Hall conductance of a non-Hermitian two-band system with k-dependent decay rates

Junjie Wang(王俊杰)[1], Fude Li(李福德)[1], and Xuexi Yi(衣学喜)[1,2,†]

[1] *Center for Quantum Sciences and School of Physics, Northeast Normal University, Changchun* 130024, *China*
[2] *Center for Advanced Optoelectronic Functional Materials Research, and Key Laboratory for UV-Emitting Materials and Technology of Ministry of Education, Northeast Normal University, Changchun* 130024, *China*

(Received 8 March 2022; revised manuscript received 17 August 2022; accepted manuscript online 8 September 2022)

Two-band model works well for Hall effect in topological insulators. It turns out to be non-Hermitian when the system is subjected to environments, and its topology characterized by Chern numbers has received extensive studies in the past decades. However, how a non-Hermitian system responses to an electric field and what is the connection of the response to the Chern number defined via the non-Hermitian Hamiltonian remains barely explored. In this paper, focusing on a k-dependent decay rate, we address this issue by studying the response of such a non-Hermitian Chern insulator to an external electric field. To this aim, we first derive an effective non-Hermitian Hamiltonian to describe the system and give a specific form of k-dependent decay rate. Then we calculate the response of the non-Hermitian system to a constant electric field. We observe that the environment leads the Hall conductance to be a weighted integration of curvature of the ground band and hence the conductance is no longer quantized in general. And the environment induces a delay in the response of the system to the electric field. A discussion on the validity of the non-Hermitian model compared with the master equation description is also presented.

Keywords: Hall conductance, non-Hermitian, topological insulators

PACS: 03.65.Vf **DOI:** 10.1088/1674-1056/ac9046

1. Introduction

Since the discovery of quantum Hall effect in the 1980s, the topological band theory has been extensively developed and applied in various systems, ranging from insulator and semimetal to superconductor.[1–4] In theory, the Chern number captures the winding of the eigenstates and is defined via the integral of the Berry curvature over the first Brillouin zone. It can not only be used to classify topological materials, but also quantify the response of the system to an external field. For example, the Hall conductance is quantized and proportional to the sum of the Thouless–Kohmoto–Nightingale–den Nijs (TKNN) invariants (or Chern numbers) of all filled bands.[5–7] This theory is previously established in closed systems, and one may wonder if it holds valid for open systems.

Open systems can be described effectively by the non-Hermitian Hamiltonian. Recently, the non-Hermitian topological theories have been introduced and experiments have been conducted in dozens of open systems.[8–47] Many interesting features are predicted and observed in the non-Hermitian systems, including the band defined on the complex plane,[11] the breakdown of the conventional bulk-boundary correspondence,[12–17] and the non-Hermitian skin effect.[13,18–25] The definition of the topological invariance for the non-Hermitian systems has also been discussed.[11,32]

From the side of response, the impact of the non-Hermitian on observable has been studied using field-theoretical techniques within the linear response theory (e.g., the Kubo formula).[35–39] They found that there is no link between the non-Hermitian topological invariants and the quantization of observables[35] and the observables are no longer quantized in general otherwise requiring more strict conditions than that of a nonzero non-Hermitian Chern number.[36] In these studies, the non-Hermitian term was introduced via self-energy in the low-energy limit,[37] or phenomenologically introduced to add into the system, so a non-Hermitian version of the TKNN could be derived to show the topological contribution.[36] This rises a question that how the decay rate depends on the momentum of the electron, and how the decay depends on the couplings between the system and the environment, and how a non-Hermitian system with k-dependent decay rate responses to an external stimulus.

In this work, we will answer these questions and shed more light on the response theory for the non-Hermitian two-band systems by the adiabatic perturbation theory.[48–50] The reminder of this paper is organized as follows. In Section 2, we introduce the system-environment couplings and derive a master equation by which we obtain a non-Hermitian Hamiltonian to describe the two-band system subjected to environments. The dependence of the decay rates on the Bloch vectors is also derived and discussed. In Section 3, we work out the response of the non-Hermitian two-band systems to a constant electric field using the adiabatic perturbation theory. The results show that the response of the system can be divided into two terms.

[†]Corresponding author. E-mail: yixx@nenu.edu.cn

© 2023 Chinese Physical Society and IOP Publishing Ltd

The first term is proportional to the non-Hermitian Chern number for two-band systems, which is reminiscent of the relationship between the Chern number and the Hall conductance of closed systems. While the second term can be treated as a correction that suggests the relationship between the Chern number and the conductance might fail for open systems. In Section 4, taking a tight-binding electron in a two-dimensional lattice as an example, we calculate and plot the complex-band structures as a function of the momentum of the electron, k_x and k_y. The Hall conductance of the system is also shown and analyzed. We find that Hall conductance depends on the strength of the decay rate and the electron occupation on the Bloch band, which no longer leads to a quantized Hall conductance in general and a delay in the response of the system to the constant electric field appears. We also compare the Hall conductance of the open systems by the non-Hermitian Hamiltonian and by master equation in this section. In Section 5, we conclude the paper with discussions.

2. Master equation and effective non-Hermitian Hamiltonian

We start with a two-band Chern insulator, whose Hamiltonian can be expressed as (setting $\hbar = 1$)

$$\hat{H}_{\rm S}(\boldsymbol{k}) = d_0(\boldsymbol{k})\sigma_0 + \boldsymbol{d}(\boldsymbol{k})\cdot\boldsymbol{\sigma}, \tag{1}$$

where σ_0 is the 2×2 identity matrix, $(\boldsymbol{\sigma}) = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli matrix and $\boldsymbol{k} = (k_x, k_y)$ represents the Bloch vector of the electron. $d_0(\boldsymbol{k})$ is a shift of zero energy level and can be ignored[51] for simplicity. $\boldsymbol{d}(\boldsymbol{k}) = (d_x(\boldsymbol{k}), d_y(\boldsymbol{k}), d_z(\boldsymbol{k}))$ are three component vectors depending on the feature of materials under research. The eigenvalues of this Hamiltonian are $\varepsilon_\pm = \pm d(\boldsymbol{k})$, with $d(\boldsymbol{k}) = \sqrt{\sum_{i=x,y,z} d_i^2(\boldsymbol{k})}$. The corresponding eigenstates of the model read

$$\begin{aligned} |k_+\rangle &= \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi}|+\rangle + \sin\left(\frac{\theta}{2}\right)|-\rangle, \\ |k_-\rangle &= \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi}|+\rangle - \cos\left(\frac{\theta}{2}\right)|-\rangle, \end{aligned} \tag{2}$$

where the parameter

$$\begin{aligned} \theta &= \arccos(d_z(\boldsymbol{k})/d(\boldsymbol{k})), \\ \phi &= \arctan(d_y(\boldsymbol{k})/d_x(\boldsymbol{k})). \end{aligned} \tag{3}$$

Considering an environment as quantized electromagnetic field of many modes coupled to the system, by Peierls substitution[49,52] $\boldsymbol{k} \to (\boldsymbol{k} + e\boldsymbol{A})$, where the elementary charge of electron is $-e$ $(e > 0)$, $\boldsymbol{A} = -\sum_{nj} \boldsymbol{e}_j g_n (b_n + b_n^\dagger)$ with $\boldsymbol{e}_j$ standing for the unit vector, $g_n = \sqrt{1/2V\varepsilon_0\omega_n}$ and V being the volume of the environment, and in the weak field limit, the Hamiltonian up to the first order in $\boldsymbol{A}$ reads

$$\begin{aligned} \hat{H}(\boldsymbol{k}) &= \hat{H}_{\rm S}(\boldsymbol{k}) + \hat{H}_{\rm R} + \hat{H}_{\rm I}(\boldsymbol{k}) \\ &\approx \boldsymbol{d}(\boldsymbol{k})\cdot\boldsymbol{\sigma} + \sum_n \omega_n \hat{b}_n^\dagger \hat{b}_n + e\sum_{i=x,y,z} (\nabla_{\boldsymbol{k}} d_i(\boldsymbol{k})\cdot\boldsymbol{A})\sigma_i, \end{aligned} \tag{4}$$

where ω_n is the eigenfrequency of the environment and ε_0 is the vacuum permittivity. $\hat{b}_n^\dagger$ and $\hat{b}_n$ are creation and annihilation operators of the nth mode of the environment, respectively. $\nabla_{\boldsymbol{k}}$ stands for the partial derivative with respect to $\boldsymbol{k}$. For simplicity, we set $\nabla_{k(t)} = \nabla_k$. Simple algebra with rotating-wave approximations[53,54] shows that the total Hamiltonian in the interaction picture takes

$$\hat{H}_{\rm I}(t, \boldsymbol{k}) = \sum_n g_n(\boldsymbol{k})\sigma_+(\boldsymbol{k})\hat{b}_n\,\mathrm{e}^{\mathrm{i}(2d(\boldsymbol{k})-\omega_n)t} + \mathrm{H.c.}, \tag{5}$$

where $g_n(\boldsymbol{k}) = -eg_n D_-$, $\sigma_+(\boldsymbol{k}) \equiv |k_+\rangle\langle k_-|$, $\sigma_-(\boldsymbol{k}) \equiv |k_-\rangle\langle k_+|$ and

$$\begin{aligned} D_+ &= \nabla_{\boldsymbol{k}} d_x(\boldsymbol{k})(\cos\phi\cos\theta + \mathrm{i}\sin\phi) \\ &\quad + \nabla_{\boldsymbol{k}} d_y(\boldsymbol{k})(\sin\phi\cos\theta - \mathrm{i}\cos\phi) \\ &\quad - \nabla_{\boldsymbol{k}} d_z(\boldsymbol{k})\sin\theta, \\ D_- &\equiv D_+^*. \end{aligned} \tag{6}$$

