Conformable fractional heat equation with fractional translation symmetry in both time and space

W S Chung[1], A Gungor[2], J Kříž[3], B C Lütfüoğlu[3,4,†], and H Hassanabadi[3,5]

[1] *Department of Physics and Research Institute of Natural Science, College of Natural Science, Gyeongsang National University, Jinju* 660-701, *Korea*
[2] *Mechanical Engineering, Akdeniz University, Antalya, Turkey*
[3] *Department of Physics, University of Hradec Králové, Rokitanského* 62, 500 03 *Hradec Králové, Czech Republic*
[4] *Department of Physics, Akdeniz University, Antalya, Turkey*
[5] *Faculty of Physics, Shahrood University of Technology, Shahrood, Iran*

(Received 18 August 2022; revised manuscript received 11 September 2022; accepted manuscript online 21 September 2022)

We investigate the fractional heat equation with fractional translation in both time and position with different fractional orders. As examples, we consider a rod and an α-disk with an initial constant temperature and discuss their cooling processes in the examined formalism.

Keywords: fractional derivative, heat equation, conformable symmetry

PACS: 02.30.Jr, 02.60.Lj, 45.10.Hj **DOI:** 10.1088/1674-1056/ac935f

1. Introduction

The theoretical investigation of fractional equations receives much attention. For linear systems, equations with fractional derivatives serve as a crucial tool in the description and analysis of kinetic and transfer processes. For instance, fractional differentiation or integration operators with non-integer exponents can spontaneously overcome the irreversibility problem in linear systems, and hence they can be employed to describe a wide range of kinetic or transfer phenomena. Thereof, there is increasing interest in the application of fractional calculus in diverse scientific fields of natural and applied science.[1–9]

Since Khalil, Al Horani, Yousef, and Sababheh were the first scientists who introduced conformable fractional derivatives (CFDs).[10] After their novel study, several applications of CFDs formalism were given in Refs. [10–26]. In a recent work, two authors of this manuscript, with their co-authors, adopted this formalism to the quantum harmonic oscillator problem.[27]

CFDs can be defined for spatial coordinates, e.g., in one dimension (1D) for x, where the spatial space receives values in the interval of $(-\infty,\infty)$. Also, it can be defined for time t, which takes only positive values. Recently, Chung and Hassanabadi showed that the CFDs are related to the fractional translation symmetry by introducing α-addition.[28] There, they used the α-translation to explain the anomalous diffusion model by establishing a relation between the fractional order and the fractional power of the anomalous diffusion.

Some studies concerning the conformable fractional heat equation were given in Refs. [14,25,26] For example, in Ref. [14] the conformable fractional heat equation with CFDs in time is examined. In Ref. [25] the conformable fractional heat equation with CFDs in time and space in the same fractional order is investigated. More precisely, there authors used the CFD in x by restricting the domain of x with $x > 0$. In another work, in Ref. [26], the two-dimensional conformable fractional heat equation is handled with the CFD only in time.

In this paper, we discuss the fractional heat equation with fractional translation in both time and position with different fractional orders. Our model is well defined in $-\infty < x < \infty$ and unifies the results of Refs. [14,25,26]. In addition, we show that the conformable fractional heat equation possesses the fractional translation symmetry. We construct the paper as follows: In Section 2, we briefly introduce the CFD with fractional translational invariance. Then, in Section 3, we present the derivation of the conformable fractional heat equation. Next, in Section 4, we derive the non-dimensional conformable fractional heat equation. For applications, we consider a rod and an α-disk at a constant temperature in Sections 5 and 6 and investigate their cooling processes by the introduced formalism. Finally, in the conclusion section, we summarize our findings.

2. CFD with fractional translational invariance

We start by considering the (1+1)-dimensional ordinary heat equation

$$\frac{\partial u(x,t)}{\partial t} = \kappa \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{1}$$

Here, u and κ denote the temperature and thermal conductivity, respectively. Equation (1) is invariant under time translation, $t \to t + \Delta t$, and space translation, $x \to x + \Delta x$. Thus, if

†Corresponding author. E-mail: bekircanlutfuoglu@gmail.com

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$u(x,t)$ is a solution of Eq. (1), then $u(x+a,t+b)$ must also be a solution of the heat equation for constants a and b.

What would happen if the translation invariance is broken and is replaced with a deformed translation? This question was first handled in Ref. [28], where the authors established the fractional translation symmetry in the CFD. In this paper we consider a more general case by adopting the different fractional order for CFD with respect to time and position.

