Riemann–Hilbert approach of the complex Sharma–Tasso–Olver equation and its N-soliton solutions

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We study the complex Sharma–Tasso–Olver equation using the Riemann–Hilbert approach. The associated Riemann–Hilbert problem for this integrable equation can be naturally constructed by considering the spectral problem of the Lax pair. Subsequently, in the case that the Riemann–Hilbert problem is irregular, the N-soliton solutions of the equation can be deduced. In addition, the three-dimensional graphic of the soliton solutions and wave propagation image are graphically depicted and further discussed.

Keywords: complex Sharma–Tasso–Olver equation, Riemann–Hilbert problem, spectral problem, soliton solutions

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1. Introduction

The Sharma–Tasso–Olver (STO) equation

$$u_y+\frac{1}{2}(u_{xxx}-3(u_xu)_x-3u^2u_x)=0 \tag{1}$$

was first recognized as an example of odd members of Burgers hierarchy.[1] In recent decades, due to the enormous application of the STO equation (1) in mathematical physics, more and more attention has been paid to it. Tasso gave the bi-Hamiltonian structure and the generalized Poisson bracket for this equation.[2] The exact solutions of this equation were obtained through the method with symmetry reduction.[3] Furthermore, with the help of various methods, researchers have presented the new solitons and kinks solutions, the multiple solitons, exact traveling solutions, non-traveling wave solutions of the STO equation.[4–14]

In the present work, we focus on the cSTO equation that was derived by Fan[12] as follows:

$$u_y+\frac{1}{2}(u_{xxx}-3\mathrm{i}(u_xu)_x-3u^2u_x)=0, \tag{2}$$

for the given initial value $u(x,0)=u_0(x)\in\mathbb{S}(\mathbb{R})$. Here, $\mathbb{S}(\mathbb{R})=\{f(x)|\int_{-\infty}^{\infty}[1+|x|^pf(x)]\mathrm{d}x<\infty,\ p>1\}$ is the Schwartz space. Moreover, if we let $\mathrm{i}\lambda\to\lambda$, the real version of the STO equation (1) can be drawn. Since the cSTO equation (2) has many applications in mathematical physics, several studies have concentrated on this equation in recent years.[15,16] In Ref. [15], the equation was analyzed by means of the Fokas method on the half line, its function representation of algebro-geometric solutions and related crucial quantities have been presented.[16] In fact, there is no report on using the Riemann–Hilbert method to construct N-soliton solutions of the cSTO equation.

Nowadays, there is an increasing number of problems concerning the solution of nonlinear equations,[17–22] and thus, a large group of scholars have been investigating them with the help of some effective tools, two of the more common methods are inverse scattering transformation and Riemann–Hilbert method, which play a great role in the research of soliton equations. The inverse scattering transformation, as a method to solve the initial value problem of the integrable equations, was first proposed. However, the calculation for this method is more complicated. As a result, based on the inverse scattering transformation method, Novikov *et al.*[23] gave a relatively direct and useful approach, which is now known as the Riemann–Hilbert method. With this approach, a number of researchers have solved a series of integrable models,[24–33] which have led to many interesting and valuable results.

The outline of this paper is divided into the following parts. In Section 2, utilizing the spectral analysis of the Lax pair of the cSTO equation, we obtain the analytical properties of Jost functions. Furthermore, based on the above conclusions, the Riemann–Hilbert problem for the cSTO equation is established. In Section 3, we compute the N-soliton solutions of the cSTO equation by solving the Riemann–Hilbert problem in the case of irregularity and reflectionless, and then based on the obtained results, we give the spatial structure and collision dynamics behaviors of the breathers and soliton solutions. A brief conclusion is presented in Section 4.