We assume that the initial states of the system and environment are separable, i.e., $\rho_{\rm SR}^{\rm I}(0) = \rho_{\rm S}(0)\otimes\rho_{\rm R}(0)$. In the Born–Markov approximation, we obtain the following master equation for the system:[55,56]

$$\dot{\rho}_{\rm S} = -\mathrm{i}\left[\hat{H}_{\rm S}(\boldsymbol{k}), \rho_{\rm S}\right] + \mathcal{U}(\rho_{\rm S}) + \mathcal{D}(\rho_{\rm S}), \tag{7}$$

where

$$\begin{aligned} \mathcal{U}(\rho_{\rm S}) &= \gamma(\boldsymbol{k})(N(\boldsymbol{k})+1)[2\sigma_-(\boldsymbol{k})\rho_{\rm S}\sigma_+(\boldsymbol{k}) \\ &\quad - \sigma_+(\boldsymbol{k})\sigma_-(\boldsymbol{k})\rho_{\rm S} - \rho_{\rm S}\sigma_+(\boldsymbol{k})\sigma_-(\boldsymbol{k})], \\ \mathcal{D}(\rho_{\rm S}) &= \gamma(\boldsymbol{k})N(\boldsymbol{k})[2\sigma_+(\boldsymbol{k})\rho_{\rm S}\sigma_-(\boldsymbol{k}) \\ &\quad - \sigma_-(\boldsymbol{k})\sigma_+(\boldsymbol{k})\rho_{\rm S} - \rho_{\rm S}\sigma_-(\boldsymbol{k})\sigma_+(\boldsymbol{k})]. \end{aligned} \tag{8}$$

Here, $\gamma(\boldsymbol{k})$ is the decay rate that depends on the Bloch vectors. $N(\omega_n) = \{\exp[\omega_n/k_{\rm B}T] - 1\}^{-1}$ represents the average number of photons in the system. In the Weisskopf–Wigner approximation, $N(\boldsymbol{k}) = \{\exp[2d(\boldsymbol{k})/k_{\rm B}T] - 1\}^{-1}$. In order to obtain an effective non-Hermitian Hamiltonian, the master equation can be written as

$$\begin{aligned} \dot{\rho}_{\rm S} &= -\mathrm{i}\left(\hat{H}_{\rm eff}\rho_{\rm S} - \rho_{\rm S}\hat{H}_{\rm eff}^\dagger\right) \\ &\quad + 2\gamma(\boldsymbol{k})[N(\boldsymbol{k})+1]\sigma_-(\boldsymbol{k})\rho_{\rm S}\sigma_+(\boldsymbol{k}) \\ &\quad + 2\gamma(\boldsymbol{k})N(\boldsymbol{k})\sigma_+(\boldsymbol{k})\rho_{\rm S}\sigma_-(\boldsymbol{k}). \end{aligned} \tag{9}$$

This type of master equation has been studied in Ref. [54], where the environmental electromagnetic field along the x-axis is assumed, $A_x = E_x t$. In the following discussion, we lift this restriction and consider the field along the x and y directions. When the quantum-jump term in Eq. (9) can be neglected, the system reduces to a system described by an effective non-Hermitian Hamiltonian

$$\hat{H}_{\rm eff} = (d(\boldsymbol{k}) - \mathrm{i}\gamma(\boldsymbol{k})[N(\boldsymbol{k})+1])|k_+\rangle\langle k_+|$$

$$-(d(\boldsymbol{k})+\mathrm{i}\gamma(\boldsymbol{k})N(\boldsymbol{k}))|k_-\rangle\langle k_-|. \tag{10}$$

This situation holds valid when we consider the dynamics in a sufficiently short period of time,[57–59] which greatly simplifies the calculation, because it suffices to study the eigenvectors and eigenvalues of an effective Hamiltonian, instead of performing a time-resolved calculation of the density matrix. Here, we consider a Bose reservoir as an environment. Similarly, we also obtain an effective non-Hermitian Hamiltonian for the Fermi reservoir (see Eq. (A5)).

In the limit $T\to 0$, we find that $N(\boldsymbol{k})\to 0$. With this consideration and a unitary transformation, we have

$$\hat{H}_{\mathrm{eff}}=\boldsymbol{h}(\boldsymbol{k})\cdot\boldsymbol{\sigma}-\mathrm{i}\Gamma_0(\boldsymbol{k})\sigma_0, \tag{11}$$

where $\boldsymbol{h}(\boldsymbol{k})=\boldsymbol{d}(\boldsymbol{k})-\mathrm{i}\boldsymbol{\Gamma}(\boldsymbol{k})$. $\boldsymbol{\Gamma}(\boldsymbol{k})\equiv\frac{1}{2}\boldsymbol{\gamma}(\boldsymbol{k})=(\Gamma_x(\boldsymbol{k}),\Gamma_y(\boldsymbol{k}),\Gamma_z(\boldsymbol{k}))$ with $\Gamma_{x,y,z}(\boldsymbol{k})=\delta\beta d_{x,y,z}(\boldsymbol{k})$ and $\Gamma_0(\boldsymbol{k})=\delta\beta d(\boldsymbol{k})$. $\delta=e^2/(2\pi\varepsilon_0 c^3)$ with c stands for the light speed and $\beta=D_+D_-$,

$$\begin{aligned}D_+D_- &= \sum_{\alpha=x,y}(\nabla_{k_\alpha}d_x(\boldsymbol{k})\cdot\cos\phi\cdot\cos\theta+\nabla_{k_\alpha}d_y(\boldsymbol{k})\\ &\quad\times\sin\phi\cdot\cos\theta-\nabla_{k_\alpha}d_z(\boldsymbol{k})\cdot\sin\theta)^2\\ &\quad+(\nabla_{k_\alpha}d_x(\boldsymbol{k})\cdot\sin\phi-\nabla_{k_\alpha}d_y(\boldsymbol{k})\cdot\cos\phi)^2.\end{aligned} \tag{12}$$

In the following we set δ to be a small constant for convenience. We find that the energy spectrum of the excited band is complex and the energy spectrum of ground band is real: $E_+=d(\boldsymbol{k})-2\mathrm{i}\Gamma(\boldsymbol{k})$ and $E_-=-d(\boldsymbol{k})$, where $\Gamma(\boldsymbol{k})=\sqrt{\sum_{i=x,y,z}\Gamma_i^2(\boldsymbol{k})}$.[60–63] It is clear that the decay rate $\gamma(\boldsymbol{k})$ depends on the Bloch-vector $\boldsymbol{k}$, which is different from the previous studies. This result suggests that the decay rate of the upper Bloch band is different at different positions in the momentum space, leading to interesting features in the Hall conductance.

In the free space, decay rate can also be expressed as $\gamma(\boldsymbol{k})=\pi|g(\boldsymbol{k})|^2\xi(\boldsymbol{k})$, where

$$\xi(\boldsymbol{k})=V(2d(\boldsymbol{k}))^2/\pi^2c^3, \tag{13}$$

is the mode density. Consider the following spectral density of the environment $J_{\mathrm{c}}(\omega)$

$$J_{\mathrm{c}}(\omega)=\alpha\omega\Big(\frac{\omega}{\omega_{\mathrm{c}}}\Big)^{s-1}\mathrm{e}^{-\omega/\omega_{\mathrm{c}}}, \tag{14}$$

where α is the dimensionless coupling strength, and ω_{c} is the hard upper cutoff. The index s accounts for various physical situations, for example for Ohmic spectrum, $s=1$. We have

$$\Gamma_{\mathrm{c}}(\boldsymbol{k}) = \frac{\pi}{2}J_{\mathrm{c}}(\boldsymbol{k})=\pi\alpha d(\boldsymbol{k})\,\mathrm{e}^{-2d(\boldsymbol{k})/\omega_{\mathrm{c}}}, \tag{15}$$

where $J_{\mathrm{c}}(\boldsymbol{k})=\sum_n|g(\boldsymbol{k})|^2\delta(2d(\boldsymbol{k})-\omega_n)$.

