Soliton molecules, T-breather molecules and some interaction solutions in the $(2+1)$-dimensional generalized KDKK equation

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By employing the complexification method and velocity resonant principle to N-solitons of the $(2+1)$-dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt (KDKK) equation, we obtain the soliton molecules, T-breather molecules, T-breather–L-soliton molecules and some interaction solutions when $N \leq 6$. Dynamical behaviors of these solutions are discussed analytically and graphically. The method adopted can be effectively used to construct soliton molecules and T-breather molecules of other nonlinear evolution equations. The results obtained may be helpful for experts to study the related phenomenon in oceanography and atmospheric science.

Keywords: soliton molecules, breather molecules, interaction solutions, velocity resonant principle, Konopelchenko–Dubrovsky–Kaup–Kupershmidt (KDKK) equation

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1. Introduction

The generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt (KDKK) equation considered in this paper takes the form of

$$u_t + h_1 u_{xxx} + h_2 u u_x + h_3 u_{xxxxx} + h_4 \partial_x^{-1} u_{yy} + h_5 u_{xxy} + h_6 (u_x \partial_x^{-1} u_y + u u_y) + h_7 (u_x u_{xx} + u u_{xxx}) + h_8 u^2 u_x = 0, \quad (1)$$

where u is a real function of x, y, t, and h_i $(i = 1, 2, \ldots, 8)$ are real constant parameters. Such an equation has been widely used in fluid mechanics, plasma physics, ocean dynamics and so on.[1–3] It is shown that different special selections on the parameters of h_i can lead to different physical models. For example, when $h_1 = h_2 = 0$, $h_3 = 1$, $h_4 = h_5 = -5$, $h_6 = -15$, $h_7 = 15$, and $h_8 = 45$, this equation becomes the B-type Kadomtsev–Petviashvili equation to describe the shallow water wave in fluids and electrostatic wave potential in plasmas.[4–8] When $h_2 = 6h_1$, $h_6 = 4h_5$, and $h_3 = h_4 = h_7 = h_8 = 0$, it turns to the breaking soliton equation between a long wave propagating along the x-axis and a Riemann wave along the y-axis.[9,10] When $h_1 = h_2 = 0$, $h_3 = -1$, $h_4 = h_8 = 5$, and $h_5 = h_6 = h_7 = -5$, the equation corresponds to the Sawada–Kotera equation and it is used to describe the long waves in the shallow water under the gravity and in one-dimensional nonlinear lattice.[11–15] Due to the wide applications of the generalized KDKK equation, it has been studied extensively and intensively in many literatures. For example, N-soliton solutions and periodic wave solutions were constructed and their asymptotic behaviors were analyzed in Ref. [16]. Lump, lumpoff and rogue waves solutions were derived in Ref. [17]. Pfaffian solutions, Wronskian solutions and breather waves were given in Ref. [18]. The general M-lumps, T-breathers and hybrid solutions were discussed in Ref. [19]. Contributions were also made by Zhang and Ma *et al.* on the higher-order lump and breathers of the generalized KDKK equation in Refs. [20,21].

Soliton molecules were introduced by Barashenkov *et al.* as a soliton complex[22–24] and experimentally observed by Stratmann *et al.* in dispersion-managed optical fibers.[25] Soliton molecules are a stable bound state which exhibit moleculelike behaviors. The study on soliton molecules provides a direct route to investigate the interactions between solitary waves. The formation and dissociation of soliton molecules are closely related to many subjects.[26] Except for the importance to bring to the fundamental understanding of soliton physics, soliton molecules provide the possibility to transfer optical data surpassing the limitation of binary coding.[27] Due to the importance of soliton molecules, they have attracted much attention in the fields of optics,[28–31] Bose–Einstein condensates,[32] fluid dynamics,[33] and plasmas physics.[34] Various theories on form soliton molecules have been established.[35,36] Recently, Lou proposed a velocity resonant principle.[37] It demonstrated that when the attraction and repulsion of the solitons are balanced with each other, the velocities of the solitons will turn to be the same to produce the soliton molecules. By using the velocity resonant principle, Lou and Li *et al.* constructed various soliton molecules of $(1+1)$-dimensional integrable models, such as dromion molecules, breather molecules and kink molecules.[38–44] To the best of our knowledge, not much work has been done on solition molecules, breather molecules of the $(2+1)$-dimensional generalized KDKK equation. Especially no work has been done on their interactions, which is due to the difficulty in choosing suitable parameters involved in the solu-

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tions. However, such interaction solutions, due to the rich dynamical behaviors, have wide applications in the real world as mentioned in Refs. [4–17]. Therefore, it constitutes our main motivation to pursue the present work to investigate soliton molecules, breather molecules and the interaction wave solutions in the $(2+1)$-dimensional generalized KDKK equation.

The outline of the paper is arranged as follows. In Section 2, a special transformation together with coefficient constraints is introduced for the KDKK Eq. (1), thereby N-soliton solutions are constructed. In Section 3, by employing the complexication method and velocity resonant principle to the related N-solitons, soliton molecules, T-breather molecules, T-breather–L-soliton molecules and various interaction solutions are obtained when $N \leq 6$. To detect dynamical behaviors that the solutions may possess, numerical simulations and theoretical analysis are presented, which fully show the selections of the parameters have a great impact on the types of solutions and their propagation properties. Finally, a short conclusion is attached.

2. N-solitons to the generalized KDKK equation

As a preliminary for deriving the soliton molecules, breather molecules and interaction solutions, we first need give N-soliton solutions to the generalized KDKK equation. Therefore, a logarithmic transformation is introduced via

$$u = 12h_1h_2^{-1}(\ln f)_{xx}, \tag{2}$$

the coefficients satisfy the following conditions:

$$2h_1h_2^{-1} = 5h_3h_7^{-1} = h_5h_6^{-1} = h_7h_8^{-1}, \ \ h_4 = \frac{-h_5^2}{5h_3}. \tag{3}$$

Computation shows that under the transformation (2), Eq. (1) can be converted into a bilinear formulism

$$\left(D_xD_t + h_1D_x^4 + h_3D_x^6 + h_4D_y^2 + h_5D_x^3D_y\right) f\cdot f = 0, \tag{4}$$

where the bilinear derivative operator D is defined by

$$\begin{aligned} D_x^\alpha D_y^\beta D_t^\gamma (f\cdot g) \ =\ & (\partial_x-\partial_{x'})^\alpha(\partial_y-\partial_{y'})^\beta(\partial_t-\partial_{t'})^\gamma \\ & \times f(x,y,t)g(x',y',t')\,|_{x'=x,y'=y,t'=t}, \end{aligned} \tag{5}$$

where α, β, and γ are nonnegative integers, which denote the orders of the bilinear derivative operators.

Based on the Hirota bilinear method,[45] the N-soliton solutions of the generalized KDKK equation are obtained via

$$\begin{aligned} u &= 12h_1h_2^{-1}(\ln f)_{xx} \\ &= 12h_1h_2^{-1}\left\{\ln\left[\sum_{\mu=0,1}\exp\left(\sum_{i=1}^{N}\mu_i\eta_i+\sum_{i<j}^{N}\mu_i\mu_jA_{ij}\right)\right]\right\}_{xx}, \end{aligned} \tag{6}$$

with

$$\eta_i = k_ix+p_iy+\omega_it+\eta_i^0, \ \ \omega_i = -k_i^2(h_1k_i+h_5p_i+h_3k_i^3) - \frac{h_4p_i^2}{k_i}, \ \ i,j=1,2,\ldots,N,$$

$$\mathrm{e}^{A_{ij}} = \frac{3h_1k_i^2k_j^2(k_i-k_j)^2+5h_3k_i^2k_j^2(k_i-k_j)^2(k_i^2-k_ik_j+k_j^2)+h_5k_ik_j(k_i-k_j)(k_i^2p_j+2k_ik_jp_i-2k_ik_jp_j-k_j^2p_i)-h_4(k_ip_j-k_jp_i)^2}{3h_1k_i^2k_j^2(k_i+k_j)^2+5h_3k_i^2k_j^2(k_i+k_j)^2(k_i^2+k_ik_j+k_j^2)+h_5k_ik_j(k_i+k_j)(k_i^2p_j+2k_ik_jp_i+2k_ik_jp_j+k_j^2p_i)-h_4(k_ip_j-k_jp_i)^2}, \tag{7}$$

where k_i, p_i, ω_i, and η_i^0 are constants. While $\sum_{\mu=0,1}$ indicates a summation over all possible combinations of $\mu_i,\mu_j=0,1$.