2. The Riemann–Hilbert problem of the cSTO equation

In this section, constructing a Riemann–Hilbert problem for the cSTO equation is at the center of our study. Initially,

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the Lax pair of the cSTO equation (2) reads

$$\begin{cases} \varphi_x = U\varphi = (-\mathrm{i}\lambda^2\sigma_3 - \frac{1}{2}\mathrm{i}u\sigma_3 + \lambda R)\varphi, \\ \varphi_y = V\varphi = (-2\mathrm{i}\lambda^6\sigma_3 + V_1 + V_2\sigma_3)\varphi, \end{cases} \tag{3}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, and

$$\begin{aligned} \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & u \\ 2 & 0 \end{pmatrix}, \\ V_1 &= 2\lambda^5 R + \lambda^3(\mathrm{i}R_x + uR) \\ &\quad + \lambda\left(-\frac{1}{2}R_{xx} + \frac{3}{8}\mathrm{i}(R^3)_x - \frac{1}{2}\mathrm{i}RR_x\sigma_- + \frac{1}{2}u^2R\right), \\ V_2 &= -2\mathrm{i}\lambda^4 u + \lambda^2(u_x - \mathrm{i}u^2) + \frac{1}{4}\mathrm{i}u_{xx} + \frac{3}{4}uu_x - \frac{1}{4}\mathrm{i}u^3, \end{aligned}$$

with $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, assuming that $R^\dagger = -R$, the superscript † represents conjugate transpose.

Afterward, the compatibility condition $U_y - V_x + [U,V] = 0$ of Eq. (3) can yield the cSTO equation (2), in this case, the square bracket indicates the usual matrix commutator defined as $[U,V] = UV - VU$.

Since the initial-value $u(x,0)$ belongs to the Schwartz space, we can then get the asymptotic condition on the basis of the matrix spectral problem of the Lax pair (3):

$$\varphi \sim \mathrm{e}^{-\mathrm{i}\lambda^2 x\sigma_3 - 2\mathrm{i}\lambda^6 y\sigma_3}, \quad x, y \to \pm\infty. \tag{4}$$

For the sake of simplicity, a new Jost function $J = J(x,y;\lambda)$ can be given as follows:

$$J = \varphi \mathrm{e}^{\mathrm{i}\lambda^2 x\sigma_3 + 2\mathrm{i}\lambda^6 y\sigma_3}, \quad x, y \to \pm\infty. \tag{5}$$

By virtue of Eq. (5), the Lax pair (3) can be rewritten in an equivalent form as

$$\begin{cases} J_x + \mathrm{i}\lambda^2[\sigma_3, J] = U'J, \\ J_y + 2\mathrm{i}\lambda^6[\sigma_3, J] = V'J, \end{cases} \tag{6}$$

where $U' = -(1/2)\mathrm{i}u\sigma_3 + \lambda R$, $V' = V_1 + V_2\sigma_3$, $[\sigma_3, J] = \sigma_3 J - J\sigma_3$.

In purpose of constructing the Riemann–Hilbert problem for the solution of the initial value problem, we aim to find solutions of the spectral problem that approach the 2×2 identity matrix when $\lambda \to \infty$. Therefore, we expand the solution of the Lax pair (6) to the Laurent series as follows:[34,35]

$$\begin{aligned} J &= D + \frac{J_1}{\lambda} + \frac{J_2}{\lambda^2} + \frac{J_3}{\lambda^3} + \frac{J_4}{\lambda^4} + \frac{J_5}{\lambda^5} + \frac{J_6}{\lambda^6} + O\left(\frac{1}{\lambda^7}\right), \\ &\quad \lambda \to \infty, \end{aligned} \tag{7}$$

where D and J_k $(k = 1,2,3,4,5,6)$ are independent of the spectral parameter λ.

Substituting Eq. (7) into Eq. (6) and then comparing the coefficients of λ of the same order on both sides of the equation, we can realize by calculation that D is a diagonal matrix and satisfies the following relations:

$$D_x = \frac{1}{2}\mathrm{i}u\sigma_3 D, \tag{8}$$

$$D_y = \left(-\frac{1}{4}\mathrm{i}u_{xx} - \frac{1}{2}uu_x + \frac{1}{4}\mathrm{i}u^3\right)\sigma_3 D. \tag{9}$$

In addition, it can be observed that the cSTO equation (2) holds the conservation law

$$(u)_y = \left(-\frac{1}{2}u_{xx} + \mathrm{i}uu_x + \frac{1}{2}u^3\right)_x. \tag{10}$$

Accordingly, we can know that Eqs. (8) and (9) are consistent for D. From the analysis, we obtain

$$D(x) = \mathrm{e}^{(1/2)\mathrm{i}\int_{-\infty}^{x} u(\tilde{x},y)\,\mathrm{d}\tilde{x}\sigma_3}. \tag{11}$$

Subsequently, we introduce a new function Ψ which allows $J = D\Psi$. With the help of Ψ, the asymptotic expansion (7) can be rewritten as

$$\Psi = I + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty, \tag{12}$$

with I being an identity matrix.

