Dynamical analysis for the sustained harvesting of microorganisms using flocculants in a random environment

Rong Liu(刘蓉)[1,2] and Wanbiao Ma(马万彪)[1,†]

[1] *School of Mathematics and Physics, University of Science and Technology Beijing, Beijing* 100083, *China*
[2] *School of Science, Chongqing University of Posts and Telecommunications, Chongqing* 400065, *China*

(Received 4 May 2022; revised manuscript received 24 August 2022; accepted manuscript online 21 September 2022)

The use of flocculants to collect/extract microorganisms is of great practical significance for the development of the application of microorganisms. In this paper, a high-dimensional nonlinear stochastic differential equation model is constructed to describe the continuous culture of microorganisms with multiple nutrients and the flocculation process of microorganisms. The study of the dynamics of this model can provide feasible control strategies for the collection/extraction of microorganisms. The main theoretical results are sufficient conditions for the permanence and extinction of the stochastic differential equation model, which are also extensions of some results in the existing literatures. In addition, through numerical simulations, we vividly demonstrate the statistical characteristics of the stochastic differential equation model.

Keywords: stochastic process, stochastically ultimately bounded, stochastic permanence, noise-induced transitions

PACS: 05.40.–a, 02.50.–r, 02.50.Ga, 02.50.Ng **DOI:** 10.1088/1674-1056/ac9367

1. Introduction

As one of the future directions of the development of eco-friendly energy, it is extremely important to extract industrial products such as biodiesel from some important microorganisms. However, during the continuous culture of microorganisms, in order to obtain more biomass, the main problem to be solved include the selection of appropriate nutrients. And, how to extract/collect microorganisms from the culture medium is also one of the problems to be solved. These are also important factors restricting the wide application of microorganism products.[1] The use of flocculation to extract/collect microorganisms is one of the main methods used at present. Though, the development of efficient, eco-friendly and low-cost flocculants remains one of the main research topics in the field of biotechnology.[2,3] In addition, in the process of continuous culture of microorganisms, various chemical regulators are disturbed by noise in regulating the growth of microorganisms.[4] Therefore, how to realize the continuous culture of microorganisms and efficient extraction/collection of microorganisms by flocculation method is an important issue to be studied.

This paper attempts to make a preliminary study of this issue by using the methods of stochastic differential equations and gives some feasible control strategies. Recently, there are some relevant studies on the use of mathematical models to describe the continuous culture of microorganisms and the flocculation process of microorganisms.[5–9] Zhang *et al.*[7] investigated a stochastic microorganism flocculation model and obtained the existence of a stationary distribution when the noise intensities are small. In Ref. [9], Guo *et al.* obtained complete conclusions on the permanence of a time-delayed microorganism flocculation model with general monotonic response functions.

Permanence is an important and well-established concept for population models, because the permanence of a model guarantees that all interacting species in the model can coexist.[10] In the microorganism flocculation model, permanence means that the microorganism populations will survive permanently. There have been numerous results on the permanence of stochastic models, see, e.g., Refs. [11–21] and the references therein. Typically, sufficient conditions for persistence in the mean can be obtained by constructing suitable auxiliary functions. However, a more meticulous theoretical analysis is needed for the stochastic permanence of high-dimensional nonlinear stochastic differential equation models.

Here, we propose the following stochastic microorganism flocculation model:

$$\begin{cases} \mathrm{d}C_i(t) = [D(C_i^0 - C_i(t)) - \dfrac{r_i}{\delta_i} f_i(C_i(t))X(t)]\,\mathrm{d}t \\ \qquad\qquad + \sigma_i C_i(t)\,\mathrm{d}B_i(t), \quad i = 1,\ldots,n, \\ \mathrm{d}X(t) = \left[\displaystyle\sum_{i=1}^{n} r_i f_i(C_i(t))X(t) - DX(t) - m_1 X(t)P(t)\right]\mathrm{d}t \\ \qquad\qquad + \sigma_{n+1} X(t)\,\mathrm{d}B_{n+1}(t), \\ \mathrm{d}P(t) = [D(P^0 - P(t)) - m_2 X(t)P(t)]\,\mathrm{d}t \\ \qquad\qquad + \sigma_{n+2} P(t)\,\mathrm{d}B_{n+2}(t), \end{cases} \tag{1}$$

where $C_i(t)$, $X(t)$ and $P(t)$ represent the concentrations of the i-th nutrient, microorganisms and flocculants at time t, respectively. C_i^0 and P^0 represent the input concentrations of the i-th nutrient and flocculants, respectively. $D > 0$ is the dilu-

†Corresponding author. E-mail: wanbiao_ma@ustb.edu.cn

© 2023 Chinese Physical Society and IOP Publishing Ltd

tion rate. The nonlinear term $\frac{r_i}{\delta_i} f_i(C_i(t))X(t)$ is the amount of the i-th nutrient that is consumed by microorganisms per unit time, where $\delta_i > 0$ represents yield coefficient, $i \in \{1,\ldots,n\}$. The term $\sum_{i=1}^{n} r_i f_i(C_i(t))X(t)$ is the amount of microorganisms growth per unit of time. Throughout this paper, we make the following assumptions on each nutrient uptake function: $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone increasing and locally Lipschitz with $f_i(0) = 0$; there exists $\zeta_i > 0$, such that $f_i(C_i) \le \zeta_i C_i$ for $C_i \in [0,\infty)$, $i \in \{1,\ldots,n\}$. The term $m_1 X(t)P(t)$ is the amount of microorganisms extracted by flocculants. The term $m_2 X(t)P(t)$ is the amount of flocculants for consumed. r_i, m_1 and m_2 are all positive constants.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions. $B_1(t)$, …, $B_{n+2}(t)$ are mutually independent standard Brownian motions defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},\mathbb{P})$. σ_i^2 denotes the intensity of white noise, $i \in \{1,\ldots,n+2\}$. Let $Z(t) = (C_1(t),\ldots,C_n(t),X(t),P(t))^{\mathrm{T}}$ be the solution of model (1) with the initial value $Z(0) = (C_1(0),\ldots,C_n(0),X(0),P(0))^{\mathrm{T}} \in \mathbb{R}_+^{n+2}$, where

$$\mathbb{R}_+^{n+2} = \{x = (x_1,\ldots,x_{n+2}) \in \mathbb{R}^{n+2} : x_i > 0,\ \ i = 1,\ldots,n+2\}.$$

Define $|Z(t)| = \sqrt{C_1^2(t) + \ldots + C_n^2(t) + X^2(t) + P^2(t)}$.