3. Soliton molecules, breather molecules and interaction wave solutions

3.1. Two resonance solutions from the three-solitons

According to Eq. (6), when $N=3$ one can easily write down the expression of the three-solitons u, where f_3 is given by

$$\begin{aligned} f_3 \ =\ & 1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+\mathrm{e}^{\eta_3}+\mathrm{e}^{\eta_1+\eta_2+A_{12}}+\mathrm{e}^{\eta_2+\eta_3+A_{23}} \\ & +\mathrm{e}^{\eta_1+\eta_3+A_{13}}+\mathrm{e}^{\eta_1+\eta_2+\eta_3+A_{123}}, \end{aligned} \tag{8}$$

where $A_{123}=A_{12}+A_{13}+A_{23}$ and η_i, A_{ij} $(i,j=1,2,3)$ are described in Eq. (7). Generally, from the three-solitons, one can obtain some interesting resonant solutions. Here, we just list the following two resonance solutions. One is a breather–soliton molecule derived via the complexification method and velocity resonant principle. The other is an interaction solution between a soliton molecule and a bell-shaped soliton only via the velocity resonant principle.

Case 1 A breather–soliton molecule.

According to the complexification method,[46,47] we set

$$\begin{aligned} k_1 &= k_2^* = k_{1\mathrm{r}}+\mathrm{i}k_{1\mathrm{i}}, \\ p_1 &= p_2^* = p_{1\mathrm{r}}+\mathrm{i}p_{1\mathrm{i}}, \\ \omega_1 &= \omega_2^* = \omega_{1\mathrm{r}}+\mathrm{i}\omega_{1\mathrm{i}}, \end{aligned} \tag{9}$$

then the two solitons in f_3 constitute a smooth breather on a zero background. If the velocities of the breather and the third soliton satisfy the module resonant principle via

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}} = \frac{\omega_3}{k_3}, \ \ \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}} = \frac{\omega_3}{p_3}, \tag{10}$$

then they form a breather–soliton molecule. In order to exhibit dynamical behaviors that the breather–soliton molecule

may possess, here we choose the parameters as

$$k_{1\mathrm{r}}=1,\ \ k_{1\mathrm{i}}=\frac{2}{3},\ \ p_{1\mathrm{r}}=\frac{1}{6},\ \ p_{1\mathrm{i}}=\frac{29}{27}+\frac{\sqrt{546}}{54},$$
$$k_3=\frac{2}{3},\ \ p_3=\frac{1}{9},\ \ \eta_{10}=\eta_{20}=8,\ \ \eta_{30}=0, \tag{11}$$

and h_i as

$$h_1=6,\ \ h_2=2,\ \ h_3=-2,\ \ h_4=-2,\ \ h_5=1. \tag{12}$$

Then, a special breather–soliton molecule is depicted in Fig. 1. From Fig. 1, one can see that the breather and the stripe soliton in the molecule parallel to each other, and the distance between them keeps unchanged with time evolutions, which fully shows that a bounded state is formed between them.

Case 2 An interaction solution between a soliton molecule and a stipe soliton.

If two solitons in f_3 satisfy the velocity resonant principle

$$\frac{\omega_1}{k_1}=\frac{\omega_2}{k_2},\ \ \frac{\omega_1}{p_1}=\frac{\omega_2}{p_2}, \tag{13}$$

which is different from Eq. (10), then a kind of interaction solution between a soliton molecule and a bell-shaped soliton can be obtained. Here, we set the parameters as

$$k_1=\frac{\sqrt{119}}{6},\ \ p_1=\frac{\sqrt{119}}{4},\ \ k_2=\frac{2}{3},\ \ p_2=1,\ \ k_3=\frac{6}{7},$$
$$p_3=\frac{1}{2},\ \ \eta_{10}=10,\ \ \eta_{20}=-5,\ \ \eta_{30}=0, \tag{14}$$

and dynamical behaviors of this interaction solution are displayed in Fig. 2. From the pictures, one can easily see that the bell-shaped soliton in the solution intersects and moves towards to the soliton molecule. With time varying, their propagation directions and velocities keep unchanged.

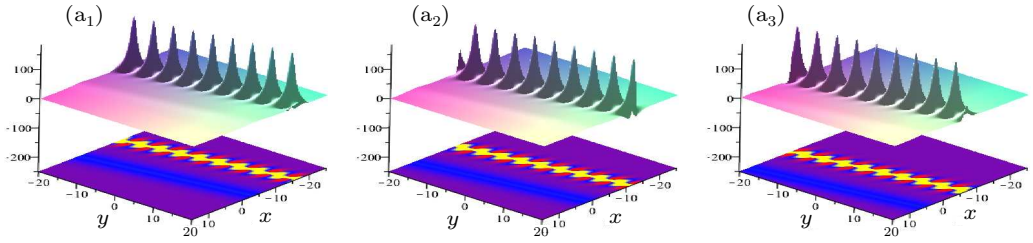


Fig. 1. Dynamical behaviors of the breather–soliton molecule. The time is chosen as $t=2i-4$ in (a_i) $(i=1,2,3)$.

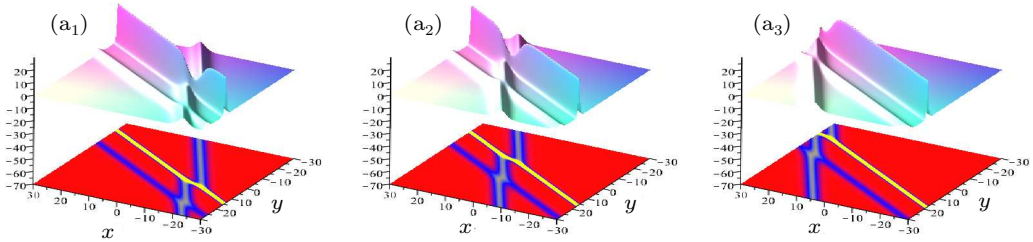


Fig. 2. Dynamical behaviors of the hybrid solution between a soliton molecule and a stripe soliton. The time is chosen as $t=4i-8$ in (a_i) $(i=1,2,3)$.

Remark We would like to point out that the nontrivial solution arising from the velocity resonant principle (13), namely,

$$p_1=-\frac{k_1[h_1+h_3(k_1^2+k_2^2)]}{h_5},\ \ p_2=-\frac{k_2[h_1+h_3(k_1^2+k_2^2)]}{h_5},$$

plays an important role in deriving the interaction solution established in Case 2. However, for the following general condition:

$$\frac{\omega_i}{k_i}=\frac{\omega_j}{k_j},\ \ \frac{\omega_i}{p_i}=\frac{\omega_j}{p_j},\ \ (i,j=1,2,3,\ \ i\neq j)$$

arising from the three solitons, no such nontrivial solution as Eq. (15) exists. Therefore, there is no three-solitons molecule for the KDKK equation.

3.2. Five special resonant solutions from the four-solitons

When $N=4$, the expression of f_4 is described as

$$\begin{aligned} f_4 = {} & 1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+\mathrm{e}^{\eta_3}+\mathrm{e}^{\eta_4}+\mathrm{e}^{\eta_1+\eta_2+A_{12}}+\mathrm{e}^{\eta_1+\eta_3+A_{13}} \\ & +\mathrm{e}^{\eta_1+\eta_4+A_{14}}+\mathrm{e}^{\eta_2+\eta_3+A_{23}}+\mathrm{e}^{\eta_2+\eta_4+A_{24}}+\mathrm{e}^{\eta_3+\eta_4+A_{34}} \\ & +\mathrm{e}^{\eta_1+\eta_2+\eta_3+A_{123}}+\mathrm{e}^{\eta_1+\eta_2+\eta_4+A_{124}}+\mathrm{e}^{\eta_1+\eta_3+\eta_4+A_{134}} \\ & +\mathrm{e}^{\eta_2+\eta_3+\eta_4+A_{234}}+\mathrm{e}^{\eta_1+\eta_2+\eta_3+\eta_4+A_{1234}}, \end{aligned} \tag{15}$$

where η_i, A_{ij} are given in Eq. (7) and $A_{mnl}=\sum_{1\leq m<i<j}^{4}A_{ij}$, $A_{1234}=\sum_{1\leq i<j}^{4}A_{ij}$. When adopt the complexification method and velocity resonant principle, we can obtain many different types of resonant solutions. Here, we mainly record the following five types of them.

Case 1 Two interaction solutions: mixed by one soliton molecule and one breather/soliton molecule.