3. Hall conductance

In order to derive the response to a constant electric field $\mathcal{E}$, one can introduce a uniform vector potential $\boldsymbol{A}(t)$ that changes in time such that $\partial_t\boldsymbol{A}(t)=-\mathcal{E}$, which can modify the Bloch vectors, i.e., $\boldsymbol{k}(t)=\boldsymbol{k}+e\boldsymbol{A}(t)$. The Hamiltonian $\hat{H}_{\mathrm{eff}}(\boldsymbol{k}(t))$ satisfies the following time-dependent Schrödinger equation:

$$\mathrm{i}\frac{\partial}{\partial t}|u(\boldsymbol{k}(t))\rangle=\hat{H}_{\mathrm{eff}}(\boldsymbol{k}(t))|u(\boldsymbol{k}(t))\rangle. \tag{16}$$

Consider a crystal under the perturbation of a weak electric field. We can write the total Hamiltonian as $\hat{H}_{\mathrm{eff}}(\boldsymbol{k}(t))=\hat{H}_{\mathrm{eff}}+\hat{H}'$, where $\hat{H}'$ stands for the perturbation Hamiltonian. If the system is initially in the ground band, it will always stay in this band if the adiabatic condition is met. But now, we consider the case that there is still a small probability for particles to move from ground to excited band. So we can use the adiabatic perturbation theory with the perturbation Hamiltonian $\hat{H}'=-\mathrm{i}\partial/\partial t$. The corresponding ground band wave function up to the first order in the field strength satisfies[49]

$$|u'_-\rangle=|u_-(\boldsymbol{k})\rangle-\mathrm{i}\frac{\langle\hat{u}_+(\boldsymbol{k})|\partial_t u_-(\boldsymbol{k})\rangle}{E_--E_+}|u_+(\boldsymbol{k})\rangle, \tag{17}$$

where $|u_\pm(\boldsymbol{k})\rangle$ are the ground band (with $-$) and the excited band (with $+$) of the Hamiltonian $\hat{H}_{\mathrm{eff}}$, respectively. $|u_n(\boldsymbol{k})\rangle$ and $\langle\hat{u}_n(\boldsymbol{k})|$ denote the right and the left eigenstates of $\hat{H}_{\mathrm{eff}}$. They satisfy $\langle\hat{u}_m(\boldsymbol{k})|u_n(\boldsymbol{k})\rangle=\delta_{nm}$ and the completeness relation

$$\sum_n|u_n(\boldsymbol{k})\rangle\langle\hat{u}_n(\boldsymbol{k})|=\sum_n|\hat{u}_n(\boldsymbol{k})\rangle\langle u_n(\boldsymbol{k})|=1. \tag{18}$$

Similarly, its biorthogonal partner reads

$$|\hat{u}'_-\rangle=|\hat{u}_-(\boldsymbol{k})\rangle-\mathrm{i}\frac{\langle u_+(\boldsymbol{k})|\partial_t\hat{u}_-(\boldsymbol{k})\rangle}{E_-^*-E_+^*}|\hat{u}_+(\boldsymbol{k})\rangle. \tag{19}$$

Next, we use biorthogonal basis vectors to study a dynamic evolution. For any observable $\hat{A}$, the generalised expectation values can be defined similarly as $\langle\hat{\phi}|\hat{A}|\phi\rangle$.[64,65] The average velocity $\hat{v}_y$ in such a state is simplified as

$$\begin{aligned}\bar{v}_y &= \langle\hat{u}'_-|\hat{v}_y|u'_-\rangle\\ &\approx \langle\hat{u}_-(\boldsymbol{k})|\hat{v}_y|u_-(\boldsymbol{k})\rangle\\ &\quad+\mathrm{i}e\mathcal{E}_x\frac{\langle\hat{u}_-(\boldsymbol{k})|\hat{v}_y|u_+(\boldsymbol{k})\rangle\langle\hat{u}_+(\boldsymbol{k})|\frac{\partial}{\partial k_x}u_-(\boldsymbol{k})\rangle}{E_--E_+}\\ &\quad-\mathrm{i}e\mathcal{E}_x\frac{\langle\frac{\partial}{\partial k_x}\hat{u}_-(\boldsymbol{k})|u_+(\boldsymbol{k})\rangle\langle\hat{u}_+(\boldsymbol{k})|\hat{v}_y|u_-(\boldsymbol{k})\rangle}{E_--E_+},\end{aligned} \tag{20}$$

where we have assumed that the electric field is along the x-axis. For linear responses, the higher-order terms of $\mathcal{E}_x$ can be ignored. With the velocity operator defined by $\hat{v}_y=\partial\hat{H}_{\mathrm{S}}(\boldsymbol{k})/\partial k_y$ and making use of the following identities:

$$\langle\hat{u}_+(\boldsymbol{k})|\frac{\partial}{\partial k_\alpha}u_-(\boldsymbol{k})\rangle=\frac{\langle\hat{u}_+(\boldsymbol{k})|\frac{\partial\hat{H}_{\mathrm{eff}}}{\partial k_\alpha}|u_-(\boldsymbol{k})\rangle}{E_--E_+},$$

$$\left\langle \frac{\partial}{\partial k_\alpha}\hat{u}_-(\boldsymbol{k})|u_+(\boldsymbol{k})\right\rangle = \frac{\langle \hat{u}_-(\boldsymbol{k})|\frac{\partial \hat{H}_{\rm eff}}{\partial k_\alpha}|u_+(\boldsymbol{k})\rangle}{E_- - E_+}, \quad (21)$$

we derive the average of v_y as follows:

$$\bar{v}_y = \frac{\partial E_-}{\partial k_y} - \frac{e\mathcal{E}_x}{2h^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{d}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{h}(\boldsymbol{k}), \quad (22)$$

where $h(\boldsymbol{k}) = (E_+ - E_-)/2$. For a filled band, the sum over the first term in the velocity is zero. The second term that contributes to the Hall current can be calculated by $\langle \hat{u}_\pm(\boldsymbol{k})|\sigma_i|u_\pm(\boldsymbol{k})\rangle = \pm h_i(\boldsymbol{k})/h(\boldsymbol{k})$. In two-dimensional $\boldsymbol{k}$ space, the Hall current follows, $-e\int \mathrm{d}k_x\mathrm{d}k_y/(2\pi)^2\bar{v}_y$. The response to the external electric field is given by

$$\sigma_{\rm H} = \frac{e^2}{h}(N + \mathrm{i}N_\Gamma), \quad (23)$$

with

$$\begin{aligned} N &= \int \frac{\mathrm{d}k_x\mathrm{d}k_y}{4\pi h^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{h}(\boldsymbol{k}),\\ N_\Gamma &= \int \frac{\mathrm{d}k_x\mathrm{d}k_y}{4\pi h^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{\Gamma}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{h}(\boldsymbol{k}), \end{aligned} \quad (24)$$

where h stands for the Planck constant and N is known as the non-Hermitian Chern number defined in Ref. [11]. We observe that this response can no longer be proportional to the Chern number of the non-Hermitian system. The response $\sigma_{\rm H}$ for the non-Hermitian system can be divided into two terms. The first term is the Chern number of the non-Hermitian system, while the second term can be treated as a correction that suggests the relationship between the Chern number and the conductance might fail for open systems, which is not quantized and a delay in the response can be found.[66] Specific details will be clearer after simplifying Eq. (24), leading to

$$\begin{aligned} \sigma_{\rm H} &= \mathrm{Re}(\sigma_{\rm H}) + \mathrm{i}\mathrm{Im}(\sigma_{\rm H})\\ &= \frac{e^2}{h}[(N - \mathrm{Im}(N_\Gamma)) + \mathrm{i}\mathrm{Re}(N_\Gamma)], \end{aligned} \quad (25)$$

with

$$\begin{aligned} \mathrm{Im}(N_\Gamma) &= \int \frac{\mathrm{d}k_x\mathrm{d}k_y}{2\pi}\frac{\Gamma^2(\boldsymbol{k})}{\mathrm{d}^2(\boldsymbol{k}) + \Gamma^2(\boldsymbol{k})}\Omega_{xy},\\ \mathrm{Re}(N_\Gamma) &= \int \frac{\mathrm{d}k_x\mathrm{d}k_y}{2\pi}\frac{\Gamma(\boldsymbol{k})\mathrm{d}(\boldsymbol{k})}{\mathrm{d}^2(\boldsymbol{k}) + \Gamma^2(\boldsymbol{k})}\Omega_{xy}, \end{aligned} \quad (26)$$

and

$$\Omega_{xy} = \frac{1}{2h^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{h}(\boldsymbol{k}), \quad (27)$$

which is the key result of this work. $\mathrm{Re}(\sigma_{\rm H})$ and $\mathrm{Im}(\sigma_{\rm H})$ are the real and imaginary parts of $\sigma_{\rm H}$, respectively. The first part $\mathrm{Re}(\sigma_{\rm H})$ is related to the non-Hermitian Chern number, and it is no longer quantized in general. The second part $\mathrm{Im}(\sigma_{\rm H})$ can be understood as the environment induced delay for the system in its response to the electric field, which is reminiscent of the complex admittance in a delay circuit with capacitor and inductor. The energy bands of the non-Hermitian systems are complex in general, which indicates that energy and particle may exchange between system and environment. Due to the decay effect, the Hall conductance is generalized to Hall admittance. So we define the real part of the $\sigma_{\rm H}$ as the Hall conductance and the imaginary part as the Hall susceptance. When $\Gamma(\boldsymbol{k}) = 0$, the Hall conductance returns to the quantized result and the delay disappears,

$$\mathrm{Re}(\sigma_{\rm H}) = \frac{e^2}{h}\int \frac{\mathrm{d}k_x\mathrm{d}k_y}{4\pi d^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{d}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{d}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{d}(\boldsymbol{k}). \quad (28)$$

