Adaptive multi-step piecewise interpolation reproducing kernel method for solving the nonlinear time-fractional partial differential equation arising from financial economics

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This paper is aimed at solving the nonlinear time-fractional partial differential equation with two small parameters arising from option pricing model in financial economics. The traditional reproducing kernel (RK) method which deals with this problem is very troublesome. This paper proposes a new method by adaptive multi-step piecewise interpolation reproducing kernel (AMPIRK) method for the first time. This method has three obvious advantages which are as follows. Firstly, the piecewise number is reduced. Secondly, the calculation accuracy is improved. Finally, the waste time caused by too many fragments is avoided. Then four numerical examples show that this new method has a higher precision and it is a more timesaving numerical method than the others. The research in this paper provides a powerful mathematical tool for solving time-fractional option pricing model which will play an important role in financial economics.

Keywords: time-fractional partial differential equation, adaptive multi-step, reproducing kernel method method, numerical solution

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1. Introduction

It is well known that the research of numerical solutions in the time-fractional option pricing model of the financial economics has crucial academic value and practical application significance. The time-fractional option pricing model is a vital option pricing model in financial economics. In today's society, the wide application of the standard Black–Scholes (B–S) option pricing model is a key factor of the booming growth in the financial derivatives market. However, the standard B–S model contains many assumptions. Through the observation and research for the stock market, it is found that the standard B–S model is too idealized to match the actual stock trend. So plenty of fractional differential equations in financial theory are put forward. It is discovered that the financial problems can be described by the time-fractional B–S model more broadly. It is a specific fractional advection–dispersion equation, in which the time-varying coefficient and time-fractional derivative are as follows:

$$P_t^{\alpha}(S,t)-(rP-rSP_S)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{\Gamma(1+\alpha)}{2}\sigma^2S^2P_{SS}=0,\quad t>0,\ \ 0<\alpha\le 1, \tag{1}$$

where P is the European option price, σ is the volatility, r is the risk-free interest rate, S is the stock price, t is the time. The fractional derivative is described in the Caputo sense, and α is its order.

It is well known that the time-fractional B–S equation is more coincident with real financial market than that of B–S equation, but there is no analytical solution for the time-fractional B–S equation. It is vital to research the numerical method for this equation. Recently, few numerical methods have been done of the numerical solution of the time-fractional B–S equation. For example, Samuel M Nuugulu derives the time-fractional B–S partial differential equation for option pricing, who proposes a robust numerical method which arises from the extension of a Crank Nicholson finite difference method.[1] Ruifang Yan uses a difference method of parallel nature to solve time–space fractional B–S model. The auther proposes a kind of difference format of parallel nature which uses an implicit format and an explicit format instead of the Saul'yev asymmetric format on the inner boundary.[2] Sunil Kumar overcomes some difficulty of traditional methods, the auther uses the homotopy analysis method and the homotopy perturbation method to solve the time-fractional B–S equation with boundary condition on an European option problem.[3] Yinghao Chen proposes an original numerical algorithm by the three layers neurons to solve the time-fractional B–S equations with the Laguerre neural network.[4] Xingyu An solves the time-fractional B–S equation with a space–time spectral method which applies Fourier-like basis functions for the spatial discretisation and the Jacobi polynomials for the temporal discretisation.[5] Pradip Roul uses a compact finite difference method for solving the time-fractional B–S option