If we set two pairs of solitons satisfy the velocity resonant

principle

$$\frac{\omega_1}{k_1}=\frac{\omega_2}{k_2},\ \frac{\omega_1}{p_1}=\frac{\omega_2}{p_2},\ \frac{\omega_3}{k_3}=\frac{\omega_4}{k_4},\ \frac{\omega_3}{p_3}=\frac{\omega_4}{p_4}, \tag{16}$$

then an interaction solution composed of two different soliton-molecules ($\{\mathrm{SM},\mathrm{SM}\}$) can be obtained. Here we take the parameters as

$$\begin{aligned} &k_1=\frac{\sqrt{530}}{12},\ p_1=\frac{3\sqrt{530}}{16},\ k_2=\frac{2}{3},\ p_2=\frac{3}{2},\\ &k_3=\frac{\sqrt{399}}{12},\ p_3=\frac{\sqrt{399}}{18},\ k_4=\frac{3}{4},\ p_4=\frac{1}{2}, \end{aligned} \tag{17}$$

and dynamical behaviors of a typical interaction solution between two soliton molecules are depicted in Fig. 3. It can be seen from Fig. 3 that there indeed exists two sets of solitons which parallel to each other to form two soliton molecules. During the interaction of the two soliton molecules, the distance between two solitons in the molecules that describes the size of the soliton molecules, obviously becomes wide after the interaction. Therefore, the interaction between two soliton molecules is inelastic under the meaning that the size of the soliton molecules is changed.

In the sequel, we will theoretically prove that the interaction between two soliton molecules is inelastic. Therefore, the asymptotic analysis approach is adopted. For the sake of expression, we record the four solitons in the two different soliton molecules (labeled as SM-I and SM-II) as 1$^{\mathrm{st}}$-soliton, 2$^{\mathrm{nd}}$-soliton, 3$^{\mathrm{rd}}$-soliton, and 4$^{\mathrm{th}}$-soliton and assume that: the 1$^{\mathrm{st}}$-soliton and 2$^{\mathrm{nd}}$-soliton form SM-I while the 3$^{\mathrm{rd}}$-soliton and 4$^{\mathrm{th}}$-soliton form SM-II. Now, we will prove that the distance change in SM-I and 3$^{\mathrm{rd}}$-soliton is unequal to the change in SM-I and 4$^{\mathrm{th}}$-soliton after the interaction to explain the interaction is inelastic. Without loss of generality, we assume $k_1,k_3>0$ and $k_3\omega_1>k_1\omega_3$. Then η_1 can be reformulated as

$$\begin{aligned} \eta_1=k_1x+p_1y+\omega_1t+\eta_1^0=\frac{k_1}{k_3}\left[\eta_3+\left(\frac{p_1k_3}{k_1}-p_3\right)y\right.\\ \left.+\left(\frac{k_3\omega_1}{k_1}-\omega_3\right)t+\frac{k_3}{k_1}\eta_1^0-\eta_3^0\right]. \end{aligned} \tag{18}$$

Along the trajectory where η_3 is a constant, it holds that

(i) When $t\to-\infty$, the relation (18) tells us that $\eta_1\to-\infty$. Accordingly, $\mathrm{e}^{\eta_1}\sim0$, $\mathrm{e}^{\eta_1+\eta_3+A_{13}}\sim0$, $1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_3}+\mathrm{e}^{\eta_1+\eta_3+A_{13}}\sim1+\mathrm{e}^{\eta_3}$. Thus, in this case, we have

$$\begin{aligned} u=u_{13}&=12h_1h_2^{-1}\left[\ln\left(1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_3}+\mathrm{e}^{\eta_1+\eta_3+A_{13}}\right)\right]_{xx}\\ &\sim 12h_1h_2^{-1}\left[\ln\left(1+\mathrm{e}^{\eta_3}\right)\right]_{xx}\\ &=3h_1h_2^{-1}k_3^2\mathrm{sech}^2\frac{\eta_3}{2},\ t\to-\infty. \end{aligned} \tag{19}$$

(ii) When $t\to+\infty$, the relation (18) reveals that $\eta_1\to+\infty$. Accordingly, $\mathrm{e}^{\eta_1}\sim+\infty$, $\mathrm{e}^{\eta_1+\eta_3+A_{13}}\sim+\infty$, $1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_3}+\mathrm{e}^{\eta_1+\eta_3+A_{13}}\sim\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_1+\eta_3+A_{13}}$. Therefore, under this case, we get

$$\begin{aligned} u=u_{13}&=12h_1h_2^{-1}[\ln(1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_3}+\mathrm{e}^{\eta_1+\eta_3+A_{13}})]_{xx}\\ &\sim 12h_1h_2^{-1}[\ln(\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_1+\eta_3+A_{13}})]_{xx}\\ &=3h_1h_2^{-1}k_3^2\mathrm{sech}^2\frac{\eta_3+A_{13}}{2},\ t\to+\infty. \end{aligned} \tag{20}$$

On making similar analysis along the trajectory where η_1 is a constant, we have

$$\begin{aligned} &u_{13}\sim3h_1h_2^{-1}k_1^2\mathrm{sech}^2\frac{\eta_1+A_{13}}{2},\ t\to-\infty,\\ &u_{13}\sim3h_1h_2^{-1}k_1^2\mathrm{sech}^2\frac{\eta_1}{2},\ t\to+\infty. \end{aligned} \tag{21}$$

Inspecting the relations of Eqs. (19)–(21) show that before and after the interaction of 1$^{\mathrm{st}}$-soliton with 3$^{\mathrm{rd}}$-soliton, the only change happens to them is the phase shift of A_{13}. Analogously, one can compute that during the interaction, the change between the 2$^{\mathrm{nd}}$-soliton and 3$^{\mathrm{rd}}$-soliton, between 1$^{\mathrm{st}}$-soliton and 4$^{\mathrm{th}}$-soliton as well as between 2$^{\mathrm{nd}}$-soliton and 4$^{\mathrm{th}}$-soliton, are A_{23}, A_{14} and A_{24}, respectively. Obviously, $A_{13}\neq A_{23}$ and $A_{14}\neq A_{24}$ in general case. In other words, the interaction between two different soliton molecules is inelastic under the meaning that the size of the soliton molecules is changed. For the present case shown in Fig. 3, it is easy to compute that $A_{13}\approx-0.0027$, $A_{23}\approx-0.0175$, $A_{14}\approx0.0115$, and $A_{24}\approx0.1483$.

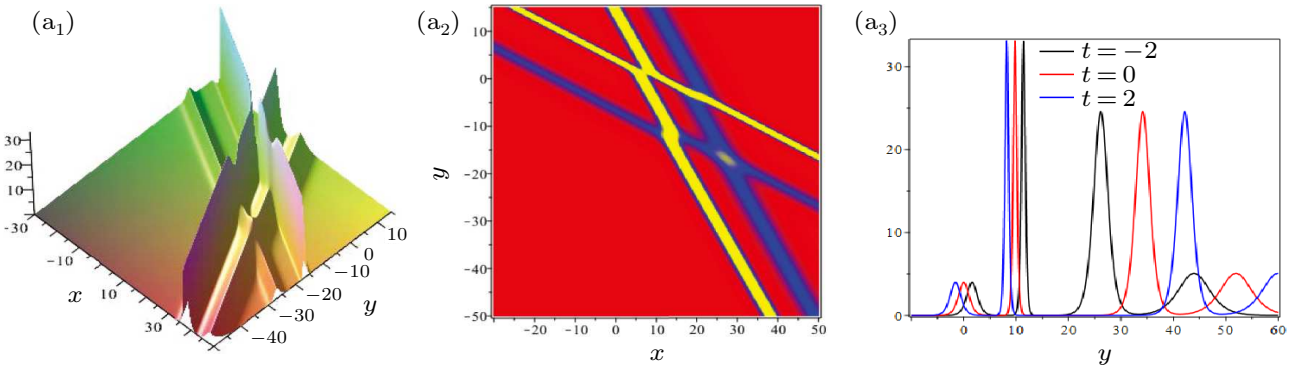


Fig. 3. Dynamical behaviors of the interaction solution between two soliton molecules. The parameters are given in Eq. (17). Panel (a_3) denotes the y-curves of the $\{\mathrm{SM},\mathrm{SM}\}$ at $x=-15$.

