Breather and its interaction with rogue wave of the coupled modified nonlinear Schrödinger equation

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We investigate the coupled modified nonlinear Schrödinger equation. Breather solutions are constructed through the traditional Darboux transformation with nonzero plane-wave solutions. To obtain the higher-order localized wave solution, the N-fold generalized Darboux transformation is given. Under the condition that the characteristic equation admits a double-root, we present the expression of the first-order interactional solution. Then we graphically analyze the dynamics of the breather and rogue wave. Due to the simultaneous existence of nonlinear and self-steepening terms in the equation, different profiles in two components for the breathers are presented.

Keywords: coupled modified nonlinear Schrödinger equation, Darboux transformation, breather, rouge wave

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1. Introduction

The nonlinear Schrödinger (NLS) equation, as one of the nonlinear partial differential equations, has been studied in such fields as nonlinear optics, plasma physics and condensed matter physics due to its potential mathematical properties and physical applications.[1–4] It has been found that the NLS equation can describe the dynamic of a nonlinear localized wave usually associated with soliton, breather and rogue wave. The analysis of the dynamic mechanism of a localized wave can be done by considering a certain kind of exact solution of the NLS equation, and has been done to the scalar, coupled and multi-dimensional NLS equation.[5–9] The nonlinear localized wave solutions of the NLS equation can be constructed by Darboux transformation and the Hirota method.[10,11] One generalization of the NLS equation is the modified NLS equation, of which the nonlinear localized wave has been investigated,[12–17] such as the fundamental and super-regular breathers,[18] the asymmetric spectra of Akhmediev breathers.[19,20] Another extension is from the single-component NLS model to the multi-component one, which opens up new fields for the study of related areas.[21]

In this paper, we will consider a coupled modified NLS equation[22]

$$\mathrm{i}u_t + u_{xx} + \mu(|u|^2+|v|^2)u + \mathrm{i}\gamma[(|u|^2+|v|^2)u]_x = 0, \tag{1a}$$

$$\mathrm{i}v_t + v_{xx} + \mu(|u|^2+|v|^2)v + \mathrm{i}\gamma[(|u|^2+|v|^2)v]_x = 0, \tag{1b}$$

where u and v are complex functions of t and x, and designate the slowly varying envelopes for the two polarizations; t and x respectively denote the normalized time and distance, μ and γ are real constants referring to the nonlinearity and self-steepening coefficients. Equation (1) is the complete integrability in view of admitting the N-soliton solution via Darboux transformation, and dark and anti-dark vector solitons through the Hirota method reported in former literature.[22–24] The double-pole soliton and first-order rogue wave for Eq. (1) have also been discussed.[25,26]

When $\mu = 0$, equation (1) reduces to the coupled derivative NLS equations associated with the polarized Alfvén waves in plasma physics.[27] The Darboux transformation and multi-soliton solution formulae, conservation laws, modulation instability and semi-rational solutions have been given for the coupled derivative NLS equations.[28,29] Equation (1) is simplified to the coupled NLS equation (or Manakov model) under the condition $\gamma = 0$. The vector semi-rational solutions and the diverse localized waves, such as the bright-dark-rogue solution, Akhmediev breathers, non-degenerate rogue waves and Kuznetsov–Ma solitons for the Manakov model have been presented in Refs. [30–37]. The main consideration in this paper is the case of $\gamma \neq 0$ and $\mu \neq 0$ for Eq. (1), which contains substantial dynamic property.

As is well known, since the breather solution plays an important part in the excitation of a rogue wave, we focus on generating the breather solution and analyzing the dynamic of a nonlinear localized wave for Eq. (1). To our knowledge, few reports exist on the dynamics of multi-breathers for the coupled modified NLS system (1) — that is the aim of this paper. With the above consideration, in Section 2, we give the multi-breather solutions by the traditional Darboux transformation and discuss the corresponding dynamics. Then the general breather is shown. In Section 3, we present the expression of the N-fold generalized Darboux transformation for Eq. (1) and construct the interaction solutions between multi-breather and higher-order rogue waves. Section 4 will provide the conclusion.

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2. Breather for the coupled modified NLS equation

In order to understand the excitation mechanism of the interactional solutions, the multi-breather solutions are firstly discussed in Eq. (1) by utilizing the traditional Darboux transformation.