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pricing model , which is applied for discretization of space derivative.[6] Mustafa Almushaira aims at proposing the improved spectral method and L1 operator splitting method for the B–S differential equations with fractional derivatives on both space and time.[7] Pradip Roul proposes a high order numerical method arising from the uniform mesh for productive numerical solution of the time-fractional B–S equation, considering the European options.[8] Ved Prakash Dubey solves the nonlinear time-fractional B–S European option pricing equation using a credible means of the residual power series (RPS) method, which is a approximative iterative mathematical scheme based on the RPS algorithm.[9] Chris Chen solves the American option fractional B–S models by a new operator splitting method, in which the fractional linear complementarity problem can be divided into two simple subproblems.[10] Recently, we attempt to solve the above equations using the RK method, but the numerical results are not satisfactory. Aiming at presenting the new method, we are going to solve the nonlinear time-fractional partial differential equation which has a wider range as following:

$$\begin{aligned}&\frac{\partial^\alpha V(S,t)}{\partial t^\alpha}-\varepsilon\delta(S,t)\frac{\partial^2 V(S,t)}{\partial S^2}-\mu\beta(S,t)\frac{\partial V(S,t)}{\partial S}\\&+\gamma(S,t)V(S,t)=f(S,t,V(S,t)),\ \ (S,t)\in\Omega,\\&V(S,0)=u_0(S),\ \ S\in\Omega_S,\ \ V(0,t)=V(1,t)=0,\ \ t\in[0,T],\end{aligned}\tag{2}$$

where the fractional derivative is defined in the Riemann–Liouville sense $0<\alpha\leq 1$. $0<\varepsilon\leq 1$ and $0<\mu\leq 1$ are two small parameters, $\delta(S,t)$, $\beta(S,t)$, $\gamma(S,t)$, and $f(S,t,V(S,t))$ are given, $\Omega=\Omega_S\times(0,T]$, $\Omega_S=(0,1)$. In previous studies (Refs. [11,12]), the authors improve the traditional RK method by the piecewise interpolation technology. Although we obtain the higher accuracy numerical solution by the piecewise interpolation reproducing kernel (PIRK) method, however, the more piecewise we choose, the higher the absolute error is obtained. Sometimes the absolute errors of dividing 50 and 500 tablets are almost the same, but the latter one requires much more work than that of the former one. In order to solve this problem, we are ready to combine the advantages of adaptive multi-step algorithm and precision of the PIRK method.

This paper is constituted as follows. In Section 2, the steps of the AMPIRK method are presented. In Section 3, the four numerical examples present that the AMPIRK method has a higher precision and it is a more timesaving numerical method than the others. In Section 4, conclusion and future directions are presented.

2. The steps of the AMPIRK method

To solving Eq. (2) in the reproducing kernel space, we can convert the nonhomogeneous initial conditions into homogeneous initial conditions for $D=[0,1]\times[0,1]$, we can also express the exact solution of the Eq. (3) by $V(S,t)$, then we can get

$$\begin{aligned}&LV(S,t)=F(S,t,V(S,t)),\ \ (S,t)\in D,\\&V(S,0)=V(0,t)=V(1,t)=0,\end{aligned}\tag{3}$$

where L is a bounded linear operator and reversible: $H(D)\to H_1(D)$, the reproducing kernel spaces $W_2^1[0,1]$, $W_2^2[0,1]$, $W_2^3[0,1]$, $H(D)=W_2^3[0,1]\otimes W_2^2[0,1]$, $H_1(D)=W_2^1[0,1]\otimes W_2^1[0,1]$ have been defined in Ref. [13]. $(L\zeta_i(S,t)|_{(S,t)=(S_j,t_j)})_{i,j=1,2,\ldots}$ is an identity matrix, the exact solution is $V(S,t)=\sum_{i=1}^{\infty}F(S_i,t_i,V(S_i,t_i))\zeta_i(S,t)$, the convergence theorem and the error estimate are given in Refs. [11,12]. When solving Eq. (3) with the PIRK method, the smaller h (h represents the inverse of the piecewise number) is, the larger the piecewise number is, and the smaller the absolute error is. Therefore, it can be seen that the fragment is effective. However, there are still some problems as follows.

(i) In order to guarantee the accuracy of calculation, the more the piecewise number is, the greater the amount of calculation demands, which requires a mass of computing nodes, time and space, even far beyond storage capacity of the computer.

(ii) Sometimes, the absolute error of 50 and 500 slices is almost the same, but the latter requires much more work than the former. So we need to think about how to choose the reasonable piecewise number, and guarantee the accuracy of the numerical results at the same time.