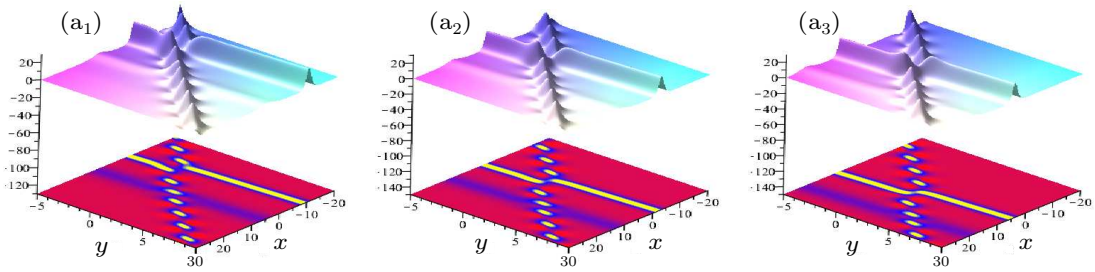


Fig. 4. Dynamical behaviors of the interaction solution between one soliton molecule and a breather. The time is chosen as $t = 3i - 6$ in (a_i) $(i = 1,2,3)$.

Interestingly, we find that different from Eq. (16), if we only require one pair of solitons to satisfy the velocity resonant condition, for example

$$\frac{\omega_3}{k_3} = \frac{\omega_4}{k_4}, \quad \frac{\omega_3}{p_3} = \frac{\omega_4}{p_4}, \tag{22}$$

when imposing the conjugate conditions on the other pair of solitons via $k_1 = k_2^* = k_{1\mathrm{r}} + \mathrm{i}k_{1\mathrm{i}}$, $p_1 = p_2^* = p_{1\mathrm{r}} + \mathrm{i}p_{1\mathrm{i}}$, then an interaction solution between one soliton molecule and a breather ($\{\mathrm{SM},\mathrm{B}\}$) can be obtained. A special $\{\mathrm{SM},\mathrm{B}\}$ solution is simulated in Fig. 4 under the suitable parameters of $k_1 = 1/2 + 2/3\mathrm{i}$, $p_1 = -3/2 + 3/2\mathrm{i}$, $p_3 = 1/9$, $p_4 = \sqrt{95}/36$, $k_3 = 2/3$, $k_4 = \sqrt{95}/6$. From Fig. 4, we can see that the breather is shown to be intersected to the soliton-molecule due to the special parameters selected. Just as the solutions given in the above, the distance in the soliton molecule keeps unchanged with time evolutions due to the condition of Eq. (22). But the distance between the soliton molecule and the breather changes all the time.

Case 2 Three resonant solutions: a 1-breather–2-soliton molecule, a 2-breather molecule and an interaction solution mixed by a breather–soliton molecule and a stipe soliton.

Now we consider $\{k_i, p_i, \omega_i\}$ that satisfy the complex conjugate conditions

$$\begin{aligned} &k_1 = k_2^* = k_{1\mathrm{r}} + \mathrm{i}k_{1\mathrm{i}}, \quad p_1 = p_2^* = p_{1\mathrm{r}} + \mathrm{i}p_{1\mathrm{i}}, \\ &\omega_1 = \omega_2^* = \omega_{1\mathrm{r}} + \mathrm{i}\omega_{1\mathrm{i}}, \end{aligned} \tag{23}$$

it is shown that under the above constraints, the two solitons form a breather. When the velocity of the breather is resonant to the third soliton via the principle

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}} = \frac{\omega_3}{k_3}, \quad \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}} = \frac{\omega_3}{p_3}, \tag{24}$$

then the breather and the third soliton constitute a breather–soliton molecule. Therefore, under the conditions (23) and (24), we obtain an interaction solution mixed by the breather–soliton molecule and a stripe soliton ($\{\mathrm{BSM},\mathrm{S}\}$).

In the above, by restricting only one of the soliton's velocity to be resonant to the breather's, we obtained the interaction solution of $\{\mathrm{BSM},\mathrm{S}\}$. Now, we would like to show that when restricting both of the solitons' velocities to be resonant to the breather's via the following conditions:

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}} = \frac{\omega_3}{k_3} = \frac{\omega_4}{k_4}, \quad \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}} = \frac{\omega_3}{p_3} = \frac{\omega_4}{p_4}, \tag{25}$$

then, another resonant solution of breather–soliton–soliton molecule ($\{\mathrm{BSSM}\}$) can be derived, which is briefly recorded as 1-breather–2-soliton molecule. Furthermore, if we additionally apply the complexification method to the above two solitons via

$$\begin{aligned} &k_3 = k_4^* = k_{3\mathrm{r}} + \mathrm{i}k_{3\mathrm{i}}, \quad p_3 = p_4^* = p_{3\mathrm{r}} + \mathrm{i}p_{3\mathrm{i}}, \\ &\omega_3 = \omega_4^* = \omega_{3\mathrm{r}} + \mathrm{i}\omega_{3\mathrm{i}}, \end{aligned}$$

and require that their velocities satisfy the following module resonant principle:

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}} = \frac{\omega_{3\mathrm{r}}}{k_{3\mathrm{r}}}, \quad \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}} = \frac{\omega_{3\mathrm{r}}}{p_{3\mathrm{r}}}, \tag{26}$$

then we can obtain a solution of breather–breather molecule, which is recorded as a 2-breather molecule ($\{\mathrm{BBM}\}$).

Below, we present some numerical simulations to study dynamical behaviors of the above three solutions. Figure 5 exhibits dynamical behaviors of the interaction solution of $\{\mathrm{BSM},\mathrm{S}\}$ with

$$\begin{aligned} &(k_1, p_1, k_3, k_4, p_3, p_4) \\ &= \left(1 + \frac{2}{3}\mathrm{i}, \ \frac{1}{6} + \frac{58 + \sqrt{546}}{54}\mathrm{i}, \ \frac{\sqrt{95}}{6}, \ \frac{3}{2}, \ \frac{\sqrt{95}}{36}, \ \frac{5}{6}\right). \end{aligned}$$

Figure 6 displays behaviors of $\{\mathrm{BSSM}\}$ with

$$\begin{aligned} &(k_1, p_1, k_3, k_4, p_3, p_4) \\ &= \left(1 + \frac{2}{3}\mathrm{i}, \ \frac{1}{6} + \frac{58 + \sqrt{546}}{54}\mathrm{i}, \ \frac{\sqrt{95}}{6}, \ \frac{2}{3}, \ \frac{\sqrt{95}}{36}, \ \frac{1}{9}\right). \end{aligned}$$

Figure 7 shows behaviors of $\{\mathrm{BBM}\}$ with

$$\begin{aligned} &(k_1, p_1, k_3, k_4, p_3, p_4) \\ &= \left(1 + \frac{2}{3}\mathrm{i}, \ \frac{1}{6} + \frac{58 + \sqrt{546}}{54}\mathrm{i}, \ \frac{5}{6} + \mathrm{i}, \ \frac{5}{6} - \mathrm{i}, \right. \\ &\left. \frac{5}{36} + \frac{67 - 2\sqrt{8418}}{36}\mathrm{i}, \ \frac{5}{36} - \frac{67 - 2\sqrt{8418}}{36}\mathrm{i}\right). \end{aligned}$$

Comparing these three figures, one will find that the soliton (s) and the breather (b) exhibit different behaviors, which fully reveals that the conditions imposed on the parameters and their categories have great impacts on the dynamical behaviors of these solutions.

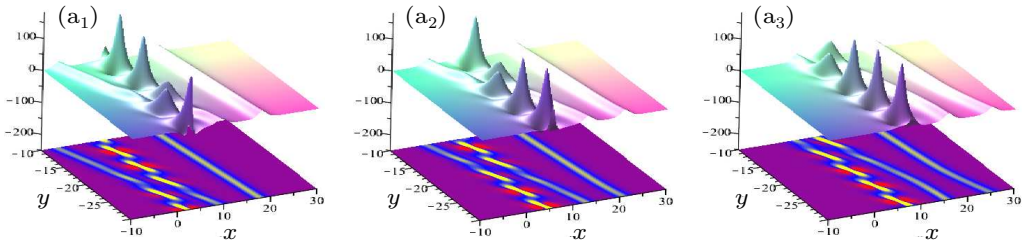


Fig. 5. Dynamical behaviors of the solution mixed by the breather–soliton molecule and stipe soliton. The time is chosen as $t = i-2$ in (a_i) $(i = 1,2,3)$.