2.1. CFD with respect to time

Following Ref. [28] we introduce the fractional time derivative as

$$D_t^\beta F(t) = \lim_{t'\to t} \frac{F(t')-F(t)}{\frac{1}{\beta}(t'^\beta - t^\beta)} = t^{1-\beta}\partial_t F(t). \tag{2}$$

Since t must always be greater than or equal to zero, we do not need to consider the absolute value of time here. Now, let us introduce the β-addition and β-subtraction for $a \geq 0, b \geq 0$ as given in Ref. [28],

$$a \oplus_\beta b = (a^\beta + b^\beta)^{1/\beta}, \tag{3}$$

$$a \ominus_\beta b = \begin{cases} (a^\beta - b^\beta)^{1/\beta}, & (a \geq b), \\ -(b^\beta - a^\beta)^{1/\beta}, & (a < b). \end{cases} \tag{4}$$

Then, we rewrite Eq. (2) as

$$D_t^\beta F(t) = \lim_{t'\to t} \frac{F(t')-F(t)}{\frac{1}{\beta}(t' \ominus_\beta t)^\beta}. \tag{5}$$

Let us introduce the β-deformed time interval

$$\Delta_\beta t = t' \ominus_\beta t. \tag{6}$$

Then, we have

$$D_t^\beta F(t) = \lim_{\Delta_\beta t\to 0} \frac{F(t\oplus_\beta \Delta_\beta t)-F(t)}{\frac{1}{\beta}(\Delta_\beta t)^\beta}. \tag{7}$$

Here, we know that the fractional time derivative is invariant under the fractional β-translation in time,

$$D^\beta_{t\oplus_\beta \tau} = D_t^\beta. \tag{8}$$

2.2. CFD with respect to position

Unlike time, position can take a negative value. Thus, the fractional position derivative can be written as

$$\begin{aligned} D_x^\alpha F(x) &= \lim_{x'\to x} \frac{F(x')-F(x)}{\frac{1}{\alpha}(|x'|^{\alpha-1}x' - |x|^{\alpha-1}x)} \\ &= |x|^{1-\alpha}\partial_x F(x). \end{aligned} \tag{9}$$

By considering the α-addition and α-subtraction for a,b as given in Ref. [28], i.e.,

$$a \oplus_\alpha b = |a|a|^{\alpha-1} + b|b|^{\alpha-1}|^{1/(\alpha-1)}(a|a|^{\alpha-1} + b|b|^{\alpha-1}), \tag{10}$$

$$a \ominus_\alpha b = |a|a|^{\alpha-1} - b|b|^{\alpha-1}|^{1/\alpha-1}(a|a|^{\alpha-1} - b|b|^{\alpha-1}), \tag{11}$$

we rewrite Eq. (9) in the following form:

$$D_x^\alpha F(x) = \lim_{x'\to x} \frac{F(x')-F(x)}{\frac{1}{\alpha}|x' \ominus_\alpha x|^{\alpha-1}(x' \ominus_\alpha x)}. \tag{12}$$

Then, we introduce the α-deformed space interval

$$\Delta_\alpha x = x' \ominus_\beta x, \tag{13}$$

and we get

$$D_x^\alpha F(x) = \lim_{\Delta_\alpha x\to 0} \frac{F(x\oplus_\alpha \Delta_\alpha x)-F(x)}{\frac{1}{\alpha}|\Delta_\alpha x|^{\alpha-1}\Delta_\alpha x}. \tag{14}$$

Here, we know that the fractional position derivative is invariant under the fractional α-translation in position,

$$D^\alpha_{x\oplus_\alpha a} = D_x^\alpha. \tag{15}$$

The conformable fractional integral is then given by

$$\int \mathrm{d}_\alpha x F(x) = \int \mathrm{d}x \alpha |x|^{\alpha-1} F(x). \tag{16}$$