Accordingly, the Lax pair (6) becomes

$$\begin{cases} \Psi_x + \mathrm{i}\lambda^2[\sigma_3, \Psi] = \tilde{U}\Psi, \\ \Psi_y + 2\mathrm{i}\lambda^6[\sigma_3, \Psi] = \tilde{V}\Psi, \end{cases} \tag{13}$$

where

$$\begin{aligned} \tilde{U} &= \mathrm{e}^{-(1/2)\mathrm{i}\int_{-\infty}^{x} u(\tilde{x},y)\,\mathrm{d}\tilde{x}\hat{\sigma}_3}(\lambda R - \mathrm{i}u\sigma_3), \\ \tilde{V} &= \mathrm{e}^{-(1/2)\mathrm{i}\int_{-\infty}^{x} u(\tilde{x},y)\,\mathrm{d}\tilde{x}\hat{\sigma}_3}\Big[2\lambda^5 R - 2\mathrm{i}\lambda^4 u\sigma_3 + \mathrm{i}\lambda^3 R_x \\ &\quad + \lambda^3 uR + \lambda^2 u_x\sigma_3 - \mathrm{i}\lambda^2 u^2\sigma_3 - \frac{1}{2}\lambda R_{xx} \\ &\quad + \frac{3}{8}\mathrm{i}(R^3)_x - \frac{1}{2}\mathrm{i}\lambda RR_x\sigma_- + \frac{1}{2}\lambda u^2 R \\ &\quad + \left(\frac{1}{2}\mathrm{i}u_{xx} + \frac{5}{4}uu_x - \frac{1}{2}\mathrm{i}u^3\right)\sigma_3\Big]. \end{aligned}$$

Here it should be noted that $\hat{\sigma}_3$ stands for a matrix commutator of σ_3, namely, $\hat{\sigma}_3\kappa = [\sigma_3, \kappa]$, and $\mathrm{e}^{\hat{\sigma}_3}\kappa = \mathrm{e}^{\sigma_3}\kappa\mathrm{e}^{-\sigma_3}$ (κ is a 2×2 matrix).

Next, we can set up two matrix Jost solutions of the first expression in Eq. (13),

$$\Psi_L = ([\Psi_L]_1, [\Psi_L]_2), \quad \Psi_R = ([\Psi_R]_1, [\Psi_R]_2), \tag{14}$$

and have the asymptotic conditions

$$\Psi_L \to I, x \to -\infty, \quad \Psi_R \to I, x \to +\infty, \tag{15}$$

where $[\Psi_L]_1$ and $[\Psi_R]_1$ are the first column of Ψ_L and Ψ_R, the second column of Ψ_L and Ψ_R are $[\Psi_L]_2$ and $[\Psi_R]_2$, respectively.

Naturally, to facilitate the study, we introduce two related Volterra integral equations, which have unique solutions Ψ_L and Ψ_R,

$$\Psi_L(x,\lambda) = I + \int_{-\infty}^{x} \mathrm{e}^{\mathrm{i}\lambda^2(x-\tau)\hat{\sigma}_3}(\tilde{U}(\tau)\Psi_L(\tau,\lambda))\,\mathrm{d}\tau, \tag{16}$$

$$\Psi_R(x,\lambda) = I - \int_x^{\infty} \mathrm{e}^{\mathrm{i}\lambda^2(x-\tau)\hat{\sigma}_3}(\tilde{U}(\tau)\Psi_R(\tau,\lambda))\,\mathrm{d}\tau. \quad (17)$$