Next we present a more detailed analysis for the Hall conductance in Eq. (25). First, for a general non-Hermitian two bands system, the Berry connection $A^i_{u_-}(\boldsymbol{k})$ for band $|u_-(\boldsymbol{k})\rangle$ with energy E_- is defined as

$$A^i_{u_-}(\boldsymbol{k}) = \mathrm{i}\left\langle \hat{u}_-(\boldsymbol{k})\left|\frac{\partial}{\partial k_i}u_-(\boldsymbol{k})\right.\right\rangle. \quad (29)$$

With Eq. (29), the Berry curvature follows

$$\begin{aligned} \Omega_{u_-}(\boldsymbol{k}) &= \nabla_{\boldsymbol{k}}\times A^i_{u_-}(\boldsymbol{k})\\ &= \mathrm{i}\left[\left\langle\frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_x}\middle|\frac{\partial u_-(\boldsymbol{k})}{\partial k_y}\right\rangle\right.\\ &\quad\left.-\left\langle\frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_y}\middle|\frac{\partial u_-(\boldsymbol{k})}{\partial k_x}\right\rangle\right]. \end{aligned} \quad (30)$$

Then, by some algebras, we can obtain (see Eq. (B5))

$$\Omega_{u_-}(\boldsymbol{k}) = \frac{1}{2h^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{h}(\boldsymbol{k}). \quad (31)$$

For system that $\boldsymbol{\Gamma}(\boldsymbol{k})$ and $\boldsymbol{d}(\boldsymbol{k})$ have the same direction, the Berry curvature $\Omega_{u_-}(\boldsymbol{k})$ for the band $|u_-(\boldsymbol{k})\rangle$ with energy E_- can be rewritten as (see Eq. (C5))

$$\Omega_{u_-}(\boldsymbol{k}) = \frac{1}{2d^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{d}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{d}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{d}(\boldsymbol{k}). \quad (32)$$

Similarly, the Berry curvature $\Omega_{u_+}(\boldsymbol{k})$ for the band $|u_+(\boldsymbol{k})\rangle$ with energy E_+ reads

$$\begin{aligned} \Omega_{u_+}(\boldsymbol{k}) &= \nabla_{\boldsymbol{k}}\times A^i_{u_+}(\boldsymbol{k})\\ &= \mathrm{i}\left[\left\langle\frac{\partial \hat{u}_+(\boldsymbol{k})}{\partial k_x}\middle|\frac{\partial u_+(\boldsymbol{k})}{\partial k_y}\right\rangle - \left\langle\frac{\partial \hat{u}_+(\boldsymbol{k})}{\partial k_y}\middle|\frac{\partial u_+(\boldsymbol{k})}{\partial k_x}\right\rangle\right]\\ &= -\frac{1}{2h^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{h}(\boldsymbol{k})\\ &= -\frac{1}{2d^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{d}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{d}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{d}(\boldsymbol{k}), \end{aligned} \quad (33)$$

which is the same as $\Omega_{u_-}(\boldsymbol{k})$ except for a manus sign. As for $\hat{H}^\dagger_{\rm eff}$, the Berry curvature $\Omega^*_{u_\pm}(\boldsymbol{k})$ for the band $|\hat{u}_\pm(\boldsymbol{k})\rangle$ with energy $E^*_\pm$ reads

$$\Omega^*_{u_\pm}(\boldsymbol{k}) = \mathrm{i}\left[\left\langle\frac{\partial u_\pm(\boldsymbol{k})}{\partial k_x}\middle|\frac{\partial \hat{u}_\pm(\boldsymbol{k})}{\partial k_y}\right\rangle - \left\langle\frac{\partial u_\pm(\boldsymbol{k})}{\partial k_y}\middle|\frac{\partial \hat{u}_\pm(\boldsymbol{k})}{\partial k_x}\right\rangle\right]$$

$$= \mp\frac{1}{2h^{*3}(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{h}^*(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{h}^*(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{h}^*(\boldsymbol{k})$$
$$= \mp\frac{1}{2d^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{d}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{d}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{d}(\boldsymbol{k}). \tag{34}$$

In this case, we prove that non-Hermitian Berry curvatures with k-dependent decay rate $\Omega_{u_\pm}(\boldsymbol{k})$ and $\Omega^*_{u_\pm}(\boldsymbol{k})$ are real and equal to the curvature of the Hermitian system due to that $\boldsymbol{\Gamma}(\boldsymbol{k})$ and $\boldsymbol{d}(\boldsymbol{k})$ have the same direction, leading to quantization of the non-Hermitian Chern number.

Meanwhile the $P(\boldsymbol{k})$ term in Eq. (25) depends on $\Gamma(\boldsymbol{k})$, with $P(\boldsymbol{k})\to 1$ when $\Gamma(\boldsymbol{k})\to 0$. Therefore $\mathrm{Re}(\sigma_{\mathrm{H}})$ is not quantized in general and it is no longer proportional to the Chern number of the non-Hermitian system. However, when $\Gamma(\boldsymbol{k})\to 0$, we can obtain a nearly quantized Hall conductance. And if the spectral density of the environment $J_\zeta(\boldsymbol{k})$ and $d(\boldsymbol{k})$ is linearly dependent such that $\Gamma_\zeta(\boldsymbol{k})=\zeta d(\boldsymbol{k})$, where ζ is a k-independent constant, we can get a Hall conductance as $\mathrm{Re}(\sigma_{\mathrm{H}})=[1/(1+\zeta^2)]n, n\in(0,\pm1)$, i.e., it is not quantized due to the rate $[1/(1+\zeta^2)]$ in the Hall conductance, but it still possesses a plateau in the dependence of the conductance on the parameter of the system.

From Eqs. (25) and (26), we can find that at $h(\boldsymbol{k})=0$ the phase transition occurs

$$h(\boldsymbol{k}) = d(\boldsymbol{k})-\mathrm{i}\Gamma(\boldsymbol{k})$$
$$= [1-\mathrm{i}\Gamma(\boldsymbol{k})/d(\boldsymbol{k})]d(\boldsymbol{k}). \tag{35}$$

Beacuse $\Gamma(\boldsymbol{k})/d(\boldsymbol{k})\ll 1$, the phase transition occurs when $d(\boldsymbol{k})=0$ (where the energy gaps close). This condition of phase transition is the same as the corresponding closed systems. Hence, the phase transition points remain unchanged comparing to the isolated system.

Assuming the decay rates are very small, we can ignore the quadratic term of the decay rates Γ^2. In this situation, the Hall conductance reads

$$\mathrm{Re}(\sigma_{\mathrm{H}})=\frac{e^2}{h}N. \tag{36}$$

In this case, it can be seen that Hall conductance ($\mathrm{Re}(\sigma_{\mathrm{H}})$) is proportional to the non-Hermitian Chern number, and it takes quantized values as the change of the parameter. In order to show the validity of the above conclusions, we compare the above results with the Hall conductance by solving the steady-state solution of the master equation,[54,67]

$$\sigma_{\mathrm{ME}} = \frac{-\mathrm{i}e^2}{2\pi h}\int \mathrm{d}k_x\mathrm{d}k_y\left[\chi(\boldsymbol{k})\left\langle k_+\middle|\frac{\partial k_-}{\partial k_x}\right\rangle\left\langle\frac{\partial k_-}{\partial k_y}\middle|k_+\right\rangle + \mathrm{H.c.}\right], \tag{37}$$

where $\chi(\boldsymbol{k})=d(\boldsymbol{k})/(d(\boldsymbol{k})-\mathrm{i}\Gamma(\boldsymbol{k}))$. To facilitate the comparison with the Hall conductance of the non-Hermitian system, we rewrite the total Hall conductance as

$$\sigma_{\mathrm{ME}}=\sigma^{(0)}_{\mathrm{ME}}+\sigma^{(1)}_{\mathrm{ME}}, \tag{38}$$

where

$$\sigma^{(0)}_{\mathrm{ME}} = \frac{e^2}{2\pi h}\int \mathrm{d}k_x\mathrm{d}k_y P(\boldsymbol{k})\mathrm{i}\left[\left\langle\frac{\partial k_-}{\partial k_x}|\frac{\partial k_-}{\partial k_y}\right\rangle-\left\langle\frac{\partial k_-}{\partial k_y}|\frac{\partial k_-}{\partial k_x}\right\rangle\right]$$
$$= \frac{e^2}{2\pi h}\int \mathrm{d}k_x\mathrm{d}k_y P(\boldsymbol{k})\Omega_{xy},$$
$$\sigma^{(1)}_{\mathrm{ME}} = \frac{e^2}{2\pi h}\int \mathrm{d}k_x\mathrm{d}k_y Q(\boldsymbol{k})\left[\left\langle\frac{\partial k_-}{\partial k_x}|k_+\right\rangle\left\langle k_+|\frac{\partial k_-}{\partial k_y}\right\rangle + \left\langle\frac{\partial k_-}{\partial k_y}|k_+\rangle\langle k_+|\frac{\partial k_-}{\partial k_x}\right\rangle\right], \tag{39}$$

with

$$P(\boldsymbol{k}) = \frac{d^2(\boldsymbol{k})}{d^2(\boldsymbol{k})+\Gamma^2(\boldsymbol{k})},$$
$$Q(\boldsymbol{k}) = \frac{\Gamma(\boldsymbol{k})d(\boldsymbol{k})}{d^2(\boldsymbol{k})+\Gamma^2(\boldsymbol{k})}. \tag{40}$$