The Lax pair for Eq. (1) can be given as[23]

$$\Psi_x = U\Psi = \sum_{k=0}^{2} \zeta^{2-k} U_k \Psi, \tag{2a}$$

$$\Psi_t = V\Psi = \sum_{j=0}^{4} \zeta^{4-k} V_j \Psi, \tag{2b}$$

where $\Psi = (\psi_1, \psi_2, \psi_3)^{\mathrm{T}}$ is the vector eigenfunction with ψ_j's $(j = 1,2,3)$ as the corresponding elements, ζ is the spectral parameter, U, V, U_k, $(k = 0,1,2)$ and V_j, $(j = 0,1,\ldots,4)$ are 3×3 matrices as follows:

$$U_0 = \frac{\mathrm{i}}{6\gamma}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_1 = \frac{1}{2}\begin{pmatrix} 0 & u & v \\ -u^* & 0 & 0 \\ -v^* & 0 & 0 \end{pmatrix},$$

$$U_2 = -2\mu U_0, \quad V_0 = \frac{1}{2\gamma}U_0, \quad V_1 = \frac{1}{2\gamma}U_1, \quad V_4 = 4\mu^2 V_0,$$

$$V_2 = -4\mu V_0 + \frac{\mathrm{i}}{4}\begin{pmatrix} |u|^2+|v|^2 & 0 & 0 \\ 0 & -|u|^2 & -u^*v \\ 0 & -uv^* & -|v|^2 \end{pmatrix},$$

$$V_3 = 2\mu V_1 + \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & u_x & v_x \\ u_x^* & 0 & 0 \\ v_x^* & 0 & 0 \end{pmatrix} + \frac{\gamma}{2}\begin{pmatrix} 0 & -u(|u|^2+|v|^2) & -u(|u|^2+|v|^2) \\ u^*(|u|^2+|v|^2) & 0 & 0 \\ v^*(|u|^2+|v|^2) & 0 & 0 \end{pmatrix},$$

the asterisk $*$ represents the complex conjugation. From the compatibility condition of Eq. (2), i.e., the zero-curvature equation

$$U_t - V_x + [U,V] = 0, \tag{3}$$

one can obtain Eq. (1), where the square brackets denote the matrix commutator.

We set the nontrivial initial solutions for Eq. (1) as

$$u_1[0] = k_1 \mathrm{e}^{\mathrm{i}\theta_1}, \quad v_1[0] = k_2 \mathrm{e}^{\mathrm{i}\theta_2},$$
$$\theta_j = m_j x + [(k_1^2+k_2^2)(\mu - \gamma m_j) - m_j^2]t, \quad j = 1,2, \tag{4}$$

where k_j are amplitudes and m_j $(j = 1,2)$ are real wave numbers. In Lax pair (2), Ψ could be difficult to solve directly based on the initial solutions (4) and $\zeta = \zeta_1$. According to the idea,[31,38] we firstly transform the Lax pair (2) into the following one through a gauge transformation $\Psi = M\Phi$:

$$\Phi_x = \widehat{U}\Phi, \tag{5a}$$

$$\Phi_t = (\mathrm{i}\widehat{U}^2 + \varpi_1\widehat{U} + \varpi_2 I)\Phi, \tag{5b}$$

with I being the 3×3 identity matrix and

$$\widehat{U} = \begin{pmatrix} -\mathrm{i}\frac{\zeta_1^2-2\mu}{3\gamma} - \mathrm{i}\frac{m_1+m_2}{3} & \frac{\zeta_1 k_1}{2} & \frac{\zeta_1 k_2}{2} \\ -\frac{\zeta_1 k_1}{2} & \mathrm{i}\frac{\zeta_1^2-2\mu}{6\gamma} + \mathrm{i}\frac{2m_1-m_2}{3} & 0 \\ -\frac{\zeta_1 k_2}{2} & 0 & \mathrm{i}\frac{\zeta_1^2-2\mu}{6\gamma} + \mathrm{i}\frac{2m_2-m_1}{3} \end{pmatrix},$$