3. Derivation of conformable fractional heat equation

First let us derive the conformable fractional heat equation [29,30] in 1D. To this end, we consider a rod of length L, and we assume that the heat flux is given by

$$J(x,t) = -\kappa D_x^\alpha u(x,t), \quad 0 \leq x \leq L. \tag{17}$$

We adopt the conformable fractional continuity equation such that it possesses the fractional translation symmetry,

$$D_t^\beta u(x,t) = -D_x^\alpha J(x,t), \tag{18}$$

which gives the conformable fractional heat equation in 1D in the form

$$D_t^\beta u(x,t) = \kappa (D_x^\alpha)^2 u(x,t). \tag{19}$$

In three dimensions, the conformable fractional heat equation reads

$$D_t^\beta u(x,t) = \kappa((D_x^\alpha)^2 + (D_y^\alpha)^2 + (D_z^\alpha)^2)u(x,t), \tag{20}$$

where

$$D_t^\beta = t^{1-\beta}\partial_t, \tag{21}$$

$$D_x^\alpha = |x|^{1-\alpha}\partial_x, \tag{22}$$

$$D_y^\alpha = |y|^{1-\alpha}\partial_y, \tag{23}$$

$$D_z^\alpha = |z|^{1-\alpha}\partial_z. \tag{24}$$

It is worth noting that the case of $\alpha = 1$ in 1D was discussed in Ref. [14], while the case of $\alpha = \beta$ in 1D with $x > 0$ was examined in Ref. [25], and the case of $\alpha = 1$ in two dimensions (2D) was explored in Ref. [26].

4. Non-dimensional conformable fractional heat equation

As it is well-known, in a metal bar with an inhomogeneous temperature, thermal energy, in other words heat, flows from regions of higher temperature to regions of lower temperature. Here, we assume that the heat transfer is governed by the conformable fractional heat equation instead of the traditional one.

Dimensional (or physical) terms in Eq. (19) or Eq. (20) can be converted to the non-dimensional ones by introducing the appropriate scale such as characteristic length, time and temperature. Then, the conformable fractional heat equation in 1D reads

$$D_t^{\beta}u(x,t) = (D_x^{\alpha})^2 u(x,t), \quad 0 \le x \le 1. \tag{25}$$

For the following boundary conditions:

$$\begin{aligned} &u(0,t) = 0, \quad u(1,t) = 0, \\ &\frac{\partial u}{\partial t}(x,0) = 0, \quad u(x,0) = u_0(x), \end{aligned} \tag{26}$$

we seek a solution to the conformable fractional heat equation by assuming the solution as

$$u(x,t) = X(x)T(t). \tag{27}$$

By employing Eq. (27) in Eq. (25), we obtain

$$\frac{(D_x^{\alpha})^2 X}{X} = \frac{D_t^{\beta} T}{T}. \tag{28}$$

First, we solve the spatial part via $X(0) = X(1) = 0$.

$$X_n(x) = \sin(n\pi x^{\alpha}), \quad n = 1,2,3,\ldots. \tag{29}$$

Then, we determine the time part,

$$T_n(t) = \mathrm{e}^{-(\alpha^2 n^2 \pi^2 t)^{\beta}}. \tag{30}$$

After we express the solution of Eq. (25) by

$$u_n(x,t) = B_n \mathrm{e}^{-(\alpha^2 n^2 \pi^2 t)^{\beta}} \sin(n\pi x^{\alpha}), \tag{31}$$

applying the principle of superposition, we obtain the general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-(\alpha^2 n^2 \pi^2 t)^{\beta}} \sin(n\pi x^{\alpha}). \tag{32}$$

Considering the initial condition $u(x,0) = u_0(x)$, we get

$$u_0(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x^{\alpha}). \tag{33}$$

From the orthogonality relation of the trigonometric functions, we can determine

$$B_n = 2\alpha \int_0^1 \sin(n\pi x^{\alpha}) u_0(x) \mathrm{d}_{\alpha}x, \tag{34}$$

where

$$\int_0^1 \sin(n\pi x^{\alpha}) \sin(m\pi x^{\alpha}) \mathrm{d}_{\alpha}x = \frac{1}{2\alpha}\delta_{nm}, \tag{35}$$

$$\int_0^1 \cos(n\pi x^{\alpha}) \cos(m\pi x^{\alpha}) \mathrm{d}_{\alpha}x = \frac{1}{2\alpha}\delta_{nm}, \tag{36}$$

$$\int_0^1 \cos(n\pi x^{\alpha}) \sin(m\pi x^{\alpha}) \mathrm{d}_{\alpha}x = 0. \tag{37}$$