By using the standard method as that in Refs. [7,8,22,23], it can be easily obtained that the well-posedness of model (1). We summarise this as the following theorem.

Theorem 1 For any given initial value $Z(0) \in \mathbb{R}_+^{n+2}$, there is a unique solution $Z(t)$ of model (1) on $t \ge 0$ and the solution will remain in $\mathbb{R}_+^{n+2}$ with probability 1.

2. Stochastically ultimate boundedness

Definition 1[12] The solution $Z(t)$ of model (1) is said to be stochastically ultimately bounded, if for any $\varepsilon \in (0,1)$, there is a positive constant $H = H(\varepsilon)$ such that for any initial value $Z(0) \in \mathbb{R}_+^{n+2}$, the solution $Z(t)$ of model (1) has the property

$$\limsup_{t\to\infty} \mathbb{P}\{|Z(t)| > H\} < \varepsilon.$$

Theorem 2 The solution $Z(t)$ of model (1) is stochastically ultimately bounded.

Proof Define

$$V(Z(t)) = X(t) + P(t) + \sum_{i=1}^{n} \delta_i C_i(t) \ \text{ for } \ t \ge 0.$$

Then, we have

$$\begin{aligned} \mathrm{d}V(Z(t)) \ &= \ LV(Z(t))\,\mathrm{d}t + \sum_{i=1}^{n} \delta_i \sigma_i C_i(t)\,\mathrm{d}B_i(t) \\ &\quad + \sigma_{n+1} X(t)\,\mathrm{d}B_{n+1}(t) + \sigma_{n+2} P(t)\,\mathrm{d}B_{n+2}(t) \\ &\le \ \left[\sum_{i=1}^{n} D\delta_i C_i^0 + DP^0 - DV(Z(t))\right]\mathrm{d}t \\ &\quad + \sum_{i=1}^{n} \delta_i \sigma_i C_i(t)\,\mathrm{d}B_i(t) + \sigma_{n+1} X(t)\,\mathrm{d}B_{n+1}(t) \\ &\quad + \sigma_{n+2} P(t)\,\mathrm{d}B_{n+2}(t). \end{aligned}$$

By Itô's formula, we have

$$\begin{aligned} &\mathrm{d}[\mathrm{e}^{Dt} V(Z(t))] \\ &= \ \mathrm{e}^{Dt}[DV(Z(t))\,\mathrm{d}t + \mathrm{d}V(Z(t))] \\ &\le \ \mathrm{e}^{Dt}\left(\sum_{i=1}^{n} D\delta_i C_i^0 + DP^0\right)\mathrm{d}t + \mathrm{e}^{Dt}\sum_{i=1}^{n} \delta_i \sigma_i C_i(t)\,\mathrm{d}B_i(t) \\ &\quad + \mathrm{e}^{Dt}\sigma_{n+1} X(t)\,\mathrm{d}B_{n+1}(t) + \mathrm{e}^{Dt}\sigma_{n+2} P(t)\,\mathrm{d}B_{n+2}(t). \end{aligned}$$

Further, we have

$$\mathrm{e}^{Dt} EV(Z(t)) \le V(Z(0)) + \left(\sum_{i=1}^{n} \delta_i C_i^0 + P^0\right)\mathrm{e}^{Dt},$$

which implies that

$$\limsup_{t\to\infty} EV(Z(t)) \le \sum_{i=1}^{n} \delta_i C_i^0 + P^0.$$

Obviously, it follows from the above inequality that

$$\limsup_{t\to\infty} E[|Z(t)|] \le \frac{\sum_{i=1}^{n} \delta_i C_i^0 + P^0}{\min\{\delta_1,\ldots,\delta_n,1\}} := \Theta.$$

For any $\varepsilon > 0$, let us define $H(\varepsilon) = 2\Theta/\varepsilon$. By Chebyshev's inequality,

$$\limsup_{t\to\infty} \mathbb{P}\{|Z(t)| > H(\varepsilon)\} \le \limsup_{t\to\infty} \frac{E[|Z(t)|]}{H(\varepsilon)} < \varepsilon.$$

Therefore, the solution $Z(t)$ of model (1) is stochastically ultimately bounded.

3. Stochastic permanence and extinction

Now, we give the definition of stochastic permanence with respect to model (1).

Definition 2[11] Model (1) is said to be stochastically permanent, if for any $\varepsilon \in (0,1)$, there exist positive constants $H = H(\varepsilon)$, $\alpha = \alpha(\varepsilon)$ such that for any initial value $Z(0) \in \mathbb{R}_+^{n+2}$, the solution $Z(t)$ of model (1) has the properties

$$\begin{aligned} &\liminf_{t\to\infty} \mathbb{P}\{X(t) \le H\} \ge 1-\varepsilon, \\ &\liminf_{t\to\infty} \mathbb{P}\{X(t) \ge \alpha\} \ge 1-\varepsilon. \end{aligned}$$

Define the following equations:

$$\begin{cases} \mathrm{d}\widetilde{C}_i(t) = [D(C_i^0 - \widetilde{C}_i(t))]\,\mathrm{d}t + \sigma_i \widetilde{C}_i(t)\,\mathrm{d}B_i(t),\ i = 1,\ldots,n, \\ \mathrm{d}\widetilde{P}(t) = [D(P^0 - \widetilde{P}(t))]\,\mathrm{d}t + \sigma_{n+2}\widetilde{P}(t)\,\mathrm{d}B_{n+2}(t). \end{cases} \tag{2}$$

For each $i \in \{1,\ldots,n\}$, let $\widetilde{C}_i(t)$ be the solution of the i-th equation in Eq. (2) with any positive initial value $\widetilde{C}_i(0)$. $\widetilde{P}(t)$ denotes the solution of the last equation in Eq. (2) with any positive initial value $\widetilde{P}(0)$. Then, based on Lemma 4.1

in Ref. [24], there exists a positive solution $C_i^*(t)$ which is globally attractive for the i-th equation. From Lemma 3.3 in Ref. [25], we can easily obtain that $\widetilde{C}_i(t)$ converges weakly to a unique ergodic stationary distribution $\pi_i(\cdot)$ for each $i \in \{1,\dots,n\}$, $\widetilde{P}(t)$ converges weakly to a unique ergodic stationary distribution $\nu(\cdot)$ and $\int_0^\infty \widetilde{P}\nu(\mathrm{d}\widetilde{P}) = P^0$. By using a thorough analytical method (see Ref. [16]), we obtain the sharp condition which guarantees the stochastic permanence of model (1).