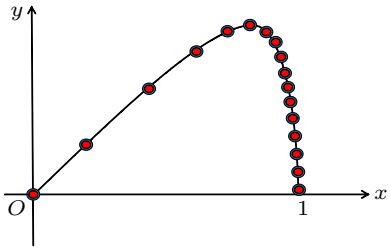


Fig. 1. Adaptive multi-step piecewise interpolation.

In order to solve the problem, this paper proposes a new numerical method called the AMPIRK method, which combines the adaptive multi-step technique with the PIRK method. As shown in Fig. 1, more points are selected in the steep region than in the gentle region. The idea of this method is to set a maximum absolute error ϖ which is set in each region $[S_{k-1},S_k]\times[0,1]$, $k=0,\ldots,n$, when the absolute errors are greater than ϖ, the region will continue to segment until the absolute errors are smaller than ϖ. Then stop segmenting, and so on, until get the numerical solution of the whole region. In this way, the piecewise number is reduced and the calculation accuracy is ensured, and the waste time caused by too many

fragments is avoided. What's more, four numerical examples in Section 3 show that AMPIRK method has a higher precision and it is a more timesaving numerical method than the others.

3. Numerical examples

Example 1 Considering the following equation:[14]

$$\frac{\partial^\alpha V(S,t)}{\partial t^\alpha} - \varepsilon a(S,t)\frac{\partial^2 V(S,t)}{\partial S^2} - \mu b(S,t)\frac{\partial V(S,t)}{\partial S} = 0, \quad (4)$$

where the initial and boundary conditions are $V(S,0) = x^2$, $V(0,t) = 0$, and $V(1,t) = \mathrm{e}^t$. The parameters $\alpha = 1$, $\varepsilon = 0.5$, $\mu = 0.1$, $a(S,t) = x^2$, $b(S,t) = 0$. The exact solution is $V(S,t) = x^2\mathrm{e}^t$. Let $N = 10$, $n = \varpi = 10^{-13}$, by mathematica 7.0, the absolute errors are given in Table 1, which compare with the absolute errors obtained by using the variational iteration method (VIM),[14] Adomian decomposition method (ADM),[14] the Sinc–Legendre collocation method (S–LCM),[14] the Ritz approximation method,[14] the RK method, and the AMPIRK method.

Table 1. Comparison of absolute errors of Example 1.

t	x	VIM and ADM[14]	S–LCM[14]	RAM[14]	RK method	AMPIRK method
0.25	0.3	1.54×10^{-5}	9.92×10^{-8}	8.92×10^{-12}	9.46×10^{-5}	2.69×10^{-14}
	0.6	6.16×10^{-5}	2.70×10^{-6}	3.79×10^{-12}	9.91×10^{-5}	3.08×10^{-14}
	0.9	1.38×10^{-4}	1.02×10^{-5}	9.01×10^{-13}	3.06×10^{-5}	1.15×10^{-14}
0.5	0.3	2.60×10^{-4}	5.56×10^{-7}	1.09×10^{-11}	1.21×10^{-4}	3.46×10^{-14}
	0.6	1.03×10^{-3}	4.87×10^{-6}	5.10×10^{-12}	1.36×10^{-4}	3.95×10^{-14}
	0.9	2.34×10^{-3}	1.30×10^{-5}	2.66×10^{-12}	4.30×10^{-5}	1.47×10^{-14}
0.75	0.3	1.39×10^{-3}	1.14×10^{-6}	2.66×10^{-11}	1.43×10^{-4}	4.45×10^{-14}
	0.6	5.56×10^{-3}	6.90×10^{-6}	3.25×10^{-13}	1.82×10^{-4}	5.08×10^{-14}
	0.9	1.25×10^{-2}	1.59×10^{-5}	3.50×10^{-12}	5.98×10^{-5}	1.91×10^{-14}
1	0.3	4.64×10^{-3}	9.83×10^{-7}	4.81×10^{-12}	2.10×10^{-4}	5.70×10^{-14}
	0.6	1.85×10^{-2}	6.40×10^{-6}	1.67×10^{-11}	2.83×10^{-4}	6.52×10^{-14}
	0.9	4.18×10^{-2}	2.58×10^{-5}	5.51×10^{-12}	9.34×10^{-5}	2.49×10^{-14}