$$M = \mathrm{diag}\left(\mathrm{e}^{\frac{\mathrm{i}}{3}(\theta_1+\theta_2)}, \mathrm{e}^{\frac{\mathrm{i}}{3}(\theta_2-2\theta_1)}, \mathrm{e}^{\frac{\mathrm{i}}{3}(\theta_1-2\theta_2)}\right), \quad \varpi_1 = \frac{\zeta_1^2-2\mu}{3\gamma} - \gamma(k_1^2+k_2^2) - \frac{2(m_1+m_2)}{3},$$

$$\varpi_2 = \frac{\mathrm{i}}{18}\left[3\zeta_1^2(k_1^2+k_2^2) + 4(m_1^2 - m_1m_2 + m_2^2) + \frac{(\zeta_1^2-2\mu)^2}{\gamma^2} + \frac{2(\zeta_1^2-2\mu)(m_1+m_2)}{\gamma}\right].$$

In order to obtain the solutions of the Lax pair (5), we make further efforts to convert the Lax pair (5) into the form with the diagonal matrix K

$$\Upsilon_x = K\Upsilon, \tag{6a}$$

$$\Upsilon_t = (\mathrm{i}K^2 + \varpi_1 K + \varpi_2 I)\Upsilon, \tag{6b}$$

by the matrix transformation $\Phi = N\Upsilon$ with

$$N = \begin{pmatrix} 1 & 1 & 1 \\ -\frac{\zeta_1 k_1}{2[\chi - \frac{2}{3}\mathrm{i}(2m_1-m_2)]} & -\frac{\zeta_1 k_1}{2[\rho - \frac{2}{3}\mathrm{i}(2m_1-m_2)]} & -\frac{\zeta_1 k_1}{2[\nu - \frac{2}{3}\mathrm{i}(2m_1-m_2)]} \\ -\frac{\zeta_1 k_2}{2[\chi - \frac{2}{3}\mathrm{i}(2m_2-m_1)]} & -\frac{\zeta_1 k_2}{2[\rho - \frac{2}{3}\mathrm{i}(2m_2-m_1)]} & -\frac{\zeta_1 k_2}{2[\nu - \frac{2}{3}\mathrm{i}(2m_2-m_1)]} \end{pmatrix}.$$

Here, the matrix $K = \mathrm{diag}\left(\chi + \mathrm{i}\frac{\zeta_1^2-2\mu}{6\gamma}, \rho + \mathrm{i}\frac{\zeta_1^2-2\mu}{6\gamma}, \nu + \mathrm{i}\frac{\zeta_1^2-2\mu}{6\gamma}\right)$, and χ, ρ, ν are three different roots of the cubic equation

$$\xi^3 + \mathrm{i}\frac{\zeta_1^2-2\mu}{2\gamma}\xi^2 + \left[\frac{\zeta_1^2}{4}(k_1^2+k_2^2) + \frac{\zeta_1^2-2\mu}{6\gamma}(m_1+m_2) + \frac{m_1^2 - m_1m_2 + m_2^2}{3}\right]\xi + \frac{\mathrm{i}\zeta_1^2}{12}\left[k_1^2(m_1-2m_2) + k_2^2(m_2-2m_1)\right]$$
$$-\mathrm{i}\frac{\zeta_1^2-2\mu}{18\gamma}(m_1-2m_2)(m_2-2m_1)$$
$$-\frac{\mathrm{i}}{27}(m_1-2m_2)(m_2-2m_1)(m_1+m_2) = 0, \tag{7}$$

where ξ is the unknown number. By the corresponding transformation and solving Eq. (6), one can obtain the eigenfunction Ψ in the Lax pair (2). Next, we will mainly discuss the dynamics of the breather for Eq. (1) under the condition $m_1 \neq m_2$. Based on the above initial solutions (4), the one-breather can be given in the following expression by setting $\zeta = \zeta_1$ via the traditional Darboux transformation:[23]

$$u = u_1[0] + 2(\zeta_1^{*2} - \zeta_1^2)\left\{\frac{\mathrm{i}\mu\psi_1[\zeta_1]\psi_2^*[\zeta_1]}{\gamma\Theta} - \left(\frac{\mathrm{i}\mu\psi_1[\zeta_1]\psi_2^*[\zeta_1]}{\gamma\Theta}\right)_x\right\}, \tag{8a}$$