Further, more attention needs to be paid to the analyticity and symmetry of the eigenfunctions Ψ_L, Ψ_R. Consequently, it can be concluded that $[\Psi_L]_1$ and $[\Psi_R]_2$ are analytic for $\lambda \in D_-$ and continuous for $\lambda \in D_- \bigcup \mathbb{R}$, while $[\Psi_R]_1$ and $[\Psi_L]_2$ are analytic for $\lambda \in D_+$ and continuous for $\lambda \in D_+ \bigcup \mathbb{R}$,

$$D_+ = \left\{\lambda \mid \arg\lambda \in \left(0,\frac{\pi}{2}\right)\bigcup\left(\pi,\frac{3\pi}{2}\right)\right\},$$
$$D_- = \left\{\lambda \mid \arg\lambda \in \left(\frac{\pi}{2},\pi\right)\bigcup\left(\frac{3\pi}{2},2\pi\right)\right\}.$$

With Abel's identity and the calculation of $\det\Psi_L$ at $x = -\infty$ and $\det\Psi_R$ at $x = +\infty$, it is natural to work out

$$\det\Psi_L = 1, \quad \det\Psi_R = 1, \quad \lambda \in \mathbb{R}. \quad (18)$$

Then, considering the spectral problem of the Lax pair (13), there exist two fundamental matrix solutions on the x-part $\Psi_L E$ and $\Psi_R E$, where $E = \mathrm{e}^{\mathrm{i}\lambda^2 x\sigma_3}$. Therefore, they are linearly related by a scattering matrix $S(\lambda) = (s_{ij})_{2\times 2}$, which can be recorded as

$$\Psi_L E = \Psi_R E \cdot S(\lambda), \quad \lambda \in \mathbb{R}. \quad (19)$$

From Eqs. (18) and (19), $\det S(\lambda) = 1$ can be determined.

In what follows, we aim to establish the Riemann–Hilbert problem for the cSTO equation, so the inverse matrices with respect to Ψ_L and Ψ_R need to be considered, assuming that

$$\Psi_L^{-1} = \begin{pmatrix} [\Psi_L^{-1}]^1 \\ [\Psi_L^{-1}]^2 \end{pmatrix}, \quad \Psi_R^{-1} = \begin{pmatrix} [\Psi_R^{-1}]^1 \\ [\Psi_R^{-1}]^2 \end{pmatrix}, \quad (20)$$

where $[\Psi_L^{-1}]^1$ and $[\Psi_R^{-1}]^1$ are the first row of Ψ_L^{-1} and Ψ_R^{-1}, the second row of Ψ_L and Ψ_R are $[\Psi_R^{-1}]^2$ and $[\Psi_R^{-1}]^2$. It is accessible that $[\Psi_L^{-1}]^1$ and $[\Psi_R^{-1}]^2$ are analytic for $\lambda \in D_+$ and continuous for $\lambda \in D_+ \bigcup \mathbb{R}$, whereas $[\Psi_R^{-1}]^1$ and $[\Psi_L^{-1}]^2$ are analytic for $\lambda \in D_-$ and continuous for $\lambda \in D_- \bigcup \mathbb{R}$.

In accordance with Eq. (19), we can obtain the analyticity of $S(\lambda)$:

$$ES(\lambda)E^{-1} = \Psi_R^{-1}\Psi_L = \begin{pmatrix} [\Psi_R^{-1}]^1[\Psi_L]_1 & [\Psi_R^{-1}]^1[\Psi_L]_2 \\ [\Psi_R^{-1}]^2[\Psi_L]_1 & [\Psi_R^{-1}]^2[\Psi_L]_2 \end{pmatrix}. \quad (21)$$

Indeed, s_{11} allows analytic extension to D_-, and s_{22} can be analytic extension to D_+.

In order to form the Riemann–Hilbert problem for the cSTO equation, we seek two analytical functions on D_- and D_+. Let

$$P_R = ([\Psi_L]_1, [\Psi_R]_2) = \Psi_L L_1 + \Psi_R L_2,$$
$$P_L = \begin{pmatrix} [\Psi_L^{-1}]^1 \\ [\Psi_R^{-1}]^2 \end{pmatrix} = L_1\Psi_L^{-1} + L_2\Psi_R^{-1},$$

where $L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $L_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Moreover, it implies that P_R is analytic in D_-, P_L is analytic in D_+, and $P_{L,R} \to I, \lambda \to \infty$.

In the context of the previous preparation knowledge of the cSTO equation, the Riemann–Hilbert problem can be raised as follows:

(1) P_R is analytic in D_-, P_L is analytic in D_+.

