_s\,{}_0D_t^{\beta(t)}q_s)\,{\rm d}t \pm \int_0^a \tilde{Q}_{s=2,3,4}\,{\rm d}t. \tag{47}$$

Here $\tilde{Q}_{s=2,3,4}$ has several forms that may be chosen by the experiment conditions with $\tilde{Q}_2 = \dot{q}_2$ or $\ddot{q}_2$, $\tilde{Q}_3 = \dot{q}_3$ or $\ddot{q}_3$, $\tilde{Q}_4 = \dot{q}_4$ or $\ddot{q}_4$ or any combination of them.

If $k_1 = {\rm const.}$, we have solutions

$$\begin{aligned}&\xi_0 = 1,\ \ \xi_1 = \dot{q}_1,\ \ \xi_2 = \dot{q}_2,\ \ \xi_3 = \dot{q}_3,\ \ \xi_4 = \dot{q}_4,\ \ \xi_5 = \dot{q}_5,\\ &\xi_6 = \dot{q}_6,\ \ \xi_7 = \dot{q}_7,\ \ \xi_8 = \dot{q}_8,\ \ \xi_9 = \dot{q}_9,\ \ \xi_{10} = \dot{q}_{10},\end{aligned} \tag{48}$$

and we can work out the gauge function

$$\begin{aligned}G_{\rm N} =\ &\frac{1}{2}k_1q_1^2 + \frac{e_C^2}{2C} - \sum_{i,s=1}^{4}\frac{1}{2}m_s({}_0D_t^{\alpha_i(t)}q_s)^2 + \frac{1}{2}k_2(q_2-q_8)^2\\ &+\frac{1}{2}k_3(q_3-q_9)^2 + \frac{1}{2}k_4(q_4-q_{10})^2.\end{aligned} \tag{49}$$

The corresponding Noether conserved quantity is trivial with $I_{\rm N} = 0$.

If $k_1 = {\rm const.}$ and the generalized nonpotential forces have forms $Q_8 = \eta_2\,{}_0D_t^{\beta(t)}q_8$, $Q_9 = \eta_3\,{}_0D_t^{\beta(t)}q_9$, $Q_{10} = \eta_4\,{}_0D_t^{\beta(t)}q_{10}$, we have solutions

$$\begin{aligned}&\xi_0 = \pm 1,\ \ \xi_1 = \pm\dot{q}_1,\ \ \xi_2 = \pm\dot{q}_2,\ \ \xi_3 = \pm\dot{q}_3,\\ &\xi_4 = \pm\dot{q}_4,\ \ \xi_5 = \dot{q}_5,\ \ \xi_6 = \dot{q}_6,\ \ \xi_7 = \dot{q}_7,\\ &\xi_8 = \pm\dot{q}_2,\ \ \xi_9 = \pm\dot{q}_3,\ \ \xi_{10} = \pm\dot{q}_4,\end{aligned} \tag{50}$$

and we can work out the gauge function

$$G_{\rm N} = \pm\frac{1}{2}k_1q_1^2 \pm \frac{e_C^2}{2C} - \sum_{i,s=1}^{4}\frac{1}{2}m_s({}_0D_t^{\alpha_i(t)}q_s)^2. \tag{51}$$

The corresponding Noether conserved quantity is

$$I_{\rm N} = \mp k_2\frac{1}{2}(q_2-q_8)^2 \mp \frac{1}{2}k_3(q_3-q_9)^2 \mp \frac{1}{2}k_4(q_4-q_{10})^2, \tag{52}$$

which denotes the micro mechanical potential energy.