5. Rod's cooling

In this section, as an application of the formalism we consider a metal rod with an initial constant temperature distribution $u_0(x)$, i.e.,

$$u_0(x) = u_0 = \text{const.} \tag{38}$$

In this case, we find

$$B_n = 2u_0\left(\frac{1-(-1)^n}{n\pi}\right), \tag{39}$$

thus, the solution reads

$$\begin{aligned} u(x,t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \mathrm{e}^{-(\alpha^2(2n-1)^2\pi^2 t)^{\beta}} \\ \times \sin[(2n-1)\pi x^{\alpha}], \end{aligned} \tag{40}$$

or

$$\begin{aligned} u(x,t) = \frac{4u_0}{\pi}\Big[\mathrm{e}^{-(\alpha^2\pi^2 t)^{\beta}} \sin(\pi x^{\alpha}) \\ + \frac{1}{3}\mathrm{e}^{-(9\alpha^2\pi^2 t)^{\beta}} \sin(3\pi x^{\alpha}) + \cdots \Big]. \end{aligned} \tag{41}$$

Then, we evaluate the ratio of the first and second terms of Eq. (41). We find

$$\begin{aligned} \frac{|\text{second term}|}{|\text{first term}|} &= \frac{1}{3}\mathrm{e}^{-(9^{\beta}-1)(\alpha^2\pi^2 t)^{\beta}} \frac{|\sin 3\pi x^{\alpha}|}{|\sin \pi x^{\alpha}|} \\ &\le \mathrm{e}^{-(9^{\beta}-1)(\alpha^2\pi^2 t)^{\beta}}, \end{aligned} \tag{42}$$

where we take

$$|\sin(nt)| \le n|\sin t|. \tag{43}$$

For $t \ge 1/(\alpha^2\pi^2)$, this ratio gives

$$\frac{|\text{second term}|}{|\text{first term}|} \le \mathrm{e}^{-(9^{\beta}-1)}. \tag{44}$$

In Fig. 1, we demonstrate the plot of $f(\beta) = \mathrm{e}^{-(9^{\beta}-1)}$ versus β for $0.9 \le \beta \le 1.1$. We notice that for small $|\beta - 1|$, the second term becomes negligible, and hence we get

$$u(x,t) \approx \frac{4u_0}{\pi} \mathrm{e}^{-(\alpha^2\pi^2 t)^{\beta}} \sin(\pi x^{\alpha}). \tag{45}$$

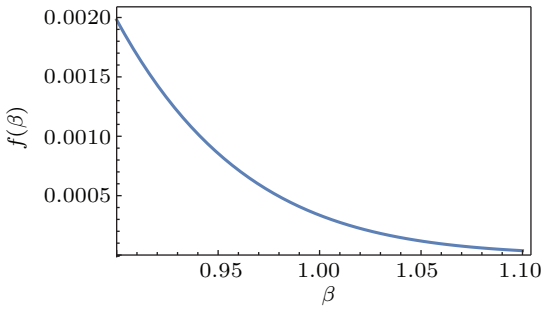


Fig. 1. Plot of $f(\beta) = \mathrm{e}^{-(9^{\beta}-1)}$ versus β for $0.9 \leq \beta \leq 1.1$.

5.1. Spatial temperature profiles

Now let us consider fixed time, for instance, $t = 2/(\alpha\pi)^2$. In this case, we have

$$u(x,t) \approx \frac{4u_0}{\pi}\,\mathrm{e}^{-(2\alpha^2)^{\beta}}\sin(\pi x^{\alpha}). \tag{46}$$

This function takes its maximum value at $x = (1/2)^{\alpha}$. Thus, the center of a rod is not a line of symmetry unless $\alpha = 1$. Figure 2 shows the plot of u versus x with $u_0 = \pi/4$ for $\alpha = \beta = 1$ (red), $\alpha = 0.9$, $\beta = 0.9$ (green), $\alpha = 0.9$, $\beta = 1.1$ (brown), $\alpha = 1.1$, $\beta = 0.9$ (pink), and $\alpha = 1.1$, $\beta = 1.1$ (purple).

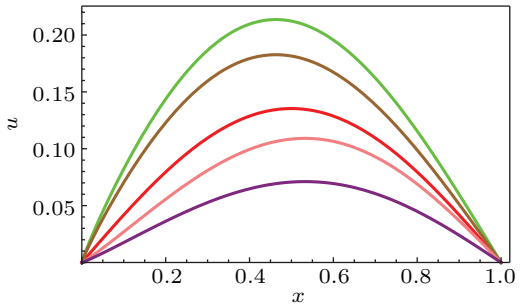


Fig. 2. Plot of u versus x with $u_0 = \pi/4$ for $\alpha = \beta = 1$ (red), $\alpha = 0.9$, $\beta = 0.9$ (green), $\alpha = 0.9$, $\beta = 1.1$ (brown), $\alpha = 1.1$, $\beta = 0.9$ (pink), and $\alpha = 1.1$, $\beta = 1.1$ (purple).