The Hall conductance for open systems by solving the steady-state solution of the master equation consists of two terms. The first term $\sigma^{(0)}_{\mathrm{ME}}$ is the weighted integration of curvature of the ground band. The second term $\sigma^{(1)}_{\mathrm{ME}}$ represents the high-order correction of the system-environment coupling to the Hall conductance. It can be found that the second term $\sigma^{(1)}_{\mathrm{ME}}$ has a small contribution to the Hall conductance and the quantum-jump term has negligible effect on the first-order steady-state solution of the master equation. The Hall conductance for the non-Hermitian systems $\mathrm{Re}(\sigma_{\mathrm{H}})$ then equals to the $\sigma^{(0)}_{\mathrm{ME}}$, which verifies that the presented theory is completely valid to ignore the quantum-jump term of the master equation for the system that we studied. When we ignore the quadratic term of the decay rates, the Hall conductance reads

$$\sigma_{\mathrm{ME}} = \frac{e^2}{2\pi h}\int \mathrm{d}k_x\mathrm{d}k_y\left[\mathrm{i}\left[\left\langle\frac{\partial k_-}{\partial k_x}\middle|\frac{\partial k_-}{\partial k_y}\right\rangle-\left\langle\frac{\partial k_-}{\partial k_y}\middle|\frac{\partial k_-}{\partial k_x}\right\rangle\right] + \frac{\Gamma(\boldsymbol{k})}{d(\boldsymbol{k})}\left[\left\langle\frac{\partial k_-}{\partial k_x}|k_+\rangle\langle k_+|\frac{\partial k_-}{\partial k_y}\right\rangle + \left\langle\frac{\partial k_-}{\partial k_y}|k_+\rangle\langle k_+|\frac{\partial k_-}{\partial k_x}\right\rangle\right]\right]. \tag{41}$$

Although the Hall conductance can be calculated by the master equation description, it can not establish a direct relationship between the Hall conductance and the topological invariant even if we ignore the quadratic term of the decay rates.

4. Example

To demonstrate the response theory of the non-Hermitian two-band system, we consider the following example of $d_\alpha(\boldsymbol{k})$

$$d_x = \sin(k_y),$$
$$d_y = -\sin(k_x),$$
$$d_z = t[2-m-\cos(k_x)-\cos(k_y)]. \tag{42}$$

This model describes a time reversal symmetry-breaking system, which might be a magnetic semiconductor with Rashba-type spin–orbit coupling, spin-dependent effective mass, and a uniform magnetization in the z direction.[68] This model can be realized in graphene with Fe atoms are adsorbed on top, which possesses quantum anomalous Hall effect in the presence of both Rashba spin–orbit coupling and an exchange field.[69]

The corresponding complex band structures $E_+ = d(\boldsymbol{k}) - 2\mathrm{i}\Gamma(\boldsymbol{k})$ and $E_- = -d(\boldsymbol{k})$, are shown in Fig. 1. Considering that the mode density is $\xi(\boldsymbol{k})$, both $\mathrm{Re}(E_\pm)$ and $\mathrm{Im}(E_\pm)$ are zero (see Figs. 1(a) and 1(b)) at $(k_x, k_y) = (0, \pi)$ in the tours surface with the parameter $m = 2$. We conclude that the phase transition point is at $m = 2$, the same as in the closed system. Real and imaginary parts of E_+ and E_- for all $\boldsymbol{k}$ with $m = 3$ (see Figs. 1(c) and 1(d)) have a gap between them and the band structures are the topological nontrivial. The imaginary part of the energy eigenvalue E_+, the decay rate, depends on the Bloch vector $\boldsymbol{k}$ (see Fig. 1(b)), which indicates that the relaxation time of the particle is different at different positions in the momentum space. Comparing $\mathrm{Re}(E_\pm)$ and $\mathrm{Im}(E_\pm)$ with the same parameters, we find that they have similar distribution in the momentum space, which is easy to understand because the relaxation time of the particle is relatively short at a higher energy Bloch band under the influence of the environment.

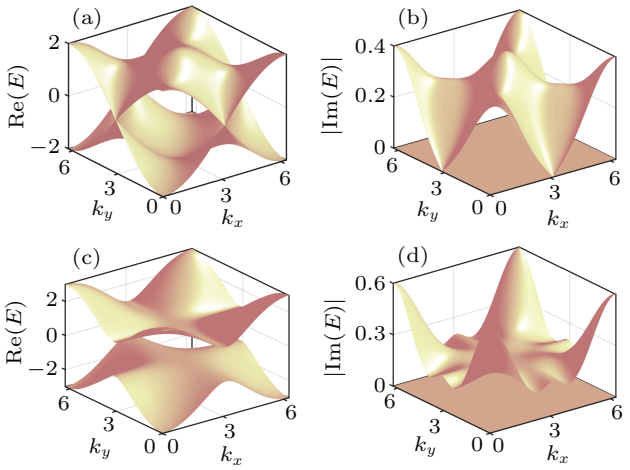


Fig. 1. Complex band structures of the non-Hermitian Chern insulator $\mathrm{Re}(E)_\pm = \pm d(\boldsymbol{k})$, $|\mathrm{Im}(E_+)| = 2\Gamma(\boldsymbol{k})$, and $|\mathrm{Im}(E_-)| = 0$. The mode density $\xi(\boldsymbol{k})$ is taken to plot this figure. (a), (b) Real and imaginary parts of the gapless bands with an exceptional point on $(k_x, k_y) = (0, \pi)$ in the tours surface with chosen parameters $m = 2$, $\delta = 0.05$, and $t = 1$. (c), (d) Real and imaginary parts of the gapped bands with $m = 3$, $\delta = 0.05$, and $t = 1$.

The Hall conductance and Hall susceptance defined in Eq. (25) are shown in Fig. 2. In the isolated system limit, i.e., $\Gamma(\boldsymbol{k}) \to 0$, the Hamiltonian model reduces to a Hermitian model. In this case, the real part of the Hall conductance $\mathrm{Re}(\sigma_\mathrm{H}) = -1$ for $0 < m < 2$, while for $2 < m < 4$, $\mathrm{Re}(\sigma_\mathrm{H}) = 1$, and $\mathrm{Im}(\sigma_\mathrm{H})$ remains zero for closed systems, namely, the Hall conductance is quantized without delay. For the open system, however, as shown in Fig. 2(a), we find that the phase transition points, i.e., $m = 0$, 2, 4, remain unchanged, but the Hall conductance is no longer quantized under the influence of the environment. Although the Hall conductance is nearly quantized with $\delta = 0.1$. The feature of non-quantization can be clearly observed when δ is increased to 0.3 and beyond. Besides, as Eq. (12) shows, the parameter β is k-dependent, which leads the Hall conductance no longer possesses a plateau in its dependence on the system parameter. The details of the dependence are closely related to the choice of the model and the spectral density of the environment. In Fig. 2(b), we show the Hall susceptance $\mathrm{Im}(\sigma_\mathrm{H})$ which can be understood a delay in the response due to the system-environment couplings. $\mathrm{Im}(\sigma_\mathrm{H})$ increases with δ (we show here from 0.1 to 0.3), while $\mathrm{Re}(\sigma_\mathrm{H})$ decreases as δ changing from 0.1 to 0.3.

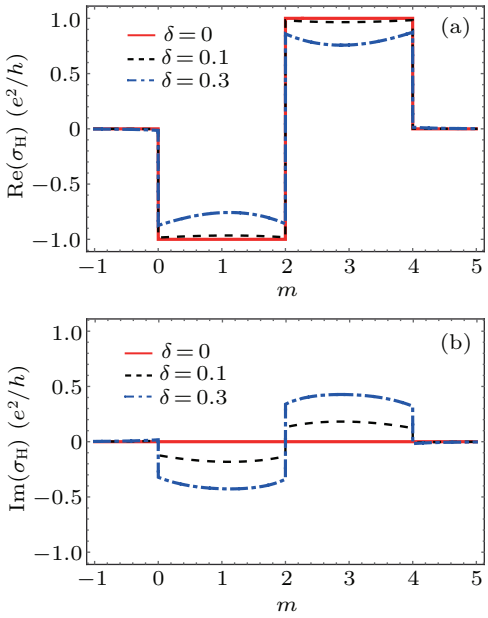


Fig. 2. The Hall conductance $\mathrm{Re}(\sigma_\mathrm{H})$ and the Hall susceptance $\mathrm{Im}(\sigma_\mathrm{H})$ (in units of e^2/h) as a function of m with different δ given by Eq. (25) and with $\xi(\boldsymbol{k})$ as the mode density. For comparison, the red solid line corresponds to the conventional Hall conductance of closed system ($t = 1$).