Example 2 Considering the following equation:

$$\frac{\partial^\alpha V(S,t)}{\partial t^\alpha} - \varepsilon\frac{\partial^2 V(S,t)}{\partial S^2} + \mu\frac{\partial V(S,t)}{\partial S} + V(S,t) = f(S,t), \quad (5)$$

where the initial and boundary conditions are $V(S,0) = V(0,t) = 0$, $V(1,t) = t^2\sin 1$. $\alpha = 0.8$, $\varepsilon = 0.01$, $\mu = 0.1$, the exact solution of the above problem is $V(S,t) = t^2\cdot\sin^2 x$. $f(S,t)$ is given by the $V(S,t)$. Using mathematica 7.0, let $N = 10$. Figure 2 shows the exact solution. Figure 3 shows the exact solution (blue line) and the numerical solutions (red line) with (a) the RK method, (b) AMPIRK method ($h = 0.1$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 0.01$, $\varpi = 0.01$) at $t = 1$. Figure 4 shows the absolute errors with (a) RK method, (b) AMPIRK method ($h = 0.1$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 0.01$, $\varpi = 0.01$) at $t = 1$.

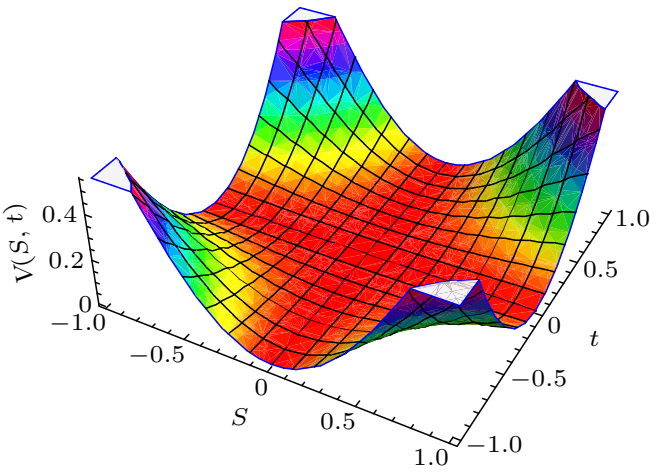


Fig. 2. Exact solution of Example 2.

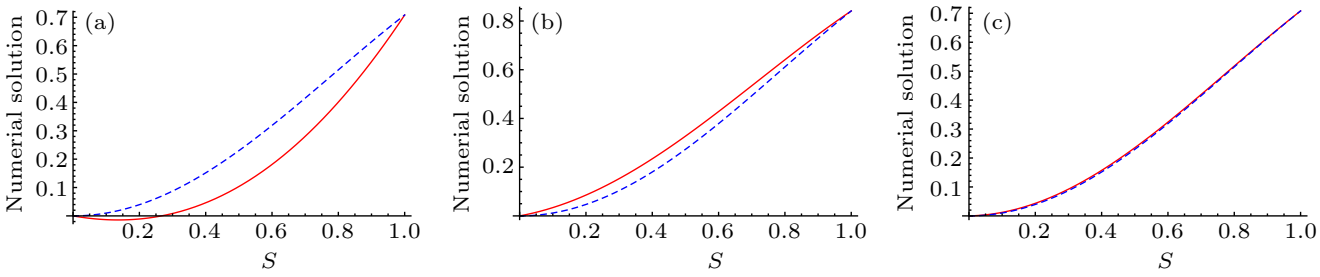


Fig. 3. Exact solution (blue line) and numerical solutions (red line) of Example 2 by (a) RK method, (b) AMPIRK method ($h = 0.1$, $\varpi = 0.1$) and (c) present method ($h = 0.01$, $\varpi = 0.01$) at $t = 1$.