(2) $P_L P_R = G(x,y;\lambda)$.

(3) $P_R \to I$, $P_L \to I$, when $\lambda \to \infty$,

where

$$G(x,y;\lambda) = \mathrm{e}^{\mathrm{i}(\lambda^2 x + 2\lambda^6 y)\sigma_3}\begin{pmatrix} 1 & h_{12} \\ s_{21} & 1 \end{pmatrix}\mathrm{e}^{-\mathrm{i}(\lambda^2 x + 2\lambda^6 y)\sigma_3}$$

is the jump matrix and $S^{-1} = H = (h_{ij})_{1\le i,j\le 2}$. On account of $\det S(\lambda) = 1$, we can obtain $h_{11}s_{11} + h_{12}s_{21} = 1$.

3. The N-soliton solutions for the cSTO equation

In this section, at the aid of the above Riemann–Hilbert problem, the soliton solutions of the cSTO equation can be calculated exactly. Assume that the Riemann–Hilbert problem is irregular, then we can see that $\det P_R$ and $\det P_L$ make sense in their own analytic areas except for a few zeros. On the basis of the definitions of P_R, P_L and the scattering relationship Eq. (19), we can directly obtain

$$\begin{cases} \det P_R = s_{11}(\lambda), & \lambda \in D_-, \\ \det P_L = h_{11}(\lambda), & \lambda \in D_+. \end{cases} \quad (22)$$

It is obvious that the zeros of $\det P_L$ and $\det P_R$ are the same as $h_{11}(\lambda)$ and $s_{11}(\lambda)$, respectively.

Due to the symmetry relation of the matrix R, the following facts can be generated:

$$\Psi_{R,L}^{\dagger}(\lambda^*) = \Psi_{R,L}^{-1}(\lambda), \quad (23)$$
$$P_R^{\dagger}(\lambda^*) = P_L(\lambda), \quad (24)$$
$$S^{\dagger}(\lambda^*) = S^{-1}(\lambda). \quad (25)$$

Furthermore, make use of Eq. (25), implying that

$$h_{11}^*(\lambda^*) = s_{11}(\lambda), \quad (26)$$

and each zero $\pm\lambda_k^*$ of h_{11} corresponds to each zero $\pm\lambda_k$ of s_{11}.

Suppose that $\det P_R$ exists N simple zeros $\lambda_1 \cdots \lambda_N \in D_-$, and $\det P_L$ exists N simple zeros $\tilde{\lambda}_1 \cdots \tilde{\lambda}_N \in D_+$. It is revealed that

$$\lambda_k^* = \tilde{\lambda}_k, \quad 1 \le k \le N. \quad (27)$$

At present, assume that η_k is nonzero column vector and η_k^* is nonzero row vector, there appear the linear equations about η_k and η_k^*:

$$\begin{cases} P_R(\lambda_k)\eta_k(\lambda_k) = 0, \\ \eta_k^*(\tilde{\lambda}_k)P_L(\tilde{\lambda}_k) = 0. \end{cases} \quad (28)$$

In view of the above formulas, we can gain the relation of the eigenvectors as follows:

$$\eta_k^* = \eta_k^\dagger, \quad 1 \le k \le N. \tag{29}$$

Taking derivatives of the first one of Eq. (28) with respect to x and t, respectively, and resorting to Eq. (13), we can obtain

$$\begin{cases} P_R(\lambda_k)(\dfrac{\partial \eta_k}{\partial x} + \mathrm{i}\lambda_k^2 \sigma_3 \eta_k) = 0, \\ P_R(\lambda_k)(\dfrac{\partial \eta_k}{\partial y} + 2\mathrm{i}\lambda_k^6 \sigma_3 \eta_k) = 0. \end{cases} \tag{30}$$

Furthermore, by comparing with Eqs. (13), (28) and (29), the expressions of η_k and η_k^* can be proposed as follows:

$$\begin{cases} \eta_k = \mathrm{e}^{-\mathrm{i}\lambda_k^2 x\sigma_3 - 2\mathrm{i}\lambda_k^6 y\sigma_3}\eta_{k,0}, & 1 \le k \le N, \\ \eta_k^* = \eta_{k,0}^\dagger \mathrm{e}^{\mathrm{i}(\lambda_k^*)^2 x\sigma_3 + 2\mathrm{i}(\lambda_k^*)^6 y\sigma_3}, & 1 \le k \le N, \end{cases} \tag{31}$$

where $\eta_{k,0} = (\alpha_k, \beta_k)^{\mathrm{T}}$ is a complex constant column vector.