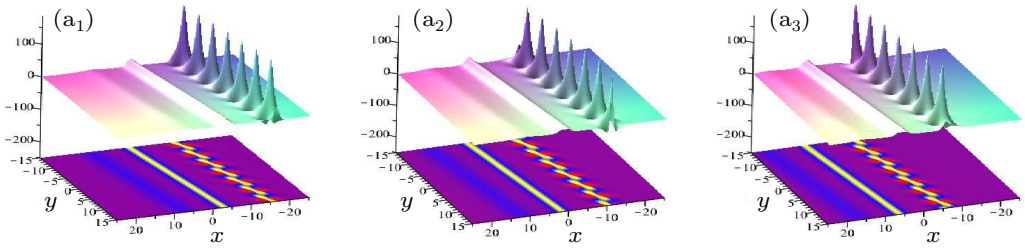


Fig. 6. Dynamical behaviors of the 1-breather–2-soliton molecule. The time is chosen as $t = 2i-4$ in (a_i) $(i = 1,2,3)$.

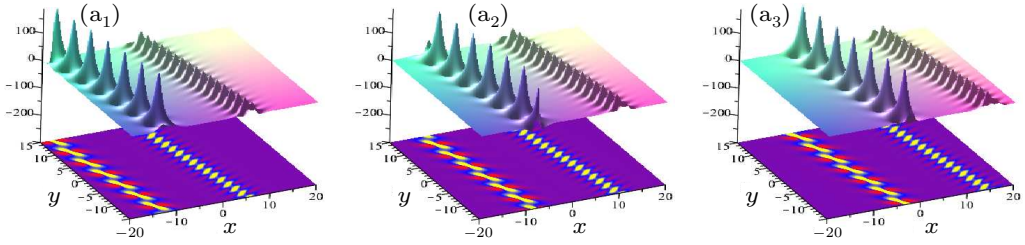


Fig. 7. Dynamical behaviors of the 2-breather molecule. The time is chosen as $t = 2i-4$ in (a_i) $(i = 1,2,3)$.

3.3. Several typical localized wave solutions for $N = 5$

When $N = 5$, by applying complex conjugate constraints and imposing the velocity resonant mechanism to the five-solitons, nine resonant solutions: (i) $\{\mathrm{SM},\mathrm{S},\mathrm{S},\mathrm{S}\}$ an interaction solution between one soliton molecule and three separated solitons; (ii) $\{\mathrm{SM},\mathrm{B},\mathrm{S}\}$ an interaction solution between one soliton molecule, a breather and soliton; (iii) $\{\mathrm{SM},\mathrm{BSM}\}$ an interaction solution between one soliton molecule and one breather–soliton molecule; (iv) $\{\mathrm{SM},\mathrm{SM},\mathrm{S}\}$ an interaction solution between two soliton molecules and a separated soliton; (v) $\{\mathrm{BSM},\mathrm{S},\mathrm{S}\}$ an interaction solution between a breather–soliton molecule and two separated solitons; (vi) $\{\mathrm{BSM},\mathrm{SM}\}$ an interaction solution between a breather–soliton molecule and a soliton molecule; (vii) $\{\mathrm{BSM},\mathrm{B}\}$ an interaction solution between a breather–soliton molecule and a breather; (viii) $\{\mathrm{BBM},\mathrm{S}\}$ an interaction solution between a 2-breather molecule and a separated soliton; (ix) $\{\mathrm{BBSM}\}$ a 2-breather–1-soliton molecule. In the following, we only discuss five special kinds of them. Details are as follows:

Case 1 Two solutions: a 2-breather–1-soliton molecule and a solution mixed by a 2-breather molecule and a stipe soliton.

Now, if we require k_i, p_i, and ω_i to satisfy the following complex conjugate constraints via:

$$k_m = k_{m+1}^* = k_{m\mathrm{r}} + \mathrm{i}k_{m\mathrm{i}},\ \ p_m = p_{m+1}^* = p_{m\mathrm{r}} + \mathrm{i}p_{m\mathrm{i}},$$
$$\omega_m = \omega_{m\mathrm{r}} + \mathrm{i}\omega_{m\mathrm{i}},\ \ (m = 1,3), \tag{27}$$

and by applying the velocity resonant conditions to the breather via

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}} = \frac{\omega_{3\mathrm{r}}}{k_{3\mathrm{r}}},\ \ \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}} = \frac{\omega_{3\mathrm{r}}}{p_{3\mathrm{r}}},$$

then we can obtain an interaction solution mixed by a 2-breather molecule and a stipe soliton ($\{\mathrm{BBM},\mathrm{S}\}$). Here the parameters are chosen as

$$k_{1\mathrm{r}} = 1,\ \ k_{1\mathrm{i}} = \frac{2}{3},\ \ p_{1\mathrm{r}} = \frac{1}{6},\ \ p_{1\mathrm{i}} = \frac{29}{27} + \frac{\sqrt{546}}{54},\ \ k_{3\mathrm{r}} = \frac{5}{6},$$
$$k_{3\mathrm{i}} = 1,\ \ p_{3\mathrm{r}} = \frac{5}{36},\ \ p_{3\mathrm{i}} = \frac{67}{36} - \frac{\sqrt{8418}}{18},\ \ k_5 = \frac{3}{4},$$
$$p_5 = -\frac{1}{6},\ \ \eta_{10} = \eta_{20} = 5,\ \ \eta_{30} = \eta_{40} = -8,\ \ \eta_{50} = 0. \tag{28}$$

The dynamical behaviors of the $\{\mathrm{BBM},\mathrm{S}\}$ solution are simulated in Fig. 8.

It is noticed that in the interaction solution of $\{\mathrm{BBM},\mathrm{S}\}$, only the velocities of the two breathers are resonant (not to that of the soliton). However, if we additionally require them to be

resonant to the soliton, namely the following velocity resonant conditions are satisfied:

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{k_{3\mathrm{r}}}=\frac{w_5}{k_5},\quad \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{p_{3\mathrm{r}}}=\frac{w_5}{p_5},\tag{29}$$

then a kind of molecule solution termed as 2-breather–1-soliton molecule ($\{\mathrm{BBSM}\}$) can be constructed, whose behaviors are simulated in Fig. 9 under the special parameters of

$$\begin{aligned}&k_{1\mathrm{r}}=1,\ \ k_{1\mathrm{i}}=\frac{2}{3},\ \ p_{1\mathrm{r}}=\frac{1}{6},\ \ p_{1\mathrm{i}}=\frac{29}{27}+\frac{\sqrt{546}}{54},\ \ k_{3\mathrm{r}}=\frac{5}{6},\\&k_{3\mathrm{i}}=1,\ \ p_{3\mathrm{r}}=\frac{5}{36},\ \ p_{3\mathrm{i}}=\frac{67}{36}-\frac{\sqrt{8418}}{18},\ \ k_5=\frac{2}{3},\ \ p_5=\frac{1}{9},\\&\eta_{10}=\eta_{20}=10,\ \ \eta_{30}=\eta_{40}=-12,\ \ \eta_{50}=0.\end{aligned}\tag{30}$$

Case 2 Three interaction solutions: mixed by a breather–soliton molecule and a soliton molecule/breather and by a 1-breather–2-soliton molecule and a stripe soliton.

Now, we set k_i, p_i, and ω_i to satisfy the conjugate conditions via $k_1=k_2^*=k_{1\mathrm{r}}+\mathrm{i}k_{1\mathrm{i}}$, $p_1=p_2^*=p_{1\mathrm{r}}+\mathrm{i}p_{1\mathrm{i}}$, $\omega_1=\omega_2^*=\omega_{1\mathrm{r}}+\mathrm{i}\omega_{1\mathrm{i}}$. By imposing the velocity resonant conditions via

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}}=\frac{\omega_3}{k_3},\ \ \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}}=\frac{\omega_3}{p_3},\ \ \frac{\omega_4}{k_4}=\frac{\omega_5}{k_5},\ \ \frac{\omega_4}{p_4}=\frac{\omega_5}{p_5},\tag{31}$$

then we can obtain an interaction solution among one breather–soliton molecule and one soliton molecule ($\{\mathrm{BSM},\mathrm{SM}\}$), whose dynamical behaviors are depicted in Fig. 10. Interestingly, we find that if the above conjugate constraints keep unchanged and the velocity resonant conditions (31) are replaced by

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}}=\frac{\omega_3}{k_3}=\frac{\omega_4}{k_4},\ \ \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}}=\frac{\omega_3}{p_3}=\frac{\omega_4}{p_4},$$

then another interaction solution mixed by a 1-breather–2-soliton molecule and one stripe soliton ($\{\mathrm{BSSM},\mathrm{S}\}$) can be constructed and its behaviors are shown in Fig. 11. In addition, if we further restrict that the following complex conjugate conditions are satisfied: $k_4=k_5^*=k_{4\mathrm{r}}+\mathrm{i}k_{4\mathrm{i}}$, $p_4=p_5^*=p_{4\mathrm{r}}+\mathrm{i}p_{4\mathrm{i}}$, $\omega_4=\omega_5^*=\omega_{4\mathrm{r}}+\mathrm{i}\omega_{4\mathrm{i}}$, and the velocity resonant conditions (31) are changed to

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}}=\frac{\omega_3}{k_3},\ \ \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}}=\frac{\omega_3}{p_3},$$

then the interaction solution consisting of one breather–soliton molecule and one breather ($\{\mathrm{BSM},\mathrm{B}\}$) can be generated, seen in Fig. 12.