5.2. Profiles of temperature over time

In order to understand how the heat dissipation changes with time in the same location, we use $x = x_0$ in Eq. (45). We find

$$u(x,t) \approx \frac{4u_0}{\pi}\,\mathrm{e}^{-(\alpha^2\pi^2 t)^{\beta}}\sin(\pi x_0^{\alpha}). \tag{47}$$

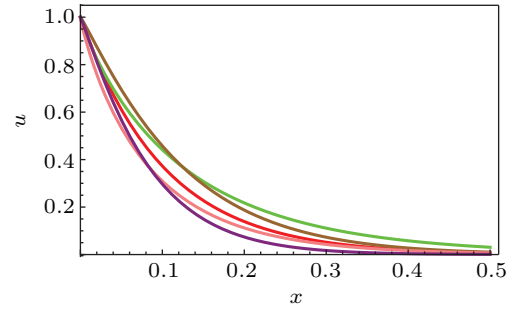


Fig. 3. Plot of u versus t with $u_0 = \pi/4, x_0 = (1/2)^{\alpha}$ for $\alpha = \beta = 1$ (red), $\alpha = 0.9$, $\beta = 0.9$ (green), $\alpha = 0.9$, $\beta = 1.1$ (brown), $\alpha = 1.1$, $\beta = 0.9$ (pink) and $\alpha = 1.1$, $\beta = 1.1$ (purple).

In Fig. 3, we present the plot of u versus t with $u_0 = \pi/4, x_0 = (1/2)^{\alpha}$ for $\alpha = \beta = 1$ (red), $\alpha = 0.9$, $\beta = 0.9$ (green), $\alpha = 0.9$, $\beta = 1.1$ (brown), $\alpha = 1.1$, $\beta = 0.9$ (pink), and $\alpha = 1.1$, $\beta = 1.1$ (purple).

6. The α-disk's cooling

As a second example, we study a 2D problem and examine cooling process of an α-disk from a constant initial temperature. We start by the following 2D conformable fractional heat equation:

$$D_t^{\beta}u(x,t) = \kappa((D_x^{\alpha})^2 + (D_y^{\alpha})^2)u(x,t). \tag{48}$$

It is worth noting that the fractional Laplacian is not invariant under the ordinary $SO(2)$ group. Instead, it is invariant under the fractional version of $SO(2)$ group which induces the invariant quantity

$$r_{\alpha} = (|x|^{2\alpha} + |y|^{2\alpha})^{1/2\alpha}. \tag{49}$$

Therefore, we consider the α-disk with α-radius, R_{α},

$$|x|^{2\alpha} + |y|^{2\alpha} = R_{\alpha}^{2\alpha}. \tag{50}$$

In Fig. 4 we present the plot of α-disk with unit α-radius for $\alpha = 1$ (red), $\alpha = 0.9$ (green), and $\alpha = 1.1$ (brown). Then, we pass through to the non-dimensional case, and write

$$D_t^{\beta}u(x,t) = ((D_x^{\alpha})^2 + (D_y^{\alpha})^2)u(x,t), \quad 0 \leq r_{\alpha} \leq 1, \tag{51}$$

where α-polar coordinates are

$$\frac{1}{\alpha}|x|^{\alpha-1}x = \frac{1}{\alpha}r_{\alpha}^{\alpha}\cos\theta, \tag{52}$$

$$\frac{1}{\alpha}|y|^{\alpha-1}y = \frac{1}{\alpha}r_{\alpha}^{\alpha}\sin\theta. \tag{53}$$

Then, Eq. (51) reads

$$D_t^{\beta}u(r_{\alpha},t) = ((D_{r_{\alpha}}^{\alpha})^2 + \frac{1}{r_{\alpha}^{\alpha}}D_{r_{\alpha}}^{\alpha})u(r_{\alpha},t). \tag{54}$$

After imposing the following initial conditions:

$$u(1,t) = 0, \quad u(r_{\alpha},0) = u_0 = \text{const.}, \tag{55}$$

we obtain the solution in the form of

$$u(r_{\alpha},t) = \sum_{n=1}^{\infty} C_n J_0(\gamma_n r_{\alpha}^{\alpha})\,\mathrm{e}^{-\gamma_n^2(t^{\beta}/\beta)}, \tag{56}$$

where γ_n is the n-th zero of $J_0(x)$. From the second relation of Eq. (54), we have

$$u_0 = \sum_{n=1}^{\infty} C_n J_0(\gamma_n r_{\alpha}^{\alpha}), \tag{57}$$

which leads to

$$C_n = \frac{2u_0}{\gamma_n J_1(\gamma_n)}. \tag{58}$$

Therefore, the solution becomes

$$u(r_{\alpha},t) = 2u_0\sum_{n=1}^{\infty}\frac{1}{\gamma_n J_1(\gamma_n)}J_0(\gamma_n r_{\alpha}^{\alpha})\,\mathrm{e}^{-\gamma_n^2(t^{\beta}/\beta)}. \tag{59}$$