Here we are mainly involved in the Riemann–Hilbert problem in the reflectionless, that is, the scattering data $s_{21} = 0$. As a result, the special Riemann–Hilbert problem with a unique solution has the expressions

$$\begin{cases} P_R(\lambda) = I - \sum\limits_{k,j=1}^{N} \dfrac{\eta_k \eta_j^* (M^{-1})_{kj}}{\lambda - \lambda_j^*}, \\ P_L(\lambda) = I + \sum\limits_{k,j=1}^{N} \dfrac{\eta_k \eta_j^* (M^{-1})_{kj}}{\lambda - \lambda_k}, \end{cases} \tag{32}$$

where M is an $N \times N$ matrix whose elements read

$$m_{kj} = \frac{\eta_k^* \eta_j}{\lambda_j - \lambda_k^*}, \quad 1 \le k, j \le N. \tag{33}$$

Moreover, $(M^{-1})_{kj}$ holds the (k, j)-entry of the inverse matrix of M.

In fact, we can express the asymptotic expansion of $P_R(\lambda)$ as

$$P_R = I + \frac{P_R^{[1]}}{\lambda} + \frac{P_R^{[2]}}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right), \quad \lambda \in D_- \to \infty, \tag{34}$$

where $P_R^{[1]}$, $P_R^{[2]}$ are 2×2 matrices. Subsequently, Eq. (34) can be substituted into the first one of Eq. (13), which leads to

$$u = 2\mathrm{i}(P_R^{[1]})_{12} \mathrm{e}^{\mathrm{i}\int_{-\infty}^{x} u(\tilde{x},y)\,\mathrm{d}\tilde{x}}, \tag{35}$$

where $(P_R^{[1]})_{12}$ is the $(1,2)$-element of $P_R^{[1]}$. Further, taking advantage of Eq. (32), we can infer

$$P_R^{[1]} = -\sum_{k=1}^{N}\sum_{j=1}^{N} \eta_k \eta_j^* (M^{-1})_{kj}. \tag{36}$$

For sake of presenting more clearly the expression of the multi-soliton solutions of the cSTO equation (2), we need to set a complex function $\vartheta_k = \mathrm{i}\lambda_k^2 x + 2\mathrm{i}\lambda_k^6 y$. By utilizing the previous results, the expression of the N-soliton solutions can be derived as

$$u(x,y) = -2\mathrm{i}\sum_{k=1}^{N}\sum_{j=1}^{N} \alpha_k \beta_j^* \mathrm{e}^{-\vartheta_k + \vartheta_j^*} (M^{-1})_{kj} \mathrm{e}^{\mathrm{i}\int_{-\infty}^{x} u(\tilde{x},y)\,\mathrm{d}\tilde{x}}. \tag{37}$$

Especially, when $N = 2$, the two-soliton solutions of the cSTO equation (2) can be written as

$$\begin{aligned} &u_2(x,y) \\ &= \frac{2\mathrm{i}}{m_{11}m_{22} - m_{12}m_{21}}(\alpha_1\beta_2^* m_{12}\mathrm{e}^{\vartheta_2^* - \vartheta_1} - \alpha_1\beta_1^* m_{22}\mathrm{e}^{\vartheta_1^* - \vartheta_1} \\ &\quad + \alpha_2\beta_1^* m_{21}\mathrm{e}^{\vartheta_1^* - \vartheta_2} - \alpha_2\beta_2^* m_{11}\mathrm{e}^{\vartheta_2^* - \vartheta_2})\mathrm{e}^{\mathrm{i}\int_{-\infty}^{x} u(\tilde{x},y)\,\mathrm{d}\tilde{x}}, \end{aligned}$$