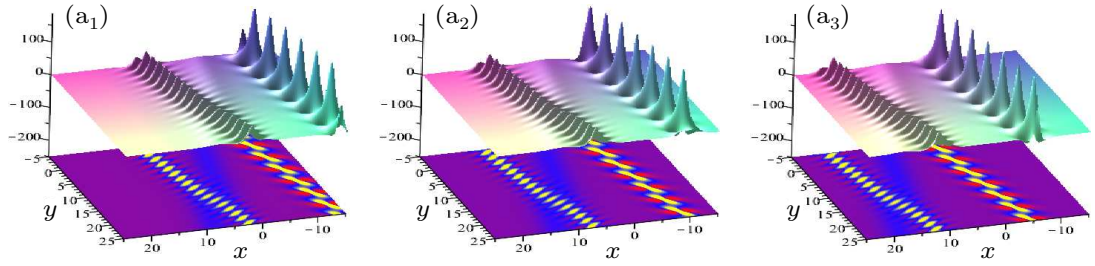


Fig. 8. Dynamical behaviors of the solution mixed by a 2-breather molecule and a stipe soliton. The time is chosen as $t=2i-4$ in (a_i) $(i=1,2,3)$.

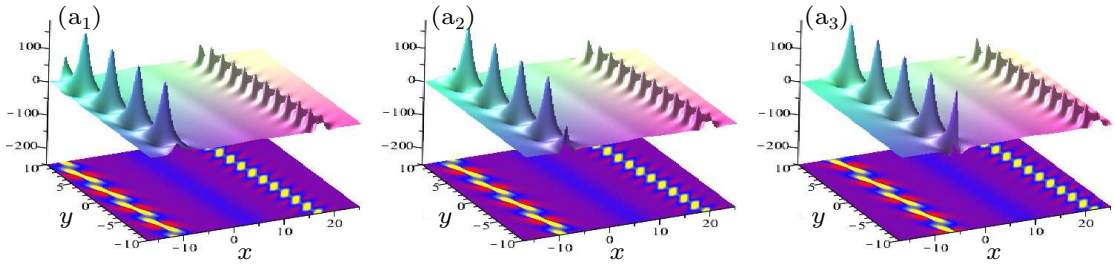


Fig. 9. Dynamical behaviors of the 2-breather–1-soliton molecule. The time is chosen as $t=i-2$ in (a_i) $(i=1,2,3)$.

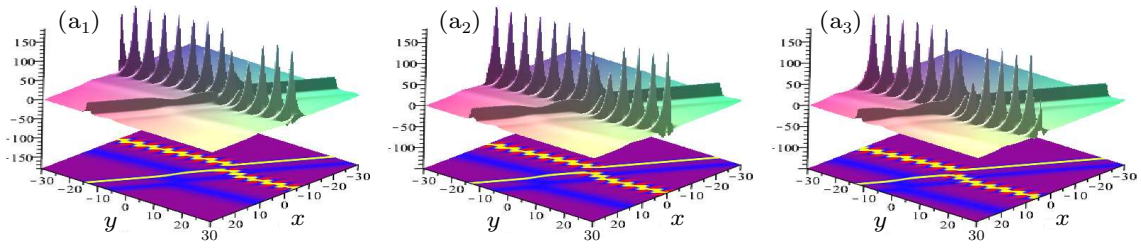


Fig. 10. Dynamical behaviors of the solution mixed by a breather–soliton molecule and a soliton molecule. The parameters are selected as $k_{1\mathrm{r}}=1$, $k_{1\mathrm{i}}=2/3$, $p_{1\mathrm{r}}=1/6$, $p_{1\mathrm{i}}=29/27+\sqrt{546}/54$, $k_3=2/3$, $p_3=1/9$, $k_4=2/3$, $p_4=1$, $k_5=\sqrt{119}/6$, $p_5=\sqrt{119}/4$, $\eta_{10}=\eta_{20}=1$, $\eta_{30}=-4$, $\eta_{40}=\eta_{50}=-5$. The time is chosen as $t=2i-4$ in (a_i) $(i=1,2,3)$.

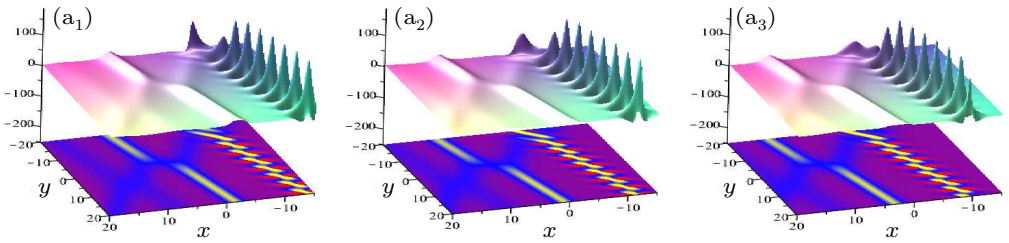


Fig. 11. Dynamical behaviors of the solution mixed by a 1-breather–2-soliton molecule and a stipe soliton. The parameters are selected as $k_{1\mathrm{r}}=1$, $k_{1\mathrm{i}}=2/3$, $p_{1\mathrm{r}}=1/6$, $p_{1\mathrm{i}}=29/27+\sqrt{546}/54$, $k_3=2/3$, $p_3=1/9$, $k_4=\sqrt{95}/6$, $p_4=\sqrt{95}/36$, $k_5=4/5$, $p_5=-1/3$, $\eta_{10}=\eta_{20}=8$, $\eta_{30}=-2$, $\eta_{40}=2$, $\eta_{50}=0$. The time is chosen as $t=i-2$ in (a_i) $(i=1,2,3)$.

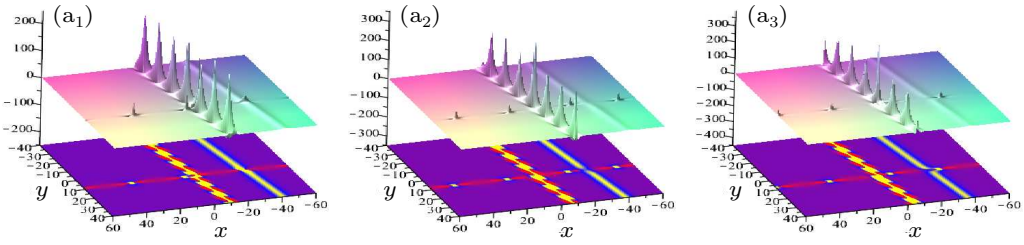


Fig. 12. Dynamical behaviors of the solution mixed by a breather–soliton molecule and a breather. The parameters are selected as $k_{1\mathrm{r}}=1$, $k_{1\mathrm{i}}=2/3$, $p_{1\mathrm{r}}=1/6$, $p_{1\mathrm{i}}=29/27-\sqrt{546}/54$, $k_3=2/3$, $p_3=1/9$, $k_{4\mathrm{r}}=0$, $k_{4\mathrm{i}}=1/5$, $p_{4\mathrm{r}}=3/4$, $p_{4\mathrm{i}}=1/15$, $\eta_{10}=\eta_{20}=3$, $\eta_{30}=20$, $\eta_{40}=\eta_{50}=-6$. The time is chosen as $t=i-2$ in (a_i) $(i=1,2,3)$.

3.4. Several typical interaction solutions for $N=6$

When $N=6$, if we take the complex conjugate constraints and impose the velocity resonant principle on the six-solitons, we can obtain seventeen resonant solutions: $\{\mathrm{SM},\mathrm{S},\mathrm{S},\mathrm{S},\mathrm{S}\}$, $\{\mathrm{SM},\mathrm{S},\mathrm{S},\mathrm{B}\}$, $\{\mathrm{SM},\mathrm{BSM},\mathrm{S}\}$, $\{\mathrm{SM},\mathrm{BSSM}\}$, $\{\mathrm{SM},\mathrm{B},\mathrm{B}\}$, $\{\mathrm{SM},\mathrm{BBM}\}$, $\{\mathrm{SM},\mathrm{SM},\mathrm{S},\mathrm{S}\}$, $\{\mathrm{SM},\mathrm{SM},\mathrm{SM}\}$, $\{\mathrm{SM},\mathrm{SM},\mathrm{B}\}$, $\{\mathrm{BSM},\mathrm{S},\mathrm{S},\mathrm{S}\}$, $\{\mathrm{BSM},\mathrm{S},\mathrm{B}\}$, $\{\mathrm{BSM},\mathrm{BSM}\}$, $\{\mathrm{BBM},\mathrm{S},\mathrm{S}\}$, $\{\mathrm{BBM},\mathrm{SM}\}$, $\{\mathrm{BBSSM}\}$, $\{\mathrm{BBM},\mathrm{B}\}$, and $\{\mathrm{BBBM}\}$. Here, we just list the following five special kinds of them.