Theorem 3 Suppose that

$$\lambda := \left(\sum_{i=1}^{n} r_i \int_0^\infty f_i(C_i^*)\pi_i(\mathrm{d}C_i^*) - D - \frac{\sigma_{n+1}^2}{2} - m_1 P^0\right) > 0$$

holds, then model (1) is stochastically permanent.

Proof By virtue of the ergodic property, we have

$$\lim_{t\to\infty}\left(\sum_{i=1}^{n}\frac{r_i}{t}\int_0^t f_i(C_i^*(s))\mathrm{d}s - D - \frac{\sigma_{n+1}^2}{2} - \frac{m_1}{t}\int_0^t \widetilde{P}(s)\mathrm{d}s\right) = \lambda.$$

Hence, there is a pair of positive constants γ and T such that for any continuous function $y_i(t)$ satisfying

$$|y_i(t) - C_i^*(t)| < \gamma, \text{ for } t > T,\ i \in \{1,\dots,n\},$$

we have

$$\liminf_{t\to\infty}\left(\sum_{i=1}^{n}\frac{r_i}{t}\int_0^t f_i(y_i(s))\mathrm{d}s - D - \frac{\sigma_{n+1}^2}{2} - \frac{m_1}{t}\int_0^t \widetilde{P}(s)\mathrm{d}s\right) > 0 \quad \text{a.s.} \tag{3}$$

It follows from Theorem 2 that, for any $\varepsilon \in (0,1)$ and each $i \in \{1,\dots,n\}$, there exits a positive constant $H(\varepsilon)$ such that

$$\begin{aligned}
&\liminf_{t\to\infty}\mathbb{P}\{C_i(t) \le H(\varepsilon)\} \ge 1 - \frac{\varepsilon}{2},\\
&\liminf_{t\to\infty}\mathbb{P}\{X(t) \le H(\varepsilon)\} \ge 1 - \frac{\varepsilon}{2},\\
&\liminf_{t\to\infty}\mathbb{P}\{P(t) \le H(\varepsilon)\} \ge 1 - \frac{\varepsilon}{2}.
\end{aligned} \tag{4}$$

Define

$$\begin{aligned}
\mathcal{M}_t^\varepsilon :=\ & \{\omega \in \Omega : C_1(t,\omega) \le H(\varepsilon),\dots,C_n(t,\omega) \le H(\varepsilon),\\
& X(t,\omega) \le H(\varepsilon), P(t,\omega) \le H(\varepsilon)\},\\
\mathcal{M}_\infty^\varepsilon :=\ & \{\omega \in \Omega : \liminf_{t\to\infty}\mathbb{P}\{\mathcal{M}_t^\varepsilon\} \ge 1-\varepsilon\}.
\end{aligned}$$

According to Eq. (4), we have $\mathbb{P}\{\mathcal{M}_\infty^\varepsilon\} \ge 1-\varepsilon$. Thus, for any $\varepsilon_0 \in (0, 1-\varepsilon)$, there exits some $T_1 > T$ satisfying $\mathbb{P}(\mathcal{M}_t^\varepsilon) \ge 1-\varepsilon-\varepsilon_0$ for $t \ge T_1$. Thus, for any $\omega \in \mathcal{M}_t^\varepsilon$, we have

$$\begin{aligned}
&0 \le C_1(t,\omega) \le H(\varepsilon),\ \dots,\ 0 \le C_n(t,\omega) \le H(\varepsilon),\\
&0 \le X(t,\omega) \le H(\varepsilon),\quad 0 \le P(t,\omega) \le H(\varepsilon)
\end{aligned} \tag{5}$$

for $t \ge T_1$.

In the following, we prove that, for any $\varepsilon \in (0,1)$, there always exists a positive constant $\alpha(\varepsilon) \in (0, H(\varepsilon))$ satisfying

$$\liminf_{t\to\infty}\mathbb{P}\{X(t) \ge \alpha(\varepsilon)\} \ge 1-\varepsilon.$$

In order to obtain the result, we first prove the following claim.

Claim For any $\omega \in \mathcal{M}_\infty^\varepsilon$, there is a positive constant $\alpha_1(\varepsilon)$ such that $\limsup_{t\to\infty} X(t,\omega) > \alpha_1(\varepsilon)$.

For any sufficiently small constant $\alpha_1 > 0$, let us consider the following equations:

$$\begin{cases}
\mathrm{d}c_1(t) = [DC_1^0 - (D + \frac{r_1\zeta_1}{\delta_1}\alpha_1)c_1(t)]\mathrm{d}t + \sigma_1 c_1(t)\mathrm{d}B_1(t),\\
\quad\vdots\\
\mathrm{d}c_n(t) = [DC_n^0 - (D + \frac{r_n\zeta_n}{\delta_n}\alpha_1)c_n(t)]\mathrm{d}t + \sigma_n c_n(t)\mathrm{d}B_n(t).
\end{cases} \tag{6}$$

By Lemma 4.1 in Ref. [24], for the i-th equation in Eq. (6), it has a unique positive solution $c_i^*(t)$ which is globally attractive. Note that, let α_1 tend to zero, $c_i^*(t) \xrightarrow{\text{a.s.}} C_i^*(t)$. Hence, there exits a positive constant $\alpha_1(\varepsilon) < \varepsilon \wedge \gamma$ such that

$$c_i^*(t) > C_i^*(t) - \alpha_1(\varepsilon)/2 \text{ for } t \ge 0,\ i \in \{1,\dots,n\},\ \text{a.s.} \tag{7}$$

If the claim would not hold, then there exist a solution $\widehat{Z}(t) := (\widehat{C}_1(t),\dots,\widehat{C}_n(t),\widehat{X}(t),\widehat{P}(t))^{\mathrm{T}}$ of model (1) with positive initial value $\widehat{Z}(0) > 0$, a nonempty set $\mathcal{M}_1 \in \mathcal{M}_t^\varepsilon$ with $\mathbb{P}(\mathcal{M}_1) > 0$, and a constant $T_2 > T_1$ such that $\widehat{X}(t,\omega) \le \alpha_1(\varepsilon)$ for all $\omega \in \mathcal{M}_1$ and $t > T_2$. Hence, it has

$$\begin{cases}
\mathrm{d}\widehat{C}_1(t) \ge [D(C_1^0 - \widehat{C}_1(t)) - \frac{r_1\zeta_1}{\delta_1}\alpha_1(\varepsilon)\widehat{C}_1(t)]\mathrm{d}t\\
\qquad\qquad + \sigma_1\widehat{C}_1(t)\mathrm{d}B_1(t),\\
\qquad\vdots\\
\mathrm{d}\widehat{C}_n(t) \ge [D(C_n^0 - \widehat{C}_n(t)) - \frac{r_n\zeta_n}{\delta_n}\alpha_1(\varepsilon)\widehat{C}_n(t)]\mathrm{d}t\\
\qquad\qquad + \sigma_n\widehat{C}_n(t)\mathrm{d}B_n(t),
\end{cases}$$

for all $\omega \in \mathcal{M}_1$ and $t > T_2$. Obviously,

$$\begin{cases}
\mathrm{d}\widehat{C}_1(t) \le [D(C_1^0 - \widehat{C}_1(t))]\mathrm{d}t + \sigma_1\widehat{C}_1(t)\mathrm{d}B_1(t),\\
\quad\vdots\\
\mathrm{d}\widehat{C}_n(t) \le [D(C_n^0 - \widehat{C}_n(t))]\mathrm{d}t + \sigma_n\widehat{C}_n(t)\mathrm{d}B_n(t).
\end{cases}$$