This result can be approximated to

$$u(r_\alpha,t) \approx 2u_0 \frac{1}{\gamma_1 J_1(\gamma_1)} J_0(\gamma_1 r_\alpha^\alpha)\, \mathrm{e}^{-\gamma_1^2 (t^\beta/\beta)}, \tag{60}$$

where $\gamma_1 \approx 2.40483$.

If we consider a fixed time value, $t = (2\beta/\gamma_1^2)^{1/\beta}$. Then, we have

$$u(r_\alpha,t) \approx 2u_0\, \mathrm{e}^{-2} \frac{1}{\gamma_1 J_1(\gamma_1)} J_0(\gamma_1 r_\alpha^\alpha). \tag{61}$$

In Fig. 5, we depict the plot of u versus r_α for $\alpha = 1$ (red), $\alpha = 0.9$ (green) and $\alpha = 1.1$ (brown). We observe that when $\alpha < 1$ (or $\alpha > 1$), the temperature decreases faster (or slower) than $\alpha = 1$ from the center to the boundary.

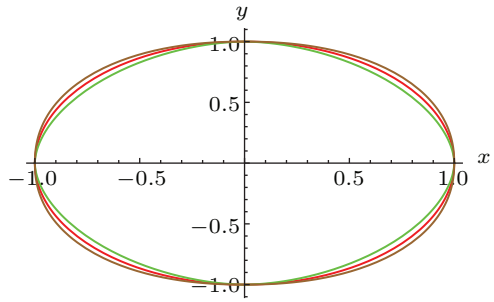


Fig. 4. Plot of α-disk with unit α-radius for $\alpha = 1$ (red), $\alpha = 0.9$ (green), and $\alpha = 1.1$ (brown).

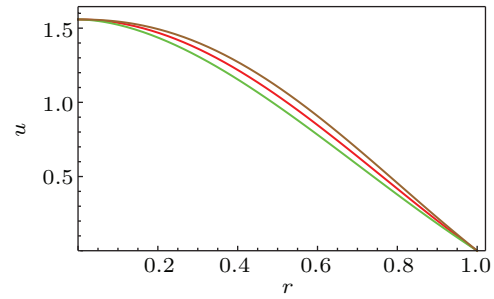


Fig. 5. Plot of u versus r_α for $\alpha = 1$ (red), $\alpha = 0.9$ (green), and $\alpha = 1.1$ (brown).

7. Conclusion

In summary, we have presented the conformable fraction derivatives for time and space which may have different fractional orders. Using these derivatives, we obtain the conformable fractional heat equation which is well defined in $-\infty < x < \infty$. We observe that the latter heat equation is the unified result of Refs. [14,25]. Moreover, we show that the conformable fractional heat equation possesses the fractional translation symmetry in both time and space. Then, as an application we consider a 1D metal rod with a constant initial temperature and study its cooling process. For the fixed time, the temperature is shown to have a maxima at $x = (1/2)^\alpha$. Thus, we conclude that the center of a rod is not a line of symmetry unless $\alpha = 1$. Finally, we consider a 2D problem and examine the cooling process of a α-disk which has a constant initial temperature. We find that for $\alpha < 1$ (or $\alpha > 1$) the temperature decreases faster (or slower) than $\alpha = 1$ from the center to the boundary.

We note that, when we take the ordinary case limits by assuming α and β go to one, then our findings, e.g., Eq. (47), reduce to the same of ordinary case. For determining the best values of α and β, we should consider a real source and we should sort the temperature of a special place at different times. Then, by sorting the temperature of the different places at the same time we should make a list of the experimental data. After all preparation, by using the scanning method we can obtain the best value of α and β with a minimum of the standard deviation.

Before ending the paper, we would like to emphasize the physical significance of the conformable fractional derivatives. This formalism gives exact solutions unlike the other fractional derivatives which give approximate solutions.

Acknowledgment

This work was supported by the Internal Project of Excellent Research of the Faculty of Science of Hradec Králové University (Grant No. 2022/2218).

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