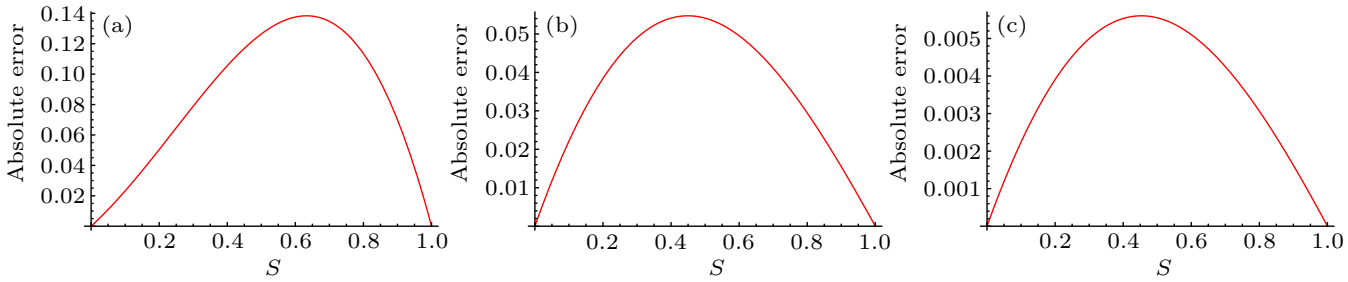


Fig. 4. Absolute errors of Example 2 by (a) RK method, (b) AMPIRK method ($h = 0.1$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 0.01$, $\varpi = 0.01$) at $t = 1$.

Example 3 Considering the following equation:

$$\frac{\partial^\alpha V(S,t)}{\partial t^\alpha} - \varepsilon \frac{\partial^2 V(S,t)}{\partial S^2} + \mu \frac{\partial V(S,t)}{\partial S} + V^2(S,t) = f(S,t), \quad (6)$$

where the initial and boundary conditions are $V(S,0) = 0$, $V(0,t) = 0$, and $V(1,t) = \sin 4\pi^2 t^2 \cdot \cos 4\pi^2 t^2$. $\alpha = 0.5$, $\varepsilon = 0.01$, $\mu = 0.05$, the exact solution of the above problem is $V(S,t) = \sin[2\pi x t]^2 \cdot \cos[2\pi x t]^2$. $f(S,t)$ is given by the $V(S,t)$. Using mathematica 7.0, let $N = 10$. Figure 5 shows the exact solution. Figure 6 shows the exact solution (blue line) and numerical solutions (red line) with (a) RK method, (b) AMPIRK method ($h = 0.01$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 10^{-6}$, $\varpi = 10^{-5}$) at $t = 0.5$. Figure 7 shows the absolute errors of Example 3 with (a) RK method, (b) AMPIRK method ($h = 0.01$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 10^{-6}$, $\varpi = 10^{-5}$) at $t = 0.5$.

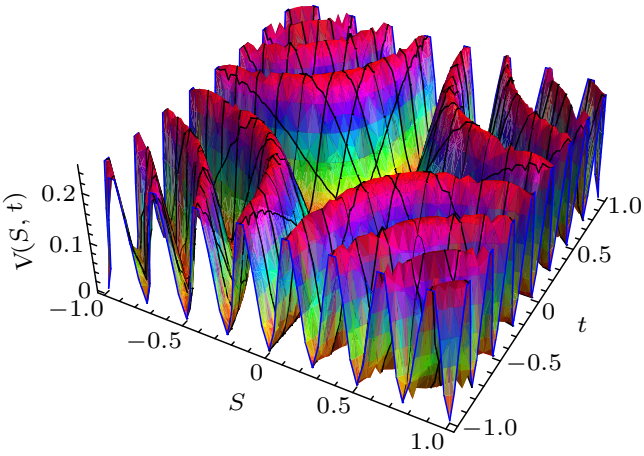


Fig. 5. Exact solution of Example 3.