Then, by the stochastic comparison theorem, the global attractivity of $C_i^*(t)$ and Eq. (7), there exists a positive constant $T_3 > T_2$ satisfying

$$|\widehat{C}_i(t) - C_i^*(t)| < \alpha_1(\varepsilon),\ i \in \{1,\dots,n\}, \tag{8}$$

for all $\omega \in \mathcal{M}_1$ and $t > T_3$. From model (1), for all $\omega \in \mathcal{M}_1$ and $t > T_3$, then we have

$$\begin{aligned}
\frac{\ln\widehat{X}(t)}{t} \ge\ & \frac{\ln\widehat{X}(T_3)}{t} + \frac{1}{t}\int_{T_3}^t\left[\sum_{i=1}^{n} r_i f_i(\widehat{C}_i(s)) - D - \frac{\sigma_{n+1}^2}{2} - m_1\widetilde{P}(s)\right]\mathrm{d}s\\
& + \frac{1}{t}\int_{T_3}^t \sigma_{n+1}\mathrm{d}B_{n+1}(s).
\end{aligned}$$

Therefore, combining Eqs. (3) and (8) yields

$$\liminf_{t\to\infty}\frac{1}{t}\ln\widehat{X}(t) \geq \liminf_{t\to\infty}\left(\sum_{i=1}^{n}\frac{r_i}{t}\int_0^t f_i(\widehat{C}_i(s))\mathrm{d}s - D - \frac{\sigma_{n+1}^2}{2} - \frac{m_1}{t}\int_0^t \widetilde{P}(s)\mathrm{d}s\right) > 0,$$

a.s. in $\mathcal{M}_1$ which contradicts to Theorem 2. Hence, the claim is proved.

Next, we will show that there exists a constant $\alpha(\varepsilon)\in(0,\alpha_1(\varepsilon)]$ such that

$$\liminf_{t\to\infty} X(t,\omega) > \alpha(\varepsilon),$$

for any $\omega\in\mathcal{M}_\infty^\varepsilon$. Assume, by contradiction, there exist a sequence $\{Z_k\}\subset\mathbb{R}_+^{n+2}$, a nonempty set $\mathcal{M}_2\subset\mathcal{M}_t^\varepsilon$ with $\mathbb{P}(\mathcal{M}_2)>0$, and a positive constant $T_4>T_1$ such that

$$\inf_{t\geq T_4}\widehat{X}(t,Z_k) < \frac{\alpha_1(\varepsilon)}{k+1}$$

for all $\omega\in\mathcal{M}_2$, where $(\widehat{C}_1(t,Z_k),\ \ldots,\ \widehat{C}_n(t,Z_k),\ \widehat{X}(t,Z_k),\ \widehat{P}(t,Z_k))$ is a solution of model (1) with initial value Z_k.

According to the claim, there exist a nonempty set $\mathcal{M}_3\subseteq\mathcal{M}_2$ with $\mathbb{P}(\mathcal{M}_3)>0$, and positive constants T_5, T_6 ($T_6>T_5>T_4$) satisfying

$$\widehat{X}(T_5,Z_k)\geq\alpha_1(\varepsilon),\quad \widehat{X}(T_6,Z_k)\leq\frac{\alpha_1(\varepsilon)}{k+1}$$

for all $\omega\in\mathcal{M}_3$. Then, for $\omega\in\mathcal{M}_3$, there exist $\tau_k\in(T_5,T_6]$ satisfying $\widehat{X}(\tau_k,Z_k)=\alpha_1(\varepsilon)/(k+1)$ and $\tau\in[T_5,\tau_k)$ satisfying $\widehat{X}(\tau,Z_k)=\alpha_1(\varepsilon)$. Moreover, when $t\in(\tau,\tau_k)$,

$$\alpha_1(\varepsilon)/(k+1) < \widehat{X}(t,Z_k) < \alpha_1(\varepsilon). \tag{9}$$

By Eq. (5), we have

$$\begin{aligned}&\frac{\mathrm{d}E[\ln\widehat{X}(t,Z_k)I_{\mathcal{M}_3}]}{\mathrm{d}t}\\ &= \left(\sum_{i=1}^{n} r_i f_i(\widehat{C}_i(t)) - D - \frac{\sigma_{n+1}^2}{2} - m_1\widehat{P}(t)\right)\mathbb{P}(\mathcal{M}_3)\\ &\geq (-D-\frac{\sigma_{n+1}^2}{2} - m_1 H(\varepsilon))\mathbb{P}(\mathcal{M}_3).\end{aligned}$$

Then, integrating the above inequality from τ to τ_k,

$$\begin{aligned}&\left(D+\frac{\sigma_{n+1}^2}{2}+m_1H(\varepsilon)\right)(\tau_k-\tau)\\ &\geq -\ln\widehat{X}(\tau_k,Z_k)+\ln\widehat{X}(\tau,Z_k) = \ln(k+1).\end{aligned}$$

Obviously, we obtain that $\tau_k-\tau\to\infty$ as $k\to\infty$.