If $k_1 = {\rm const.}$, $C = {\rm const.}$, we have solutions

$$\begin{aligned}&\xi_0 = \pm 1,\ \ \xi_1 = \pm\dot{q}_1,\ \ \xi_2 = \pm\dot{q}_2,\ \ \xi_3 = \pm\dot{q}_3,\\ &\xi_4 = \pm\dot{q}_4,\ \ \xi_5 = \dot{q}_5,\ \ \xi_6 = \dot{q}_6,\ \ \xi_7 = \dot{q}_7,\\ &\xi_8 = \pm\dot{q}_2,\ \ \xi_9 = \pm\dot{q}_3,\ \ \xi_{10} = \pm\dot{q}_4,\end{aligned} \tag{53}$$

$$\begin{aligned}&\xi_0 = \pm 1,\ \ \xi_1 = \pm\dot{q}_1,\ \ \xi_2 = \pm\dot{q}_2,\ \ \xi_3 = \pm\dot{q}_3,\\ &\xi_4 = \pm\dot{q}_4,\ \ \xi_5 = \dot{q}_5,\ \ \xi_6 = \dot{q}_6,\ \ \xi_7 = \dot{q}_7,\\ &\xi_8 = \pm\dot{q}_8,\ \ \xi_9 = \pm\dot{q}_9,\ \ \xi_{10} = \pm\dot{q}_{10},\end{aligned} \tag{54}$$

and we can work out gauge functions respectively

$$G_{\rm N} = \pm\frac{1}{2}k_1q_1^2 \mp \sum_{i,s=1}^{4}\frac{1}{2}m_s({}_0D_t^{\alpha_i(t)}q_s)^2, \tag{55}$$

$$\begin{aligned}G_{\rm N} =\ &\pm\frac{1}{2}k_1q_1^2 \mp \sum_{i,s=1}^{4}\frac{1}{2}m_s({}_0D_t^{\alpha_i(t)}q_s)^2 \pm \frac{1}{2}k_2(q_2^2+q_8^2)\\ &\pm\frac{1}{2}k_3(q_3^2+q_9^2) \pm \frac{1}{2}k_4(q_4^2+q_{10}^2).\end{aligned} \tag{56}$$

The corresponding conserved quantities are

$$\begin{aligned}I_{\rm N} =\ &\mp\frac{e_C^2}{2C} \mp \frac{1}{2}k_2(q_2-q_8)^2 \mp \frac{1}{2}k_3(q_3-q_9)^2\\ &\mp\frac{1}{2}k_4(q_4-q_{10})^2,\end{aligned} \tag{57}$$

which denote the sum of the electric energy and the micro mechanical potential energy.

$$I_{\rm N} = \mp\frac{e_C^2}{2C} \mp k_2q_2q_8 \mp k_3q_3q_9 \mp k_4q_4q_{10}. \tag{58}$$

In this section we have discussed the effects of parameters $k(q_2,q_3,q_4), C(q_1)$ and non-potential forces Q_s on the forms of Noether conserved quantities. From the above calculations we can conclude that the Noether symmetry and Noether conserved quantities are strongly determined by nonpotential forces and the material parameters. We did not give all the conserved quantities for the neuron membrane dynamics model, because we can see the conserved quantities are strongly dependent on the external non-conservative forces and characteristic parameters of the neuron membrane. But we give general formulations (41) and (42) to calculate its conserved quantities.

5. Conclusion

Considering a neuron axon without myelin as an axis-symmetrical cylinder with homogeneous deformation in a neuron membrane, we unified the deformation and electrophysiological multiscale coupling process through Hamilton's principle in generalized coordinates with fractional derivative of variable orders. The Lagrange equation (10) of the variable order fractional multiscale mechano-electrophysiological model of the neuron membrane is given through which we can deduce the variable order fractional differential equations of motion found in Ref. [7] and the variable order fractional Hodgkin–Huxley equation. The variable order fractional Noether symmetry criterion (37) and (41) and Noether conserved quantities (38) and (42) are given under Lie group transformations. Through the variable order fractional Noether criterion we work out some solutions that correspond to Noether conserved quantities under different external stimuli. The results show that Noether conserved quantities closely depend on the external nonconservative forces and material parameters of the neuron, which need to be further verified by experiments. Other symmetrical methods, like Lie symmetry and Mei symmetry can also be applied to this system, and these will be studied in the future.

Acknowledgment

Project supported by the National Natural Science Foundation of China (Grant Nos. 12272148 and 11772141).

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