$$v = u_1[0] + 2(\zeta_1^{*2} - \zeta_1^2)\left\{\frac{\mathrm{i}\mu\psi_1[\zeta_1]\psi_3^*[\zeta_1]}{\gamma\Theta} - \left(\frac{\mathrm{i}\mu\psi_1[\zeta_1]\psi_3^*[\zeta_1]}{\gamma\Theta}\right)_x\right\}, \tag{8b}$$

with

$$\begin{aligned}
&\Theta = |\zeta_1|^2\{\zeta_1|\psi_1[\zeta_1]|^2 + \zeta_1^*(|\psi_2[\zeta_1]|^2 + |\psi_3[\zeta_1]|^2)\},\\
&\psi_1[\zeta_1] = l_1\mathrm{e}^{\frac{\mathrm{i}}{3}(\theta_1+\theta_2)+\eta(\chi)} + l_2\mathrm{e}^{\frac{\mathrm{i}}{3}(\theta_1+\theta_2)+\eta(\rho)} + l_3\mathrm{e}^{\frac{\mathrm{i}}{3}(\theta_1+\theta_2)+\eta(\nu)},\\
&\psi_2[\zeta_1] = -\frac{\zeta_1 k_1 l_1}{2[\mathrm{ch}\,\mathrm{i} - \frac{2}{3}\mathrm{i}(2m_1 - m_2)]}\mathrm{e}^{\frac{\mathrm{i}}{3}(2\theta_2-\theta_1)+\eta(\chi)} - \frac{\zeta_1 k_1 l_2}{2[\rho - \frac{2}{3}\mathrm{i}(2m_1 - m_2)]}\mathrm{e}^{\frac{\mathrm{i}}{3}(2\theta_2-\theta_1)+\eta(\rho)} - \frac{\zeta_1 k_1 l_3}{2[\nu - \frac{2}{3}\mathrm{i}(2m_1 - m_2)]}\mathrm{e}^{\frac{\mathrm{i}}{3}(2\theta_2-\theta_1)+\eta(\nu)},\\
&\psi_3[\zeta_1] = -\frac{\zeta_1 k_2 l_1}{2[\chi - \frac{2}{3}\mathrm{i}(2m_2 - m_1)]}\mathrm{e}^{\frac{\mathrm{i}}{3}(2\theta_1-\theta_2)+\eta(\chi)} - \frac{\zeta_1 k_2 l_2}{2[\rho - \frac{2}{3}\mathrm{i}(2m_2 - m_1)]}\mathrm{e}^{\frac{\mathrm{i}}{3}(2\theta_1-\theta_2)+\eta(\rho)} - \frac{\zeta_1 k_2 l_3}{2[\nu - \frac{2}{3}\mathrm{i}(2m_2 - m_1)]}\mathrm{e}^{\frac{\mathrm{i}}{3}(2\theta_1-\theta_2)+\eta(\nu)},\\
&\eta(z) = \mathrm{i}tz^2 + \left(x + \varpi_1 t - \frac{\zeta_1^2 - 2\mu}{3\gamma}t\right)z + \mathrm{i}\frac{\zeta_1^2 - 2\mu}{6\gamma}(x + \varpi_1 t) - \mathrm{i}\frac{(\zeta_1^2 - 2\mu)^2}{36\gamma^2}t + \varpi_2 t,
\end{aligned} \tag{9}$$

where l_1, l_2, and l_3 are arbitrary constants. By choosing the proper parameters, one can observe the general breather, which is periodic in both space and time, as shown in Fig. 1. Meanwhile, the two-breather solution can be derived by setting $N = 2$ in the traditional Darboux transformation and choosing a different spectral parameter. The interaction between two general breathers is depicted in Fig. 2, where two breathers keep their profile propagating periodically except the increase of amplitude in the crossing area.

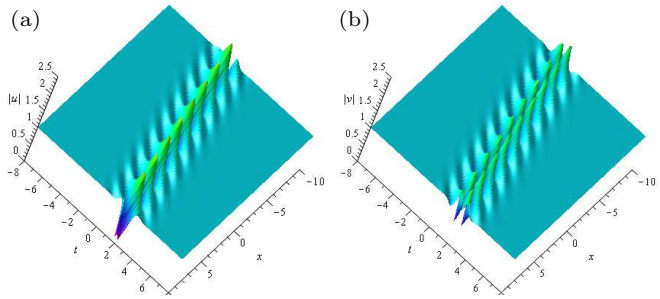


Fig. 1. General breather for two components with parameters $\mu = 2$, $\gamma = 1$, $\zeta_1 = 1 - \mathrm{i}$, $l_1 = 1$, $l_2 = 0$, $l_3 = 2$, $m_1 = 1$, $m_2 = -2$, $k_1 = 1$, and $k_2 = -1$.