By $\lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_{n+1}\mathrm{d}B_{n+1}(s)=0$ a.s. and Eq. (3), there exist positive constants ε_1, ς and T_0 such that for any $\tilde{t}\geq 0$ and $t-\tilde{t}\geq T_0$, we have

$$\left|\frac{1}{t}\int_{\tilde{t}}^t\sigma_{n+1}\mathrm{d}B_{n+1}(s)\right| < \varepsilon_1 \text{ a.s.}, \tag{10}$$

$$\frac{1}{t}\int_{\tilde{t}}^t\left(\sum_{i=1}^{n} r_i f_i(y_i(s)) - D - \frac{\sigma_{n+1}^2}{2} - m_1\widetilde{P}(s)\right)\mathrm{d}s - \varepsilon_1 > \varsigma \text{ a.s.} \tag{11}$$

For any $t\in[\tau,\tau_k]$ and $\omega\in\mathcal{M}_3$, it has

$$\begin{cases}\mathrm{d}\widehat{C}_1(t,Z_k)\geq[D(C_1^0-\widehat{C}_1(t,Z_k))-\dfrac{r_1\zeta_1}{\delta_1}\alpha_1(\varepsilon)\widehat{C}_1(t,Z_k)]\mathrm{d}t\\ \qquad\qquad +\sigma_1\widehat{C}_1(t,Z_k)\mathrm{d}B_1(t),\\ \qquad\qquad\vdots\\ \mathrm{d}\widehat{C}_n(t,Z_k)\geq[D(C_n^0-\widehat{C}_n(t,Z_k))-\dfrac{r_n\zeta_n}{\delta_n}\alpha_1(\varepsilon)\widehat{C}_n(t,Z_k)]\mathrm{d}t\\ \qquad\qquad +\sigma_n\widehat{C}_n(t,Z_k)\mathrm{d}B_n(t),\end{cases}$$

$$\begin{cases}\mathrm{d}\widehat{C}_1(t,Z_k)\leq[D(C_1^0-\widehat{C}_1(t,Z_k))]\mathrm{d}t+\sigma_1\widehat{C}_1(t,Z_k)\mathrm{d}B_1(t),\\ \qquad\vdots\\ \mathrm{d}\widehat{C}_n(t,Z_k)\leq[D(C_n^0-\widehat{C}_n(t,Z_k))]\mathrm{d}t+\sigma_n\widehat{C}_n(t,Z_k)\mathrm{d}B_n(t).\end{cases}$$

Similar to the argument in the proof of claim, we can show that, there exists $T_0'>T_0$ such that

$$|\widehat{C}_i(t)-C_i^*(t)|<\alpha_1(\varepsilon),\ i\in\{1,\ldots,n\},$$

for $t\geq T_0'$ and $\omega\in\mathcal{M}_3$. And, there exists some $K>0$ such that $\tau_k>2T_0'+\tau$ for all $k\geq K$. Thus, for all $t\in[T_0'+\tau,\tau_k]$ and $\omega\in\mathcal{M}_3$,

$$|\widehat{C}_i(t)-C_i^*(t)|<\alpha_1(\varepsilon),\quad i\in\{1,\ldots,n\}. \tag{12}$$

Further, we have

$$\begin{aligned}\ln\widehat{X}(\tau_k,Z_k) \geq\ & \ln\widehat{X}(T_0'+\tau,Z_k)+\int_{T_0'+\tau}^{\tau_k}\left(\sum_{i=1}^{n} r_i f_i(\widehat{C}_i(s))\right.\\ &\left. - D - \frac{\sigma_{n+1}^2}{2} - m_1\widetilde{P}(s)\right)\mathrm{d}s\\ &-\left|\int_{T_0'+\tau}^{\tau_k}\sigma_{n+1}\mathrm{d}B_{n+1}(s)\right|.\end{aligned}$$

By Eq. (9), we have

$$\begin{aligned}\frac{\alpha_1(\varepsilon)}{k+1} \geq\ & \frac{\alpha_1(\varepsilon)}{k+1}\exp\left\{\int_{T_0'+\tau}^{\tau_k}\left(\sum_{i=1}^{n} r_i f_i(\widehat{C}_i(s))\right.\right.\\ &\left. - D - \frac{\sigma_{n+1}^2}{2} - m_1\widetilde{P}(s)\right)\mathrm{d}s\\ &\left.-\left|\int_{T_0'+\tau}^{\tau_k}\sigma_{n+1}\mathrm{d}B_{n+1}(s)\right|\right\}.\end{aligned}$$

Hence, according to Eqs. (10) and (11), we have

$$\frac{\alpha_1(\varepsilon)}{k+1}\geq\frac{\alpha_1(\varepsilon)}{k+1}\exp\{\varsigma T_0'\}>\frac{\alpha_1(\varepsilon)}{k+1},$$

which is a contradiction. Thus, there exists a constant $\alpha(\varepsilon)\in(0,\alpha_1(\varepsilon)]$ such that $\liminf_{t\to\infty}X(t,\omega)>\alpha(\varepsilon)$ for any $\omega\in\mathcal{M}_\infty^\varepsilon$. Finally, from the claim and Fatou's Lemma, we can obtain

$$\liminf_{t\to\infty}\mathbb{P}\{\alpha(\varepsilon)<X(t)\leq H(\varepsilon)\}\geq 1-\varepsilon.$$

Remark 1 Theorem 3 shows that model (1) is stochastically permanent only under the condition $\lambda > 0$. From a biological point of view, the existence of the ergodic stationary distribution of a stochastic model implies that the stochastic model is persistent. We can similarly obtain sufficient conditions for the existence of the ergodic stationary distribution of model (1) by using a similar approach to that in Ref. [26]. However, for technical reasons in the proof process, obtaining sufficient conditions for the existence of the ergodic stationary distribution of model (1) requires extra constraints on the form of each nutrient uptake function f_i.

Theorem 4 If

$$\widetilde{\lambda} := \left(\sum_{i=1}^{n} r_i \int_0^{\infty} f_i(C_i^*)\pi_i(\mathrm{d}C_i^*) - D - \frac{\sigma_{n+1}^2}{2}\right) < 0,$$

for any given initial value $Z(0) \in \mathbb{R}_+^{n+2}$, the solution $Z(t)$ of model (1) has the property

$$\limsup_{t\to\infty} \frac{1}{t}\ln X(t) \le \widetilde{\lambda} < 0 \text{ a.s.}$$

By a method similar to that used in Ref. [8], the extinction of microorganisms of model (1) can be obtained, so the proof of the above theorem is omitted here.

Define

$$R_0 = \frac{\sum_{i=1}^{n} r_i f_i(C_i^0)}{D + m_1 P^0}, \quad \widehat{R}_0 = \frac{\sum_{i=1}^{n} r_i f_i(C_i^0)}{D}.$$

Remark 2 If $\sigma_i = 0$, $i \in \{1,\ldots,n+2\}$, then $\lambda = \sum_{i=1}^{n} r_i f_i(C_i^0) - D - m_1 P^0 > 0$ implies that $R_0 > 1$ and the corresponding deterministic model is uniformly persistent. In addition, $\widetilde{\lambda} = \sum_{i=1}^{n} r_i f_i(C_i^0) - D < 0$ implies that $\widehat{R}_0 < 1$ and the boundary equilibrium of the corresponding deterministic model is at least locally asymptotically stable. Hence, Theorem 3 and Theorem 4 cover and improve the relevant results in Refs. [6–8].