where

$$\begin{aligned} m_{11} &= \frac{1}{\lambda_1 - \lambda_1^*}(|\alpha_1|^2 \mathrm{e}^{-\vartheta_1^* - \vartheta_1} + |\beta_1|^2 \mathrm{e}^{\vartheta_1^* + \vartheta_1}), \\ m_{12} &= \frac{1}{\lambda_2 - \lambda_1^*}(\alpha_1^*\alpha_2 \mathrm{e}^{-\vartheta_1^* - \vartheta_2} + \beta_1^*\beta_2 \mathrm{e}^{\vartheta_1^* + \vartheta_2}), \\ m_{21} &= \frac{1}{\lambda_1 - \lambda_2^*}(\alpha_2^*\alpha_1 \mathrm{e}^{-\vartheta_2^* - \vartheta_1} + \beta_2^*\beta_1 \mathrm{e}^{\vartheta_2^* + \vartheta_1}), \\ m_{22} &= \frac{1}{\lambda_2 - \lambda_2^*}(|\alpha_2|^2 \mathrm{e}^{-\vartheta_2^* - \vartheta_2} + |\beta_2|^2 \mathrm{e}^{\vartheta_2^* + \vartheta_2}), \end{aligned}$$

and $\vartheta_1 = \mathrm{i}\lambda_1^2 x + 2\mathrm{i}\lambda_1^6 y$, $\vartheta_2 = \mathrm{i}\lambda_2^2 x + 2\mathrm{i}\lambda_2^6 y$.

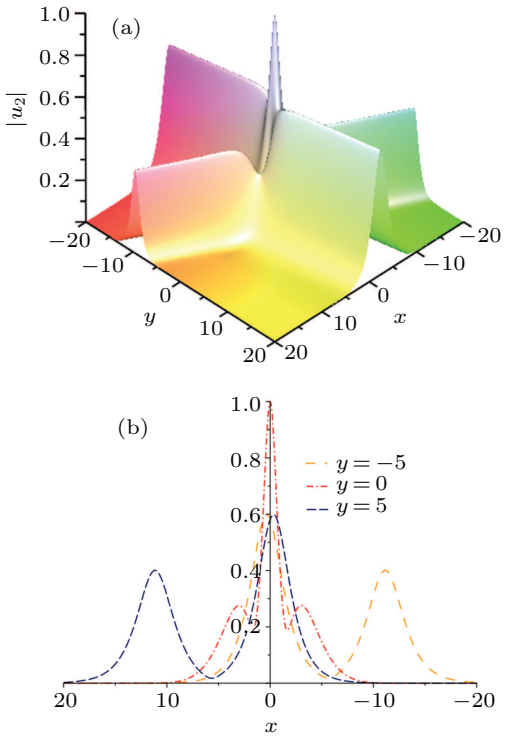


Fig. 1. The two-soliton solutions of the cSTO equation with parameters chosen as $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = \mathrm{i}$, $\alpha = 1$, $\lambda_1 = -4/5 - 1/5\mathrm{i}$, $\lambda_2 = -3/5 - 3/10\mathrm{i}$. (a) The modulus of $u_2(x,y)$ indicating the fundamental two-soliton solutions. (b) The two-soliton propagating along the x-axis with different y.

Next, by choosing a series of appropriate parameters, the spatial structure of the two-soliton solutions can be clearly portrayed in Fig. 1. The modulus of $u_2(x,y)$ represents the fundamental two-soliton solutions. We can clearly see in Fig. 1(a)

the interaction at the origin of the coordinates and their maximum magnitude. Furthermore, the general two-soliton represents the nonlinear superposition between these two fundamental solitons. The propagation of the wave on the x-axis at $y=-5$, $y=0$, $y=5$ is shown in Fig. 1(b).

4. Conclusion and discussion

In summary, with the aid of the Riemann–Hilbert method, we have systematically investigated the N-soliton solutions of the cSTO equation. Firstly, based on the analysis of spectral problem for Lax pair of the cSTO equation, the associated Riemann–Hilbert problem for the cSTO equation is constructed. Subsequently, on the premise that the Riemann–Hilbert problem is irregular, the N-soliton solutions of the cSTO equation are reduced. In addition, with the given parameters, some three-dimensional images of the cSTO equation are graphically visualized, and we can draw some interesting and meaningful conclusions from the graphics, namely, two-order solitons and breathers maintain stable velocities, periods and propagation orbit with the time evolution. Therefore, the present results will enrich research of the cSTO equation.

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