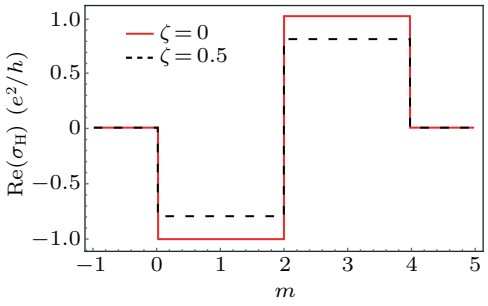


Fig. 3. The Hall conductance $\mathrm{Re}(\sigma_\mathrm{H})$ (in units of e^2/h) as a function of m with different ζ given by Eq. (25) with spectral density of the environment $J_\zeta(\boldsymbol{k})$. For comparison, the red solid line corresponds to the conventional Hall conductance of the corresponding closed system ($t = 1$).

In Fig. 3, we show the Hall conductance of the non-

Hermitian two-band systems $\mathrm{Re}(\sigma_{\mathrm{H}})$ for different ζ. We consider the spectral density of the environment $J_\zeta(\boldsymbol{k})$ and $d(\boldsymbol{k})$ are linear. One can find that the Hall conductance still possesses a plateau as the change of m, but its value is no longer an integer, for instance, $\mathrm{Re}(\sigma_{\mathrm{H}}) = 0.8$ when $\zeta = 0.5$.

In Fig. 4, we show the Hall conductance for the non-Hermitian two-band systems $\mathrm{Re}(\sigma_{\mathrm{H}})$ and the Hall conductance for open systems by solving the master equation σ_{ME}. The same mode density $\xi(\boldsymbol{k})$ as that used in Fig. 4(a) has been taken to calculate the results in Fig. 4(a), while Fig. 4(b) for the Ohmic spectral density. One can see that the Hall conductance obtained by the two methods is almost the same, because the quantum-jump term has negligible effect on the first-order steady-state solution of the master equation. Hence in this case, it is a good approximation to approximate the model by a non-Hermitian Hamiltonian. The non-Hermitian Hamiltonian is obviously more convenient to describe such a system, because it is easy to study the eigenvectors and eigenvalues of an effective Hamiltonian and the calculation is straightforward.

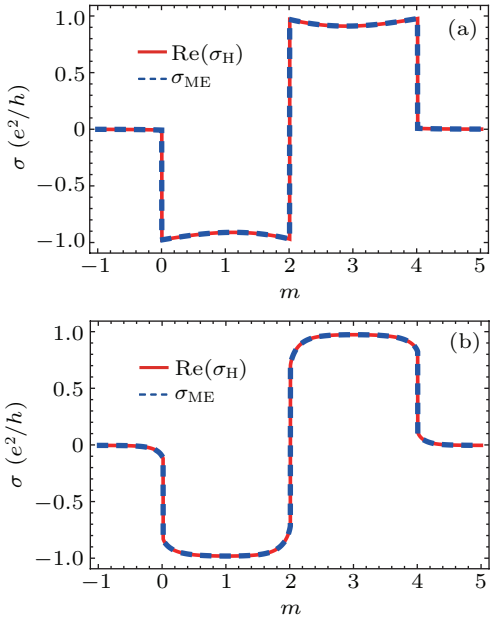


Fig. 4. The Hall conductance $\mathrm{Re}(\sigma_{\mathrm{H}})$ and the Hall conductance σ_{ME} (in units of e^2/h) as a function of m given by Eqs. (25) and (38), respectively. The red solid line is the Hall conductance for the non-Hermitian two-band systems, while the blue dashed line is for the Hall conductance obtained by solving the master equation. Panel (a) is plotted for mode density $\xi(\boldsymbol{k})$ with $\delta = 0.2$, $t = 1$, while (b) is for the Ohmic spectral density of the environment with $\alpha = 0.2$, $\omega_{\mathrm{c}} = 1$, and $t = 1$.

5. Conclusions

We have developed the response theory for the non-Hermitian two-band systems and applied it into topological insulators subjected to environments. An effective non-Hermitian system is obtained by simplifying the Markov master equation by ignoring the jump terms and a decay rate that depends on the Bloch vector is given. Based on this formalism, the Hall conductance is calculated by the adiabatic perturbation theory. We find that although the phase transition point does not changes, the Hall conductance that depends on the strength of the decay rate and distribution of the electron on the Bloch band is a weighted integration of curvature of the ground band and it is not quantized in general. In addition, the system-environment coupling induces a delay in the response of the topological insulator to the constant electric field. Finally, comparing the Hall conductance obtained by the non-Hermitian two-band model with that by solving the master equation, we claim that the non-Hermitian Hamiltonian description is a good approximation and it can provide insight understanding for the response of open systems more than that of the master equation description.

Appendix A: System coupled to Fermi reservoir

Similarly, the Hamiltonian of Fermi reservoir reads $\hat{H}'_{\mathrm{R}} = \sum_n \omega_n \hat{C}_n^\dagger \hat{C}_n$ with $\{C_n^\dagger, C_n\} = 1$. In the Born–Markov approximation, we obtain the following master equation:

$$\dot{\rho}'_{\mathrm{S}} = -\mathrm{i}\left[\hat{H}_{\mathrm{S}}(\boldsymbol{k}), \rho_{\mathrm{S}}\right] + \mathcal{U}'(\rho_{\mathrm{S}}) + \mathcal{D}'(\rho_{\mathrm{S}}), \tag{A1}$$

where

$$\begin{aligned}\mathcal{U}'(\rho_{\mathrm{S}}) &= \gamma(\boldsymbol{k})(1 - N'(\boldsymbol{k}))[2\sigma_-(\boldsymbol{k})\rho_{\mathrm{S}}\sigma_+(\boldsymbol{k}) \\ &\quad - \sigma_+(\boldsymbol{k})\sigma_-(\boldsymbol{k})\rho_{\mathrm{S}} - \rho_{\mathrm{S}}\sigma_+(\boldsymbol{k})\sigma_-(\boldsymbol{k})], \\ \mathcal{D}'(\rho_{\mathrm{S}}) &= \gamma(\boldsymbol{k})N'(\boldsymbol{k})[2\sigma_+(\boldsymbol{k})\rho_{\mathrm{S}}\sigma_-(\boldsymbol{k}) \\ &\quad - \sigma_-(\boldsymbol{k})\sigma_+(\boldsymbol{k})\rho_{\mathrm{S}} - \rho_{\mathrm{S}}\sigma_-(\boldsymbol{k})\sigma_+(\boldsymbol{k})].\end{aligned} \tag{A2}$$

Here, $N'(\boldsymbol{k}) = \{\exp[2d(\boldsymbol{k})/k_{\mathrm{B}}T] + 1\}^{-1}$. The effective non-Hermitian Hamiltonian for the Fermi reservoir can be written as

$$\begin{aligned}\hat{H}'_{\mathrm{eff}} &= (d(\boldsymbol{k}) - \mathrm{i}\gamma(\boldsymbol{k})[1 - N'(\boldsymbol{k})])|k_+\rangle\langle k_+| \\ &\quad - (d(\boldsymbol{k}) + \mathrm{i}\gamma(\boldsymbol{k})N'(\boldsymbol{k}))|k_-\rangle\langle k_-|.\end{aligned} \tag{A3}$$

Note that in the limit $T \to 0$, $N'(\boldsymbol{k}) = N(\boldsymbol{k}) = 0$. Hence, the effective non-Hermitian Hamiltonians for Fermi reservoir and Bose reservoir are the same. On the other hand, for finite-temperature open quantum systems, $N'(\boldsymbol{k}) \neq N(\boldsymbol{k})$, $1 - N'(\boldsymbol{k}) \neq 1 + N(\boldsymbol{k})$, the relaxation time of the particles is notably different in both cases.

Appendix B: Derivation of Eq. (31) from Eq. (30)

For a general non-Hermitian two bands system, the Berry curvature $\Omega_{u_-}(\boldsymbol{k})$ for the band $|u_-(\boldsymbol{k})\rangle$ with energy E_- follows

$$\begin{aligned}\Omega_{u_-}(\boldsymbol{k}) &= \nabla_{\boldsymbol{k}} \times A^i_{u_-}(\boldsymbol{k}) \\ &= \mathrm{i}\left[\left\langle \frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_x} \middle| \frac{\partial u_-(\boldsymbol{k})}{\partial k_y} \right\rangle - \left\langle \frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_y} \middle| \frac{\partial u_-(\boldsymbol{k})}{\partial k_x} \right\rangle\right].\end{aligned} \tag{B1}$$