Example Consider the following stochastic differential equation model with two nutrients and Monod-type functional reactions:

$$\begin{cases} \mathrm{d}C_1(t) = \left[D(C_1^0 - C_1(t)) - \dfrac{r_1 C_1(t)X(t)}{\delta_1(K_1 + C_1(t))}\right]\mathrm{d}t \\ \qquad\quad +\sigma_1 C_1(t)\mathrm{d}B_1(t), \\ \mathrm{d}C_2(t) = \left[D(C_2^0 - C_2(t)) - \dfrac{r_2 C_2(t)X(t)}{\delta_2(K_2 + C_2(t))}\right]\mathrm{d}t \\ \qquad\quad +\sigma_2 C_2(t)\mathrm{d}B_2(t), \\ \mathrm{d}X(t) = \left[\displaystyle\sum_{i=1}^{2} \frac{r_i C_i(t)X(t)}{(K_i + C_i(t))} - DX(t) - m_1 X(t)P(t)\right]\mathrm{d}t \\ \qquad\quad +\sigma_3 X(t)\mathrm{d}B_3(t), \\ \mathrm{d}P(t) = \left[D(P^0 - P(t)) - m_2 X(t)P(t)\right]\mathrm{d}t \\ \qquad\quad +\sigma_4 P(t)\mathrm{d}B_4(t). \end{cases} \tag{A}$$

We fix some of the parameters as follows:

$$D = 1, \quad C_1^0 = 5, \quad C_2^0 = 2, \quad P^0 = 1, \quad \delta_1 = \delta_2 = 2,$$
$$K_1 = 1.5, \quad K_2 = 0.05, \quad m_1 = 1, \quad m_2 = 2.5.$$

Let

$$r_1 = 2.5, \quad r_2 = 0.3, \quad \sigma_1 = 0.1, \quad \sigma_2 = 0.05,$$
$$\sigma_3 = 0.2, \quad \sigma_4 = 0.15.$$

We can obtain

$$\lambda = \left(\sum_{i=1}^{2} r_i \int_0^{\infty} \frac{C_i^*}{K_i + C_i^*}\pi_i(\mathrm{d}C_i^*) - D - \frac{\sigma_3^2}{2} - m_1 P^0\right)$$
$$\approx 0.194 > 0.$$

It follows from Theorem 3 that the model (A) is stochastically permanent. In Fig. 1, for any given initial value $Z(0) \in \mathbb{R}_+^4$, the sample paths fall above the blue line with 80% probability and below the red line with 80% probability. In Fig. 2, for any given initial value $Z(0) \in \mathbb{R}_+^4$, the sample paths fall above the dark blue line with 99% probability and below the dark red line with 99% probability.

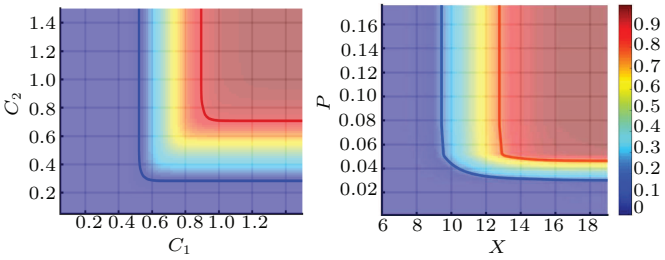


Fig. 1. Feasible domain diagram of the sample paths for model (A), where $\sigma_1 = 0.1$, $\sigma_2 = 0.05$, $\sigma_3 = 0.2$, $\sigma_4 = 0.15$. $\lambda \approx 0.194 > 0$.

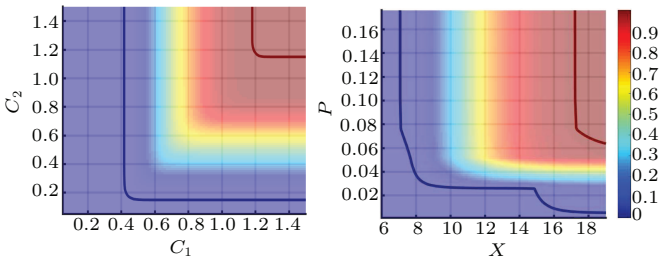


Fig. 2. Feasible domain diagram of the sample paths for model (A), where $\sigma_1 = 0.1$, $\sigma_2 = 0.05$, $\sigma_3 = 0.2$, $\sigma_4 = 0.15$. $\lambda \approx 0.194 > 0$.

Next, let

$$r_1 = 2, \quad r_2 = 0.6, \quad \sigma_1 = 0.1, \quad \sigma_2 = 0.05,$$
$$\sigma_3 = 1.5, \quad \sigma_4 = 0.15.$$

We can obtain

$$\widetilde{\lambda} = \left(\sum_{i=1}^{2} r_i \int_0^{\infty} \frac{C_i^*}{K_i + C_i^*}\pi_i(\mathrm{d}C_i^*) - D - \frac{\sigma_3^2}{2}\right) \approx -0.003 < 0.$$

Thus, it follows from Theorem 4 that the microorganisms in chemostat will decay exponentially to zero (see Fig. 3).

Now, let

$$r_1 = 1.2, \quad r_2 = 0.3, \quad \sigma_1 = \sigma_2 = \sigma_4 = 0.1, \quad \sigma_3 = 0.03.$$

The calculation gives that $\lambda \approx -0.786 < 0$ and $\widetilde{\lambda} \approx 0.214 > 0$, therefore, the conditions of Theorems 3 and 4 are not satisfied. It is important to note that the existence of flocculants may cause the backward bifurcation of the equilibrium in the corresponding deterministic model.[5,6,8] From a mathematical and biological point of view, the backward bifurcation implies that its corresponding deterministic model has complex and rich dynamical behaviors, and whether microorganism populations will persist or tend to become extinct will depend on the value of their initial conditions. For the stochastic model (A), our theoretical results reveal that the evolutionary patterns of persistence or extinction of microorganism populations do not change under moderate noise excitations. However, numerical simulations also reveal that a shift in the evolutionary patterns can occur when the intensity of the noise excitation is varied, which is very interesting.