Case 1 Two molecule solutions: one is the 2-breather–2-soliton molecule and the other is 3-breather molecule.

For the parameters $\{k_i,p_i,\omega_i\}$ $(i=1,2,\ldots,6)$, if we require that two pairs of them satisfy the complex conjugate conditions

$$\begin{aligned}&k_1=k_2^*=k_{1\mathrm{r}}+\mathrm{i}k_{1\mathrm{i}},\ \ p_1=p_2^*=p_{1\mathrm{r}}+\mathrm{i}p_{1\mathrm{i}},\\&\omega_1=\omega_2^*=\omega_{1\mathrm{r}}+\mathrm{i}\omega_{1\mathrm{i}},\ \ k_3=k_4^*=k_{3\mathrm{r}}+\mathrm{i}k_{3\mathrm{i}},\\&p_3=p_4^*=p_{3\mathrm{r}}+\mathrm{i}p_{3\mathrm{i}},\ \ \omega_3=\omega_4^*=\omega_{3\mathrm{r}}+\mathrm{i}\omega_{3\mathrm{i}},\end{aligned}\tag{32}$$

then under the above conditions, the four solitons constitute two breathers. By imposing the following velocity resonant principles:

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{k_{3\mathrm{r}}}=\frac{\omega_5}{k_5}=\frac{\omega_6}{k_6},\ \ \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{p_{3\mathrm{r}}}=\frac{\omega_5}{p_5}=\frac{\omega_6}{p_6},\tag{33}$$

on the breathers, we can obtain a 2-breather–2-soliton molecule ($\{\mathrm{BBSSM}\}$).

Interestingly, it is noticed that if we further require the parameters $\{k_5,p_5,\omega_5\}$ and $\{k_6,p_6,\omega_6\}$ also satisfy the conjugate conditions via $k_5=k_6^*=k_{5\mathrm{r}}+\mathrm{i}k_{5\mathrm{i}}$, $p_5=p_6^*=p_{5\mathrm{r}}+\mathrm{i}p_{5\mathrm{i}}$, $\omega_5=\omega_6^*=\omega_{5\mathrm{r}}+\mathrm{i}\omega_{5\mathrm{i}}$, then it produces a third breather. By taking the following velocities resonant conditions:

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{k_{3\mathrm{r}}}=\frac{\omega_{5\mathrm{r}}}{k_{5\mathrm{r}}},\ \ \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{p_{3\mathrm{r}}}=\frac{\omega_{5\mathrm{r}}}{p_{5\mathrm{r}}},\tag{34}$$

we can obtain a 3-breather molecule ($\{\mathrm{BBBM}\}$). In order to shed lights on the dynamical behaviors of these two molecule solutions, here we undertake some illustrative numerical simulations. Figure 13 exhibits the dynamical behaviors of a typical $\{\mathrm{BBSSM}\}$ and Fig. 14 displays the dynamical behaviors of a typical $\{\mathrm{BBBM}\}$. From the above two pictures, one can easily see the dynamical behaviors of $\{\mathrm{BBSSM}\}$ are quite different from that of $\{\mathrm{BBBM}\}$ due to the special conditions imposed.

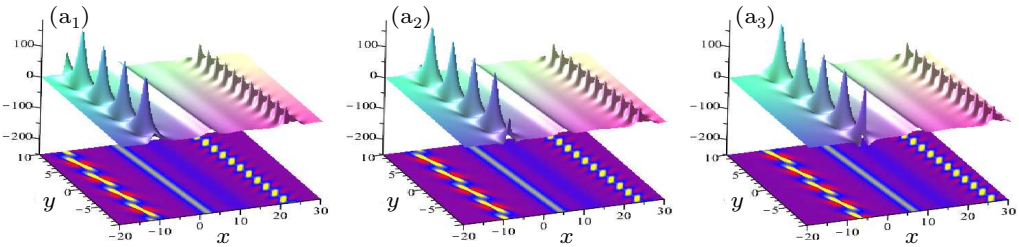


Fig. 13. Dynamical behaviors of the 2-breather–3-soliton molecule. The parameters are selected as $k_{1\mathrm{r}}=1$, $k_{1\mathrm{i}}=2/3$, $p_{1\mathrm{r}}=1/6$, $p_{1\mathrm{i}}=29/27+\sqrt{546}/54$, $k_{3\mathrm{r}}=5/6$, $k_{3\mathrm{i}}=1$, $p_{3\mathrm{r}}=5/36$, $p_{3\mathrm{i}}=67/36-\sqrt{8418}/18$, $k_5=2/3$, $p_5=1/9$, $k_6=\sqrt{95}/6$, $p_6=\sqrt{95}/36$, $\eta_{10}=\eta_{20}=10$, $\eta_{30}=\eta_{40}=-11$, $\eta_{50}=0$, $\eta_{60}=4$. The time is chosen as $t=i-2$ in (a_i) $(i=1,2,3)$.

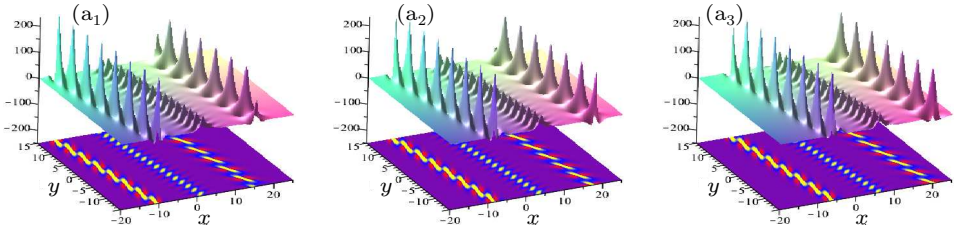


Fig. 14. Dynamical behaviors of the 3-breather molecule. The suitable parameters are set as $k_{1\mathrm{r}}=1$, $k_{1\mathrm{i}}=2/3$, $p_{1\mathrm{r}}=1/6$, $p_{1\mathrm{i}}\approx 1.51$, $k_{3\mathrm{r}}=5/6$, $k_{3\mathrm{i}}=1$, $p_{3\mathrm{r}}=5/36$, $p_{3\mathrm{i}}\approx -3.24$, $k_{5\mathrm{r}}=3/2$, $k_{5\mathrm{i}}=34/20$, $p_{5\mathrm{r}}=1/4$, $p_{5\mathrm{i}}\approx -1.22$, $\eta_{10}=\eta_{20}=10$, $\eta_{30}=\eta_{40}=-10$, $\eta_{50}=\eta_{60}=-2$. The time is chosen as $t=i-2$ in (a_i) $(i=1,2,3)$.

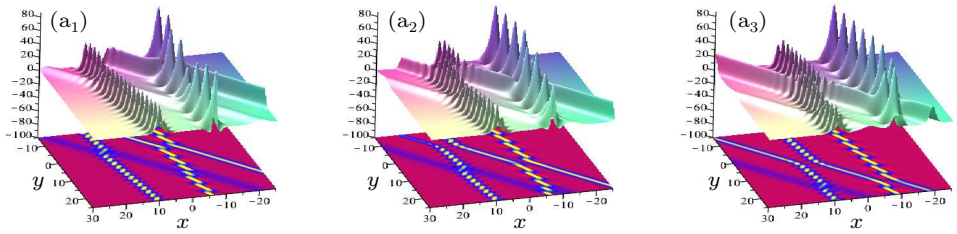


Fig. 15. Dynamical behaviors of the solution mixed by a breather molecule and a soliton molecule. The parameters are chosen by $k_{1\mathrm{r}}=1$, $k_{1\mathrm{i}}=4/5$, $p_{1\mathrm{r}}=1/5$, $p_{1\mathrm{i}}=3/2$, $k_{3\mathrm{r}}=5/6$, $k_{3\mathrm{i}}=1$, $p_{3\mathrm{r}}=1/6$, $p_{3\mathrm{i}}\approx -2.8$, $k_5\approx 1.69$, $p_5\approx 1.69$, $k_6=4/5$, $p_6=4/5$, $\eta_{10}=\eta_{20}=6$, $\eta_{30}=\eta_{40}=-6$, $\eta_{50}=3$, $\eta_{60}=-4$. The time is chosen as $t=3i-6$ in (a_i) $(i=1,2,3)$.