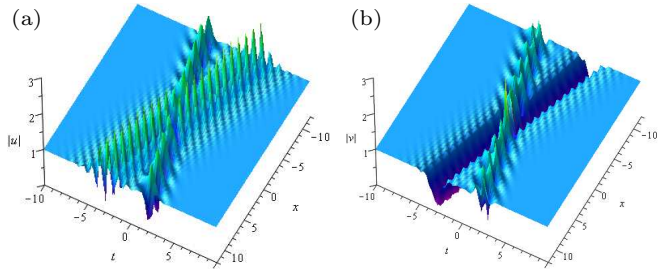


Fig. 2. Two breathers for two components with parameters $\mu = 2$, $\gamma = 1$, $\zeta_1 = 1 - \mathrm{i}$, $\zeta_2 = 2 - \mathrm{i}$, $l_1 = 1$, $l_2 = 0$, $l_3 = 2$, $m_1 = 1$, $m_2 = -2$, $k_1 = 1$, and $k_2 = -1$.

3. Interaction of multi-breather and higher-order rogue waves

In the following section, we will consider the interaction of the breather and rogue wave for Eq. (1) by the generalized Darboux transformation.

For simplicity, the N-fold generalized Darboux transformation of Eq. (1) will be directly presented in the following form:

$$\Psi[N] = T[N]T[N-1]\cdots T[1]\Psi_1[0],$$

$$T[k] = \zeta^2 P_2[k] + \zeta P_1[k] - I, \quad (k = 1,2,\ldots,N), \tag{10a}$$

$$u[N] = u[N-1] + 2(\zeta_1^{*2} - \zeta_1^2)\left\{\frac{\mathrm{i}\mu\psi_1[N-1]\psi_2[N-1]^*}{\gamma\Gamma_1[N]} - \left(\frac{\psi_1[N-1]\psi_2[N-1]^*}{\Gamma_1[N]}\right)_x\right\}, \tag{10b}$$

$$v[N] = v[N-1] + 2(\zeta_1^{*2} - \zeta_1^2)\left\{\frac{\mathrm{i}\mu\psi_1[N-1]\psi_3[N-1]^*}{\gamma\Gamma_1[N]} - \left(\frac{\psi_1[N-1]\psi_3[N-1]^*}{\Gamma_1[N]}\right)_x\right\}, \tag{10c}$$

where

$$\begin{aligned}
&\Psi_1[0] = \Psi_1^{[0]} = (\psi_1[0], \psi_2[0], \psi_3[0])^{\mathrm{T}},\\
&\Psi_1[j-1] = \lim_{\varepsilon\to 0}\frac{\Psi[j-1]|_{\zeta=\zeta_1+\varepsilon}}{\varepsilon^{j-1}}\\
&\qquad = \lim_{\varepsilon\to 0}\frac{T[j-1]T[j-2]\cdots T[1]\Psi_1[0]|_{\zeta=\zeta_1+\varepsilon}}{\varepsilon^{j-1}},\\
&\Psi_1[j-1] = (\psi_1[j-1], \psi_2[j-1], \psi_3[j-1])^{\mathrm{T}}, \ (j = 2,3,\ldots,N),
\end{aligned}$$

and the matrices $P_1[k], P_2[k]$, $(k = 1,2,\ldots,N)$ are given as

$$P_1[k]=\begin{pmatrix} 0 & \frac{(\zeta_1^{*2}-\zeta_1^2)\psi_1[k-1]\psi_2[k-1]^*}{\Gamma_1[k]} & \frac{(\zeta_1^{*2}-\zeta_1^2)\psi_1[k-1]\psi_3[k-1]^*}{\Gamma_1[k]} \\ \frac{(\zeta_1^{*2}-\zeta_1^2)\psi_2[k-1]\psi_1[k-1]^*}{\Gamma_1[k]^*} & 0 & 0 \\ \frac{(\zeta_1^{*2}-\zeta_1^2)\psi_3[k-1]\psi_1[k-1]^*}{\Gamma_1[k]^*} & 0 & 0 \end{pmatrix},$$