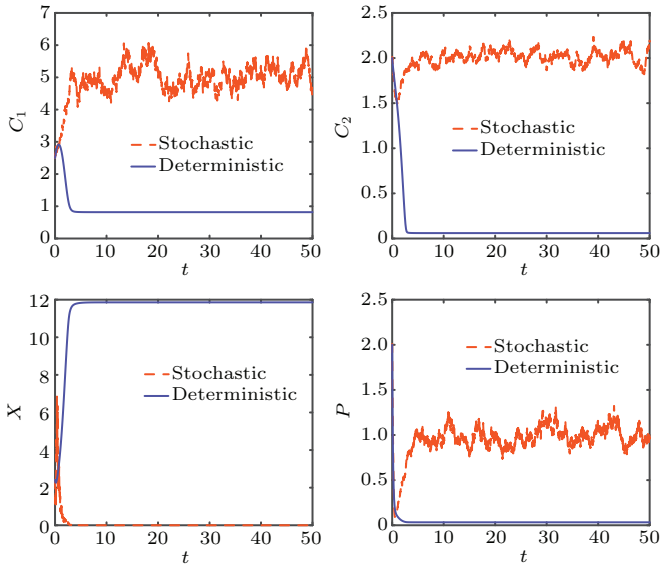


Fig. 3. Time evolutions of $C_1(t)$, $C_2(t)$, $X(t)$, $P(t)$ of stochastic model (A) and the solution curves of corresponding deterministic model, where $\sigma_1 = 0.1$, $\sigma_2 = 0.05$, $\sigma_3 = 1.5$, $\sigma_4 = 0.15$. $\widetilde{\lambda} \approx -0.003 < 0$. The initial value is taken as $(C_1(0), C_2(0), X(0), P(0)) = (2.5, 2, 2.42, 2)$.

In Fig. 4, we fix $D = 1$, $C_1^0 = 5$, $C_2^0 = 2$, $P^0 = 1$, $\delta_1 = \delta_2 = 2$, $K_1 = 1.5$, $K_2 = 0.05$, $m_1 = 1$, $m_2 = 2.5$ $r_1 = 1.2$, $r_2 = 0.3$, $\sigma_1 = \sigma_2 = \sigma_4 = 0.1$ and $\sigma_3 = 0.03$. And, we fix $C_2(0) = P(0) = 2$. By numerical simulations, we can divide (C_1, X) space roughly into 3 regions. If the initial value $(C_1(0), 2, X(0), 2)$ falls in region I, the abundance of microorganism populations converges to a sustainable level, which also means that microorganisms can be harvested sustainably. If the initial value $(C_1(0), 2, X(0), 2)$ falls in critical region II, the abundance of microorganism populations converges to a sustainable level with a certain probability. However, If the initial value $(C_1(0), 2, X(0), 2)$ falls in region III, microorganism populations will tend to become extinct, which also means that sustainable harvesting of microorganisms will fail. For instance, in Fig. 5, the initial value $(2.5, 2, 2.42, 2)$ falls in critical region II. We can see that for the corresponding deterministic model, the trajectory starting from $(2.5, 2, 2.42, 2)$ converges to the boundary equilibrium, but for the stochastic model, the sample path starting from $(2.5, 2, 2.42, 2)$ converges to the positive equilibrium.

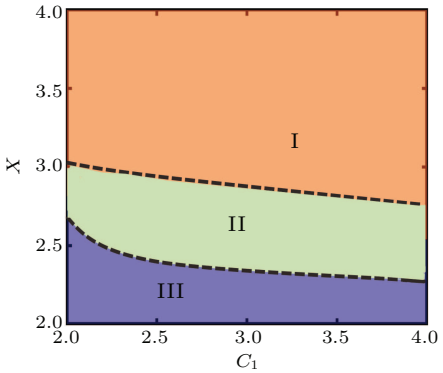


Fig. 4. Different regions in (C_1, X) space, where $\sigma_1 = \sigma_2 = \sigma_4 = 0.1$, $\sigma_3 = 0.03$. $\lambda \approx -0.786 < 0$ and $\widetilde{\lambda} \approx 0.214 > 0$.

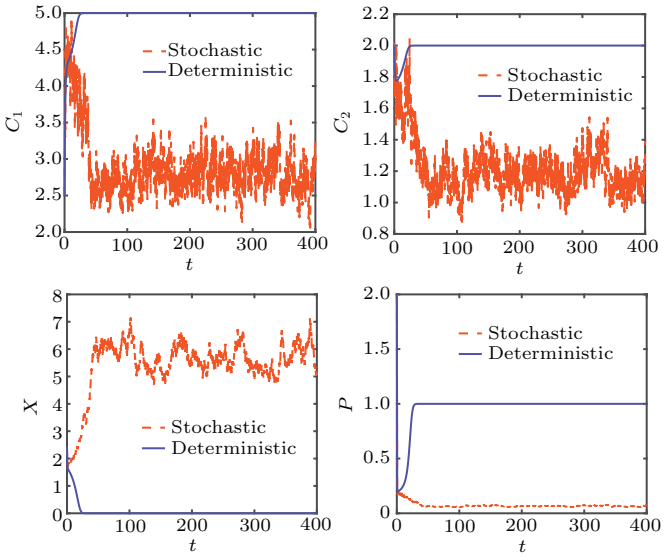


Fig. 5. Time evolutions of $C_1(t)$, $C_2(t)$, $X(t)$, $P(t)$ of stochastic model (A) and the solution curves of the corresponding deterministic model, where $\sigma_1 = \sigma_2 = \sigma_4 = 0.1$, $\sigma_3 = 0.03$. $\lambda \approx -0.786 < 0$ and $\widetilde{\lambda} \approx 0.214 > 0$. The initial value is taken as $(C_1(0), C_2(0), X(0), P(0)) = (2.5, 2, 2.42, 2)$.

Note that in Fig. 6, the initial value $(4.4, 0.5, 4.1, 6.5)$ is not in regions I–III, we can still observe a similar phenomenon, i.e, for the corresponding deterministic model the sample path starting from $(4.4, 0.5, 4.1, 6.5)$ converges to the boundary equilibrium, but for the stochastic model, the sample path starting from $(4.4, 0.5, 4.1, 6.5)$ converges to the positive equilibrium. From a biological point of view, stochastic perturbations can lead to a shift in the population status, that is, from the state of unsustainable harvest to the state of sustainable harvest. The mechanism behind this phenomenon is the emergence of noise-induced transitions in a strongly nonlinear model under noise excitation.[27–35] Nonlinear models with bifurcation can exhibit rich dynamic behaviors and there are many technical difficulties in the theoretical analysis of the

global dynamics of the models.[28–30] With the set of parameters we have chosen, the backward bifurcation can occur in the corresponding deterministic model. When there is noise excitation, then the sample paths from the attractive basin of one attractor of the deterministic model can oscillate and switch to some regions of another attractor of the deterministic model. The theoretical analysis of the mechanism of this switching phenomenon is also one of the topics for our further research.