Making use of the the completeness relation

$$\sum_n |u_n(\boldsymbol{k})\rangle\langle \hat{u}_n(\boldsymbol{k})| = 1, \tag{B2}$$

we can get

$$\begin{aligned}\Omega_{u_-}(\boldsymbol{k}) = \mathrm{i}\Bigg[&\left\langle \frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_x}|u_+(\boldsymbol{k})\rangle\langle\hat{u}_+(\boldsymbol{k})|\frac{\partial u_-(\boldsymbol{k})}{\partial k_y}\right\rangle \\ &+\left\langle \frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_x}|u_-(\boldsymbol{k})\rangle\langle\hat{u}_-(\boldsymbol{k})|\frac{\partial u_-(\boldsymbol{k})}{\partial k_y}\right\rangle \\ &-\left\langle \frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_y}|u_+(\boldsymbol{k})\rangle\langle\hat{u}_+(\boldsymbol{k})|\frac{\partial u_-(\boldsymbol{k})}{\partial k_x}\right\rangle \\ &-\left\langle \frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_y}|u_-(\boldsymbol{k})\rangle\langle\hat{u}_-(\boldsymbol{k})|\frac{\partial u_-(\boldsymbol{k})}{\partial k_x}\right\rangle\Bigg], \end{aligned} \tag{B3}$$

where the second term add the forth term equals to zero. Then we have

$$\begin{aligned}\Omega_{u_-}(\boldsymbol{k}) = \mathrm{i}\Bigg[&\left\langle \frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_x}|u_+(\boldsymbol{k})\rangle\langle\hat{u}_+(\boldsymbol{k})|\frac{\partial u_-(\boldsymbol{k})}{\partial k_y}\right\rangle \\ &-\left\langle \frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_y}|u_+(\boldsymbol{k})\rangle\langle\hat{u}_+(\boldsymbol{k})|\frac{\partial u_-(\boldsymbol{k})}{\partial k_x}\right\rangle\Bigg]. \end{aligned} \tag{B4}$$

With $\langle\hat{u}_\pm(\boldsymbol{k})|\hat{H}_{\mathrm{eff}} = \langle\hat{u}_\pm(\boldsymbol{k})|E_\pm$ we can get

$$\begin{aligned}\left\langle \frac{\partial \hat{u}_-(\boldsymbol{k})}{\partial k_{x(y)}}\middle|u_+(\boldsymbol{k})\right\rangle &= \frac{\left\langle \hat{u}_-(\boldsymbol{k})\middle|\frac{\partial \hat{H}_{\mathrm{eff}}}{\partial k_{x(y)}}\middle|u_+(\boldsymbol{k})\right\rangle}{E_- - E_+}, \\ \left\langle \frac{\partial \hat{u}_+(\boldsymbol{k})}{\partial k_{x(y)}}\middle|u_-(\boldsymbol{k})\right\rangle &= \frac{\left\langle \hat{u}_+(\boldsymbol{k})\middle|\frac{\partial \hat{H}_{\mathrm{eff}}}{\partial k_{x(y)}}\middle|u_-(\boldsymbol{k})\right\rangle}{E_+ - E_-}. \end{aligned} \tag{B5}$$

By Eq. (B5), and $\sum_n |u_n(\boldsymbol{k})\rangle\langle\hat{u}_n(\boldsymbol{k})| = 1$, we can obtain

$$\Omega_{u_-}(\boldsymbol{k}) = \mathrm{i}\frac{\left\langle \hat{u}_-(\boldsymbol{k})\middle|\frac{\partial \hat{H}_{\mathrm{eff}}}{\partial k_x}\frac{\partial \hat{H}_{\mathrm{eff}}}{\partial k_y} - \frac{\partial \hat{H}_{\mathrm{eff}}}{\partial k_y}\frac{\partial \hat{H}_{\mathrm{eff}}}{\partial k_x}\middle|u_-(\boldsymbol{k})\right\rangle}{(E_- - E_+)^2}. \tag{B6}$$

For a general non-Hermitian two-band system, the Hamiltonian reads

$$\hat{H}_{\mathrm{eff}} = \sum_{i=x,y,z} h_i(\boldsymbol{k})\cdot\sigma_i + h_0(\boldsymbol{k})\sigma_0, \tag{B7}$$

where $\boldsymbol{h}(\boldsymbol{k}) = \boldsymbol{d}(\boldsymbol{k}) - \mathrm{i}\boldsymbol{\Gamma}(\boldsymbol{k})$, $h_0(\boldsymbol{k}) = d_0(\boldsymbol{k}) - \mathrm{i}\Gamma_0(\boldsymbol{k})$ and σ_0 is the 2×2 identity matrix. $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is Pauli matrix. The eigenvalues are

$$E_\pm = \pm\sqrt{d^2(\boldsymbol{k}) - \Gamma^2(\boldsymbol{k}) - 2\mathrm{i}\boldsymbol{d}(\boldsymbol{k})\cdot\boldsymbol{\Gamma}(\boldsymbol{k})},$$

and we define $h(\boldsymbol{k}) \equiv E_+$. Furthermore, with

$$\begin{aligned}\frac{\partial \hat{H}_{\mathrm{eff}}}{\partial k_x} &= \sum_{\alpha=x,y,z}\frac{\partial h_\alpha(\boldsymbol{k})}{\partial k_x}\sigma_\alpha + \frac{\partial h_0(\boldsymbol{k})}{\partial k_x}\sigma_0, \\ \frac{\partial \hat{H}_{\mathrm{eff}}}{\partial k_y} &= \sum_{\beta=x,y,z}\frac{\partial h_\beta(\boldsymbol{k})}{\partial k_y}\sigma_\beta + \frac{\partial h_0(\boldsymbol{k})}{\partial k_y}\sigma_0, \end{aligned} \tag{B8}$$

and $[\sigma_i, \sigma_0] = 0$, we have

$$\Omega_{u_-}(\boldsymbol{k}) = \mathrm{i}\frac{\sum_{\alpha,\beta}\frac{\partial h_\alpha(\boldsymbol{k})}{\partial k_x}\frac{\partial h_\beta(\boldsymbol{k})}{\partial k_y}\langle\hat{u}_-(\boldsymbol{k})|\sigma_\alpha\sigma_\beta - \sigma_\beta\sigma_\alpha|u_-(\boldsymbol{k})\rangle}{(E_- - E_+)^2}. \tag{B9}$$

Then with $[\sigma_\alpha, \sigma_\beta] = 2\mathrm{i}\sigma_\gamma$, where $\gamma = x, y, z$ and $\langle\hat{u}_\pm(\boldsymbol{k})|\sigma_\gamma|u_\pm(\boldsymbol{k})\rangle = \pm h_\gamma(\boldsymbol{k})/h(\boldsymbol{k})$, we finally obtain

$$\Omega_{u_-}(\boldsymbol{k}) = \frac{1}{2h^3(\boldsymbol{k})}\left(\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial \boldsymbol{h}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{h}(\boldsymbol{k}). \tag{B10}$$

Appendix C: Derivation of Eq. (C1)

For the system that $\boldsymbol{\Gamma}(\boldsymbol{k})$ and $\boldsymbol{d}(\boldsymbol{k})$ are proportional, the Berry curvature $\Omega_{u_-}(\boldsymbol{k})$ for the band $|u_-(\boldsymbol{k})\rangle$ with energy E_- can be written as

$$\Omega_{u_-}(\boldsymbol{k}) = \frac{\left(\frac{\partial[\boldsymbol{d}(\boldsymbol{k})f(\boldsymbol{k})]}{\partial k_x}\times\frac{\partial[\boldsymbol{d}(\boldsymbol{k})f(\boldsymbol{k})]}{\partial k_y}\right)\cdot[\boldsymbol{d}(\boldsymbol{k})f(\boldsymbol{k})]}{2[d(\boldsymbol{k})f(\boldsymbol{k})]^3}, \tag{C1}$$

where $f(\boldsymbol{k}) = 1 - \mathrm{i}\Gamma(\boldsymbol{k})/d(\boldsymbol{k})$. And by simplifying calculation, we can obtain

$$\begin{aligned}\Omega_{u_-}(\boldsymbol{k}) &= \frac{\left(\frac{f(\boldsymbol{k})\partial\boldsymbol{d}(\boldsymbol{k})}{\partial k_x}\times\frac{f(\boldsymbol{k})\partial\boldsymbol{d}(\boldsymbol{k})}{\partial k_y}\right)\cdot[\boldsymbol{d}(\boldsymbol{k})f(\boldsymbol{k})]}{2[d(\boldsymbol{k})f(\boldsymbol{k})]^3} \\ &= \frac{\left(\frac{f(\boldsymbol{k})\partial\boldsymbol{d}(\boldsymbol{k})}{\partial k_x}\times\frac{\boldsymbol{d}(\boldsymbol{k})\partial f(\boldsymbol{k})}{\partial k_y}\right)\cdot[\boldsymbol{d}(\boldsymbol{k})f(\boldsymbol{k})]}{2[d(\boldsymbol{k})f(\boldsymbol{k})]^3} \\ &= \frac{\left(\frac{\boldsymbol{d}(\boldsymbol{k})\partial f(\boldsymbol{k})}{\partial k_x}\times\frac{f(\boldsymbol{k})\partial\boldsymbol{d}(\boldsymbol{k})}{\partial k_y}\right)\cdot[\boldsymbol{d}(\boldsymbol{k})f(\boldsymbol{k})]}{2[d(\boldsymbol{k})f(\boldsymbol{k})]^3} \\ &= \frac{\left(\frac{\boldsymbol{d}(\boldsymbol{k})\partial f(\boldsymbol{k})}{\partial k_x}\times\frac{\boldsymbol{d}(\boldsymbol{k})\partial f(\boldsymbol{k})}{\partial k_y}\right)\cdot[\boldsymbol{d}(\boldsymbol{k})f(\boldsymbol{k})]}{2[d(\boldsymbol{k})f(\boldsymbol{k})]^3}. \end{aligned} \tag{C2}$$

It can be seen that the last three terms of the above formula are all zero. Hence, the Berry curvature $\Omega_{u_-}(\boldsymbol{k})$ for the band $|u_-(\boldsymbol{k})\rangle$ with energy E_- can be rewritten as

$$\Omega_{u_-}(\boldsymbol{k}) = \frac{1}{2d^3(\boldsymbol{k})}\left(\frac{\partial\boldsymbol{d}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial\boldsymbol{d}(\boldsymbol{k})}{\partial k_y}\right)\cdot\boldsymbol{d}(\boldsymbol{k}). \tag{C3}$$

Acknowledgements

The authors acknowledge Hongzhi Shen and Weijun Cheng for helpful comments. This work was supported by the National Natural Science Foundation of China (Grant Nos. 12175033 and 12147206).