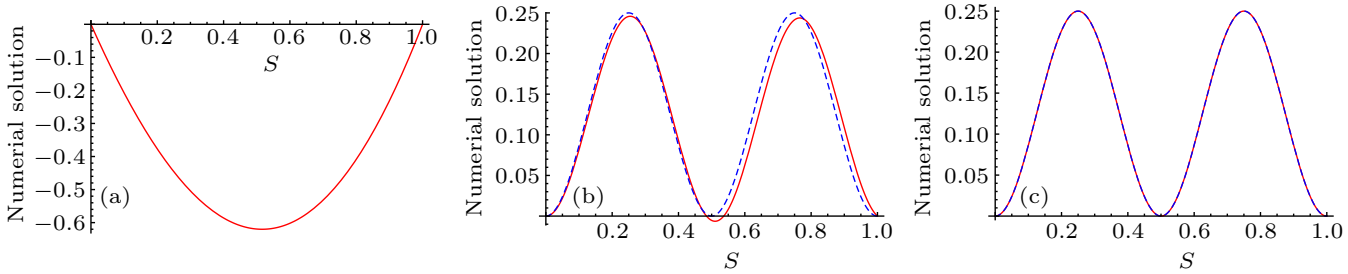


Fig. 6. Exact solution (blue line) and numerical solutions (red line) of Example 3 by (a) RK method, (b) AMPIRK method ($h = 0.01$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 10^{-6}$, $\varpi = 10^{-5}$) at $t = 0.5$.

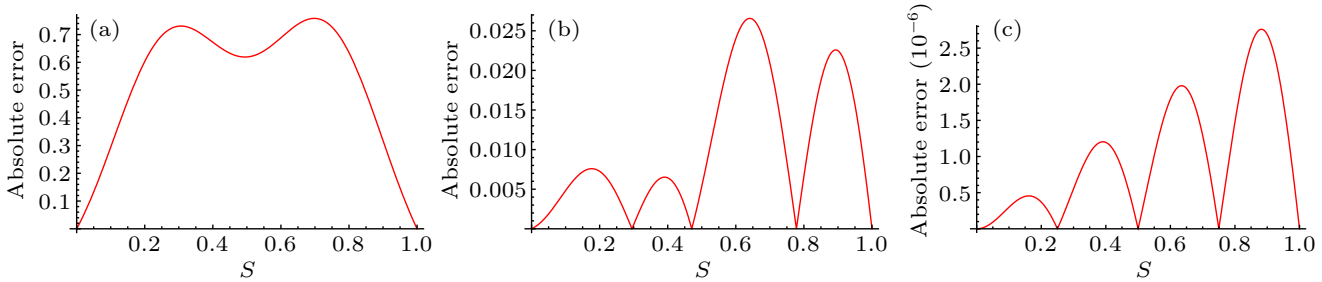


Fig. 7. Absolute errors of Example 3 by (a) RK method, (b) AMPIRK method ($h = 0.01$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 10^{-6}$, $\varpi = 10^{-5}$) at $t = 0.5$.

Example 4 Considering the following equation:[8]

$$\frac{\partial^\alpha V(S,t)}{\partial t^\alpha} - \varepsilon \frac{\partial^2 V(S,t)}{\partial S^2} + \mu \frac{\partial V(S,t)}{\partial S} + \kappa V(S,t) = f(S,t), \quad (7)$$

where the initial and boundary conditions are $V(S,0) = S^3 + S^2 + 1$, $V(0,t) = (t+1)^2$ and $V(1,t) = 3(t+1)^3$,

$$f(S,t) = \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)}\right)(S^3 + S^2 + 1) - (t+1)^2\left[\varepsilon(2+6S) + \mu(2S+3S^2) - \kappa(S^3 + S^2 + 1)\right],$$