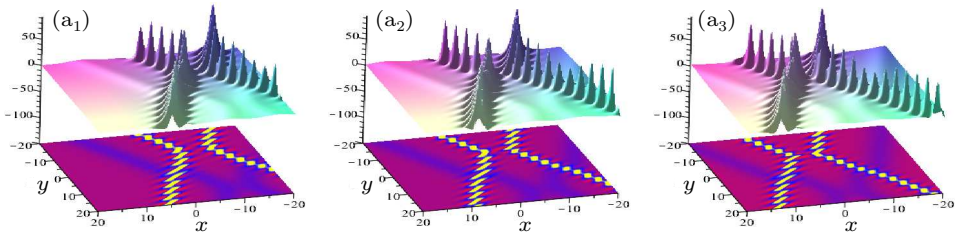


Fig. 16. Dynamical behaviors of the interaction solution between two different breather–soliton molecules. The parameters are given by $k_{1\mathrm{r}}=1$, $k_{1\mathrm{i}}=2/3$, $p_{1\mathrm{r}}=1/6$, $p_{1\mathrm{i}}\approx 1.51$, $k_5=2/3$, $p_5=1/9$, $k_{3\mathrm{r}}=-1$, $k_{3\mathrm{i}}=1$, $p_{3\mathrm{r}}=1/3$, $p_{3\mathrm{i}}\approx -2.2$, $k_6=2/3$, $p_6=-2/9$, $\eta_{10}=\eta_{20}=5$, $\eta_{30}=\eta_{40}=-5$, $\eta_{50}=-2$, $\eta_{60}=-2$. The time is chosen as $t=2i-4$ in (a_i) $(i=1,2,3)$.

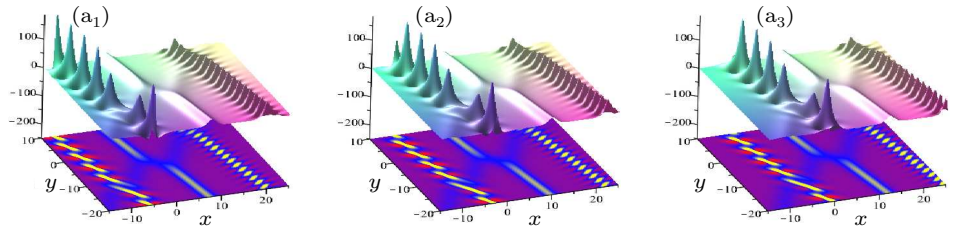


Fig. 17. Dynamical behaviors of the interaction solution between a 2-breather–1-soliton molecule and a stipe soliton. The parameters are selected as $k_{1\mathrm{r}}=1$, $k_{1\mathrm{i}}=2/3$, $p_{1\mathrm{r}}=1/6$, $p_{1\mathrm{i}}=29/27+\sqrt{546}/54$, $k_{3\mathrm{r}}=5/6$, $k_{3\mathrm{i}}=1$, $p_{3\mathrm{r}}=5/36$, $p_{3\mathrm{i}}=67/36-\sqrt{8418}/18$, $k_5=\sqrt{95}/6$, $p_5=\sqrt{95}/36$, $k_6=3/4$, $p_6=-1/2$, $\eta_{10}=\eta_{20}=10$, $\eta_{30}=\eta_{40}=-5$, $\eta_{50}=4$, $\eta_{60}=0$. The time is chosen as $t=i-2$ in (a_i) $(i=1,2,3)$.

Case 2 Three interaction solutions: mixed by a breather molecule and a soliton molecule, by a 2-breather–1-soliton molecule and a stripe soliton and by two different breather–soliton molecules.

In the above, we have shown that the complex conjugate constraints (32) can lead to two different kinds of molecule solutions. In the following, we shall show that such conjugate constrains can also produce interaction solutions if appropriate resonant conditions are introduced.

Here, we still go with the constraints (32). On introducing the following resonant conditions:

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{k_{3\mathrm{r}}},\quad \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{p_{3\mathrm{r}}},\quad \frac{\omega_5}{k_5}=\frac{\omega_6}{k_6},\quad \frac{\omega_5}{p_5}=\frac{\omega_6}{p_6},\tag{35}$$

we can obtain a mixed interaction solution between a breather molecule and a soliton molecule ($\{\mathrm{BBM},\mathrm{SM}\}$) whose dynamical behaviors are exhibited in Fig. 15. Interestingly, if we change the above resonant conditions (35) to the following form:

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}}=\frac{\omega_5}{k_5},\quad \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}}=\frac{\omega_5}{p_5},\quad \frac{\omega_{3\mathrm{r}}}{k_{3\mathrm{r}}}=\frac{\omega_6}{k_6},\quad \frac{\omega_{3\mathrm{r}}}{p_{3\mathrm{r}}}=\frac{\omega_6}{p_6},$$

then an interaction solution mixed by two different breather–soliton molecules ($\{\mathrm{BSM},\mathrm{BSM}\}$) is generated and its dynamical behaviors are simulated in Fig. 16. Furthermore, if we replace the resonant conditions (35) by

$$\frac{\omega_{1\mathrm{r}}}{k_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{k_{3\mathrm{r}}}=\frac{\omega_5}{k_5},\quad \frac{\omega_{1\mathrm{r}}}{p_{1\mathrm{r}}}=\frac{\omega_{3\mathrm{r}}}{p_{3\mathrm{r}}}=\frac{\omega_5}{p_5},$$

then an interaction solution consisting of a 2-breather–1-soliton molecule and stripe soliton ($\{\mathrm{BBSM,S}\}$) is constructed and its numerical simulations can be seen in Fig. 17.

Remark It states that by taking the conjugate restricts $k_j = k_j^* = k_{j\mathrm{r}} + \mathrm{i}k_{j\mathrm{i}}$, $p_j = p_j^* = p_{j\mathrm{r}} + \mathrm{i}p_{j\mathrm{i}}$, $j = 1,3,\ldots,2T-1$ and imposing resonant conditions

$$\frac{\omega_{1\mathrm{r}}}{k_{3\mathrm{r}}} = \frac{\omega_{3\mathrm{r}}}{k_{3\mathrm{r}}} = \cdots = \frac{\omega_{(2T-1)\mathrm{r}}}{k_{(2T-1)\mathrm{r}}} = \frac{\omega_{2T+1}}{k_{2T+1}} = \frac{\omega_{2T+2}}{k_{T+2}} = \cdots = \frac{\omega_{2T+L}}{k_{2T+L}},$$

to the $(2T+L)$-solitons, one can obtain a kind of molecule solution termed as T-breather–L-soliton molecule.

4. Conclusion

The $(2+1)$-dimensional generalized KDKK equation is an important physical model, which has been widely used in fluid mechanics, plasma physics, ocean dynamics, *et al.* In the present work, we introduce a special transformation to the KDKK model and thereby derive the N-soliton solutions. Then by imposing the complexication method and the velocity resonant principle to the related N-solitons, we obtain many different types of molecule solutions and interaction solutions, including breather–soliton molecule (see Fig. 3), breather–soliton–soliton molecule (see Fig. 6), 2-breather molecule (see Fig. 7), 2-breather–1-soliton molecule (see Fig. 9), 2-breather–2-soliton molecule (see Fig. 13), 3-breather molecule (see Fig. 14) as well as interaction solutions mixed by a breather–soliton molecule and a soliton/breather/another breather–soliton molecule (see Figs. 5, 12, and 16), mixed by soliton molecule and a soliton/breather/soliton molecule (see Figs. 2, 3, and 4) and so on. Dynamical behaviors of these solutions are analyzed theoretically and numerically, which shows that the restrictions imposed and the selections of the parameters can affect the types of the solutions as well as their corresponding propagation properties. However, when $N > 6$, due to the complexity in selecting the parameters, it becomes very difficult to study such interaction solutions. But they are worthy of deep considerations. In addition, how to use these solutions to explain (or predict) some related physical phenomena deserves consideration. Moreover, it is known that deep integrable learning is a hot topic nowadays, whether such solutions can be investigated by deep integrable learning is also an interesting question to investigate. Based on the importance and applications of the KDKK equation and breather molecules, all the questions mentioned above will be further studied in the future.

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