References

[1] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
[2] Yang Z, Chiu C K, Fang C and Hu J 2020 *Phys. Rev. Lett.* **124** 186402
[3] Bernevig B A and Hughes T L 2013 *Topological insulators and topeological superconductors* (Princeton: Princeton University Press)
[4] Qi X L and Zhang S C 2011 *Rev. Mod. Phys.* **83** 1057
[5] Thouless D J, Kohmoto M, Nightingale M P and Nijs M den 1982 *Phys. Rev. Lett.* **49** 405
[6] Klitzing K V, Dorda G and Pepper M 1980 *Phys. Rev. Lett.* **45** 494
[7] Haldane F D M 1980 *Phys. Rev. Lett.* **45** 494
[8] San-Jose P, Cayao J, Prada E and Aguado R 2016 *Sci. Rep.* **6** 21427
[9] Leykam D, Bliokh K Y, Huang C, Chong Y D and Nori F 2017 *Phys. Rev. Lett.* **118** 040401
[10] Rudner M S and Levitov L S 2009 *Phys. Rev. Lett.* **102** 065703
[11] Shen H, Zhen B and Fu L 2018 *Phys. Rev. Lett.* **120** 146402
[12] Kunst F K, Edvardsson E, Budich J C and Bergholtz E J 2018 *Phys. Rev. Lett.* **121** 026808
[13] Yao S Y and Wang Z 2018 *Phys. Rev. Lett.* **121** 086803
[14] Yao S Y, Song F and Wang Z 2018 *Phys. Rev. Lett.* **121** 136802
[15] Xiong Y 2018 *J. Phys. Commun.* **2** 035043

[16] Helbig T, Hofmann T, Imhof S, Abdelghany M, Kiessling T, Molenkamp L W, Lee C H, Szameit A, Greiter M and Thomale R 2020 *Nat. Phys.* **10** 1038
[17] Yang Z S, Zhang K, Fang C and Hu J P 2020 *Phys. Rev. Lett.* **125** 226402
[18] Okuma N, Kawabata K, Shiozaki K and Sato M 2020 *Phys. Rev. Lett.* **124** 086801
[19] Martinez Alvarez V M, Barrios Vargas J E and Foa Torres L E F 2018 *Phys. Rev. B* **97** 121401(R)
[20] Lee C H and Thomale R 2019 *Phys. Rev. B* **99** 201103(R)
[21] Li L H, Lee C H and Gong J B 2020 *Phys. Rev. Lett.* **124** 250402
[22] Hofmann T, Helbig T, Schindler F, Salgo N, Brzezińska M, Greiter M, Kiessling T, Wolf D, Vollhardt A, Kabaši A, Lee Ching Hua, Bilušić A, Thomale R and Neupert T 2020 *Phys. Rev. Res.* **2** 023265
[23] Torres Luis E F Foa 2020 *J. Phys. Mater.* **3** 014002
[24] Zhang K, Yang Z S and Fang C 2020 *Phys. Rev. Lett.* **125** 126402
[25] Yi Y F and Yang Z S 2020 *Phys. Rev. Lett.* **125** 186802
[26] Lee T E 2016 *Phys. Rev. Lett.* **116** 133903
[27] Kawabata K, Shiozaki K and Ueda M 2018 *Phys. Rev. B* **98** 165148
[28] Lieu S 2018 *Phys. Rev. B* **97** 045106
[29] Lieu S 2018 *Phys. Rev. B* **98** 115135
[30] Gong Z, Ashida Y, Kawabata K, Takasan K, Higashikawa S and Ueda M 2018 *Phys. Rev. X* **8** 031079
[31] Kawabata K, Higashikawa S, Gong Z, Ashida Y and Ueda M 2019 *Nat. Commun.* **10** 297
[32] Liang S D and Huang G Y 2013 *Phys. Rev. A* **87** 012118
[33] Wang J H, Tao Y L and Xu Y 2022 *Chin. Phys. Lett.* **39** 010301
[34] Liu H F, Su Z X, Zhang Z Q and Jiang H 2020 *Chin. Phys. B* **29** 050502
[35] Hirsbrunner M R, Philip T M and Gilbert M J 2019 *Phys. Rev. B* **100** 081104(R)
[36] Chen Y and Zhai H 2018 *Phys. Rev. B* **98** 245130
[37] Philip T M, Hirsbrunner M R and Gilbert M J 2018 *Phys. Rev. B* **98** 155430
[38] Pan L, Chen X, Chen Y and Zhai H 2020 *Nat. Phys.* **16** 767
[39] Schomerus H 2020 *Phys. Rev. Res.* **2** 013058
[40] Xiao L, Zhan X, Bian Z H, Wang K K, Zhang X, Wang X P, Li J, Mochizuki K, Kim D, Kawakami N, Yi W, Obuse H, Sanders B C and Xue P 2017 *Nat. Phys.* **13** 1117
[41] Lee C H, Li L H, Thomale Ronny and Gong J B 2020 *Phys. Rev. B* **102** 085151
[42] Parto M, Wittek S, Hodaei H, Harari G, Bandres M A, Ren J, Rechtsman M C, Segev M, Christodoulides D N and Khajavikhan M 2018 *Phys. Rev. Lett.* **120** 113901
[43] Zhou H, Peng C, Yoon Y, Hsu C W, Nelson K A, Fu L, Joannopoulos J D, Soljačić M and Zhen B 2018 *Science* **359** 1009
[44] Rechtsman M C, Plotnik Y, Lumer Y, Nolte S, Rudner M S, Segev M and Szameit A 2015 *Phys. Rev. Lett.* **115** 040402
[45] Zhao H, Miao P, Teimourpour M H, Malzard S, El-Ganainy R, Schomerus H and Feng L 2018 *Nat. Commun.* **9** 981
[46] Pan M, Zhao H, Miao P, Longhi S and Feng L 2018 *Nat. Commun.* **9** 1308
[47] Weimann S, Kremer M, Plotnik Y, Lumer Y, Nolte S, Makris K G, Segev M, Rechtsman M C and Szameit A 2017 *Nat. Mater.* **16** 433
[48] Rigolin G, Ortiz G and Ponce V H 2008 *Phys. Rev. A* **78** 052508
[49] Bohm A, Mostafazadeh A, Koizumi H, Niu Q and Zwanziger J 2003 *The geometric phase in quantum systems* (Berlin: Springer)
[50] Ibáñez S and Muga J G 2014 *Phys. Rev. A* **89** 033403
[51] Weng H W, Yu R, Hu X, Dai X and Fang Z 2015 *Adv. Phys.* **64** 227
[52] Peierls R 1993 *Z. Physik* **80** 763
[53] Zhang W Q, Shen H Z and Yi X X 2018 *Sci. Rep.* **8** 1475
[54] Shen H Z, Zhang S S, Dai C M and Yi X X 2018 *J. Phys. A: Math. Theor.* **51** 065302
[55] Orszag M 2000 *Quantum Optics* (Berlin: Springer)
[56] Breuer H P and Petruccione F 2007 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
[57] Dürr S, Garcia-Ripoll J J, Syassen N, Bauer D M, Lettner M, Cirac J I and Rempe G 2009 *Phys. Rev. A* **79** 023614
[58] Reiter F and Sørensen A S 2012 *Phys. Rev. A* **85** 032111
[59] Reiter F, Kastoryano M J and Sørensen A S 2012 *New J. Phys.* **14** 053022
[60] Lee T E and Chan C K 2014 *Phys. Rev. X* **4** 041001
[61] Ashida Y, Furukawa S and Ueda M 2016 *Phys. Rev. A* **94** 053615
[62] Ashida Y, Furukawa S and Ueda M 2017 *Nat. Commun.* **8** 15791
[63] Yamamoto K, Nakagawa M, Adachi K, Takasan K, Ueda M and Kawakami N 2019 *Phys. Rev. Lett.* **123** 123601
[64] Ziesche P 1987 *J. Phys. A: Math. Gen.* **20** 2859
[65] Dattoli G and Torre A 1990 *Phys. Rev. A* **42** 1467
[66] Li D X, Shen H Z, Liu H D and Yi X X *Sci. Rep.* **8** 1475
[67] Shen H Z, Wang W and Yi X X 2014 *Sci. Rep.* **10** 1038
[68] Qi L X, Wu Y S and Zhang S C 2006 *Phys. Rev. B* **74** 085308
[69] Qiao Z, Yang S A, Feng W, Tse W K, Ding J, Yao Y, Wang J and Niu Q 2010 *Phys. Rev. B* **82** 161414(R)