where $\varepsilon = 1$, $\mu = 0.5$, $\kappa = 0.5$, the exact solution of the above

problem is $V(S,t) = (t+1)^2(S^3+S^2+1)$. Using mathematica 7.0, let $\alpha = 0.9$, $N = 6$. Figure 8 shows the exact solution (blue line) and the numerical solutions (red line) with (a) the RK method, (b) AMPIRK method ($h = 0.01$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 10^{-8}$, $\varpi = 10^{-7}$) at $t = 1$. Figure 9 shows the absolute errors with (a) RK method, (b) AMPIRK method ($h = 0.01$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 10^{-8}$, $\varpi = 10^{-7}$) at $t = 1$.

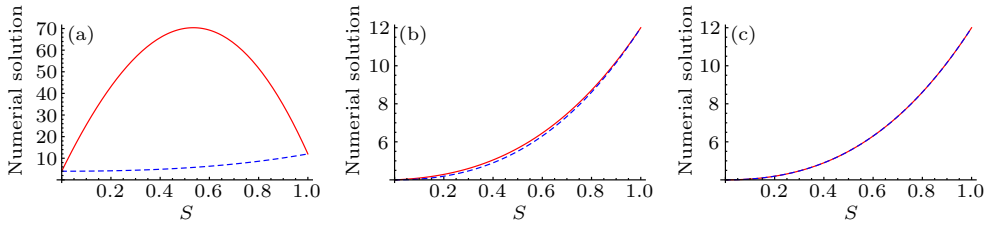


Fig. 8. Exact solution (blue line) and numerical solutions (red line) of Example 4 by (a) RK method, (b) AMPIRK method ($h = 0.01$, $\varpi = 0.1$) and (c) present method ($h = 10^{-8}$, $\varpi = 10^{-7}$) at $t = 1$.

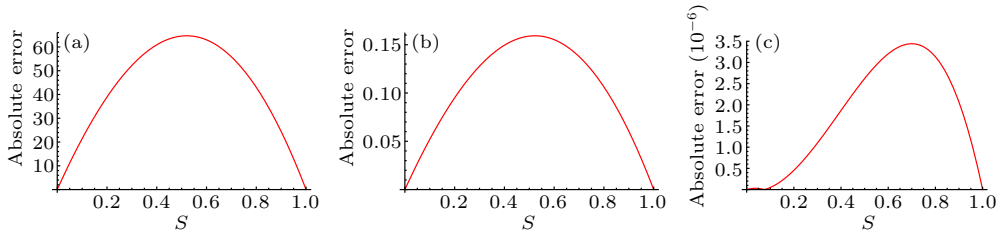


Fig. 9. Absolute errors of Example 4 by (a) RK method, (b) AMPIRK method ($h = 0.01$, $\varpi = 0.1$) and (c) AMPIRK method ($h = 10^{-8}$, $\varpi = 10^{-7}$) at $t = 1$.

4. Conclusion

In this paper, the AMPIRK method is proposed for solving the nonlinear time-fractional partial differential equations arising from option pricing model in financial economics, which combines the advantages of adaptive multi-step algorithm and precision of the piecewise interpolation technology. By properly choosing piecewise number, we can reduce the workload and ensure the accuracy of calculation. The four numerical examples indicate that the new method is a higher-precision and more time-saving numerical method compared with other numerical methods. The new method in this paper provides a powerful mathematical tool for the field of financial economics and can help investors and regulators better understand these financial derivatives. This study has important research significance and development prospect in the field of financial economics.

However, we also find a problem in the numerical calculation: for some more complex equations, the calculation process is very long. So it is not enough to slice one interval, and the calculation results will not reach the expected effect. We consider that if we divide the two intervals into the solution region at the same time, adopting the technique of stretching and then restoring, the problem will become gentle and the numerical solutions will be obtained more accurately. This will be the next problem we should break through.

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