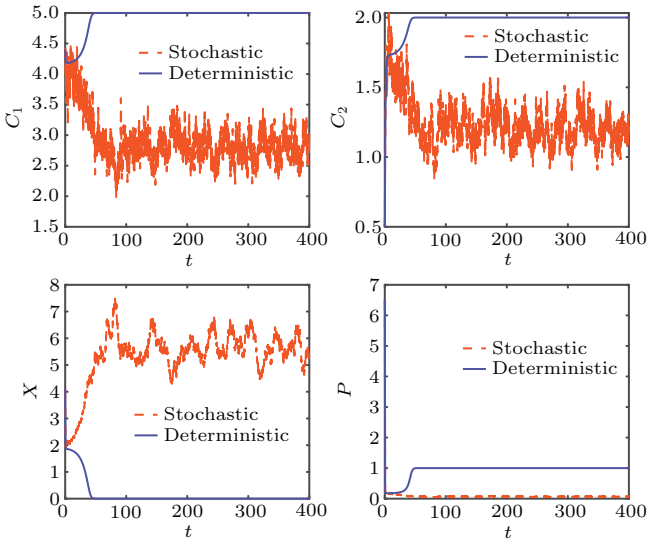


Fig. 6. Time evolutions of $C_1(t)$, $C_2(t)$, $X(t)$, $P(t)$ of stochastic model (A) and the solution curves of the corresponding deterministic model, where $\sigma_1 = \sigma_2 = \sigma_4 = 0.1$, $\sigma_3 = 0.03$. $\lambda \approx -0.786 < 0$ and $\widetilde{\lambda} \approx 0.214 > 0$. The initial value is taken as $(C_1(0), C_2(0), X(0), P(0)) = (4.4, 0.5, 4.1, 6.5)$.

4. Conclusion

In this paper, we have investigated a class of high-dimensional stochastic microorganism flocculation model with multiple substitutable nutrients and general functional response functions. By intensively analyzing the asymptotic behavior of solutions of the stochastic model, we obtained sufficient conditions for the stochastic permanence and extinction of the stochastic model. From a biological point of view, the stochastic permanence of the stochastic model indicates the sustainability of the collection of microorganisms. In addition, according to the numerical simulations, a shift in the evolutionary patterns of persistence or extinction of microorganism populations can occur when the intensity of the noise excitation is varied. The mechanism behind this phenomenon deserves our further consideration.

Acknowledgements

Project supported by the National Natural Science Foundation of China (Grant No. 11971055) and the Beijing Natural Science Foundation, China (Grant No. 1202019).

References

[1] Ananthi V, Balaji P, Sindhu R, Kim S H, Pugazhendhi A and Arun A 2021 *Sci. Total Environ.* **780** 146467
[2] Wang S K, Stiles A R, Guo C and Liu C Z 2015 *Algal Res.* **9** 178
[3] Li S, Hu T, Xu Y, Wang J, Chu R, Yin Z, Mo F and Zhu L 2020 *Renew. Sustain. Energy Rev.* **131** 110005
[4] Yamada T, Furusawa C, Nagahisa K, Kashiwagi A, Yomo T and Shimizu H 2007 *Biosystems* **90** 614
[5] Tai X, Ma W, Guo S, Yan H and Yin C 2015 *Math. Pract. Theory.* **45** 198 (in Chinese)
[6] Wang W, Ma W and Yan H 2016 *Appl. Sci.* **6** 221
[7] Zhang H and Zhang T 2020 *Appl. Math. Lett.* **103** 106217
[8] Liu R and Ma W 2021 *Chaos, Solitons & Fractals* **147** 110951
[9] Guo S, Cui J A and Ma W 2021 *DCDS-B* **27** 3831
[10] Sikder A and Roy A B 1993 *Biosystems* **31** 39
[11] Jiang D, Shi N and Li X 2008 *J. Math. Anal. Appl.* **340** 588
[12] Li X and Mao X 2009 *Discrete Contin Dyn. Syst.* **24** 523
[13] Liu M, Wang K and Wu Q 2011 *Bull Math Biol* **73** 1969
[14] Lv J, Wang K and Chen D 2015 *Methodol Comput Appl Probab* **17** 403
[15] Xu W, Wang X Y and Liu X Z 2015 *Chin. Phys. B* **24** 050204
[16] Yang J and Wang W 2017 *Appl. Math. Lett.* **63** 59
[17] Wei F and Wang C 2020 *Appl. Math. Model.* **81** 113
[18] Roy J, Barman D and Alam S 2020 *Biosystems* **197** 104176
[19] Li D X and Zhang N 2020 *Chin. Phys. B* **29** 090201
[20] Huo L A, Dong Y F and Lin T T 2021 *Chin. Phys. B* **30** 080201
[21] Xu C, Yu Y, Ren G, Hai X and Lu Z 2021 *J. Comput. Nonlinear Dyn.* **16** 111004
[22] Mao X 2008 *Stochastic Differential Equations and Applications* 2nd edn. (Chichester: Horwood)
[23] Khasminskii R 2012 *Stochastic Stability of Differential Equations* (Berlin: Springer)
[24] Wang L, Jiang D and O'Regan D 2016 *Commun. Nonlinear Sci. Numer. Simul.* **37** 1
[25] Zhang Q and Jiang D 2016 *J Math Chem* **54** 777
[26] Wang L and Jiang D 2017 *Appl. Math. Lett.* **73** 22
[27] Doering C R 1986 *Phys. Rev. A* **34** 2564
[28] Wang C J and Mei D C 2008 *Chin. Phys. B* **17** 479
[29] Qin Y H, Luo X S and Wei D Q 2010 *Chin. Phys. B* **19** 050511
[30] Su M B and Rong H W 2011 *Chin. Phys. B* **20** 060501
[31] Gu R C, Xu Y, Zhang H Q and Sun Z K 2011 *Acta Phys. Sin.* **60** 110514 (in Chinese)
[32] Wei Y G, Zeng C H, Wang H, Li K Z and Hu J H 2013 *Chin. Phys. B* **22** 060503
[33] Hailong Z, Enrong W, Fuhong M and Ning Z 2016 *Chin. Phys. B* **25** 030503
[34] Yuan S, Wu D, Lan G and Wang H 2020 *Bull Math Biol* **82** 55
[35] Yang A, Wang H, Zhang T and Yuan S 2022 *Chaos* **32** 043116