$$P_2[k]=\begin{pmatrix} \Lambda_1[k] & 0 & 0 \\ 0 & \Lambda_2[k] & \frac{(\zeta_1^{*2}-\zeta_1^2)\psi_2[k-1]\psi_3[k-1]^*}{\Gamma_2[k]} \\ 0 & \frac{(\zeta_1^{*2}-\zeta_1^2)\psi_3[k-1]\psi_2[k-1]^*}{\Gamma_2[k]} & \Lambda_3[k] \end{pmatrix},$$

with

$$\Lambda_1[k]=\frac{\zeta_1^*|\psi_1[k-1]|^2+\zeta_1(|\psi_2[k-1]|^2+|\psi_3[k-1]|^2)}{\Gamma_1[k]},$$

$$\Lambda_2[k]=\frac{|\zeta_1|^2|\psi_1[k-1]|^2+\zeta_1^{*2}|\psi_2[k-1]|^2+\zeta_1^2|\psi_3[k-1]|^2}{\Gamma_2[k]},$$

$$\Lambda_3[k]=\frac{|\zeta_1|^2|\psi_1[k-1]|^2+\zeta_1^2|\psi_2[k-1]|^2+\zeta_1^{*2}|\psi_3[k-1]|^2}{\Gamma_2[k]},$$

$$\Gamma_1[k]=|\zeta_1|^2\left\{\zeta_1|\psi_1[k-1]|^2+\zeta_1^*(|\psi_2[k-1]|^2+|\psi_3[k-1]|^2)\right\},$$

$$\Gamma_2[k]=|\zeta_1|^2\left\{\zeta_1^{*2}|\psi_1[k-1]|^2 +|\zeta_1|^2(|\psi_2[k-1]|^2+|\psi_3|^2[k-1])\right\},\quad (k=1,2,\ldots,N).$$

Based on the N-fold generalized Darboux transformation (10) and choosing the proper eigenfunction Ψ_1, one can derive the N-order localized wave solution through N times iteration. Here, we choose the following seed solution to derive the interaction solution:

$$u[0]=k_1\,\mathrm{e}^{2\mathrm{i}(k_1^2+k_2^2)t},\quad v[0]=k_2\,\mathrm{e}^{2\mathrm{i}(k_1^2+k_2^2)t}. \tag{11}$$

Through the nontrivial transformation $\Psi=\widetilde{M}\widetilde{\Psi}$, the original Lax pair (2) can be transformed into the following one:

$$\widetilde{\Psi}_x=(\widetilde{M}^{-1}U\widetilde{M}-\widetilde{M}^{-1}\widetilde{M}_x)\widetilde{\Psi}\equiv\widetilde{U}\widetilde{\Psi}, \tag{12a}$$

$$\widetilde{\Psi}_t=(\widetilde{M}^{-1}V\widetilde{M}-\widetilde{M}^{-1}\widetilde{M}_t)\widetilde{\Psi}\equiv\widetilde{V}\widetilde{\Psi}, \tag{12b}$$

with

$$\widetilde{M}=\mathrm{diag}\left(\mathrm{e}^{\frac{2\mathrm{i}}{3}(k_1^2+k_2^2)t},\mathrm{e}^{-\frac{\mathrm{i}}{3}(k_1^2+k_2^2)t},\mathrm{e}^{-\frac{\mathrm{i}}{3}(k_1^2+k_2^2)t}\right),\quad \widetilde{U}=\begin{pmatrix} -\frac{\mathrm{i}(\zeta^2-2\mu)}{3\gamma} & \frac{1}{2}\zeta k_1 & \frac{1}{2}\zeta k_2 \\ -\frac{1}{2}\zeta k_1 & \frac{\mathrm{i}(\zeta^2-2\mu)}{6\gamma} & 0 \\ -\frac{1}{2}\zeta k_2 & 0 & \frac{\mathrm{i}(\zeta^2-2\mu)}{6\gamma} \end{pmatrix},$$

$$\widetilde{V}=\begin{pmatrix} \Lambda_{11} & -\frac{\zeta k_1\left[2\zeta^2(k_1^2+k_2^2)-\zeta^2+2\mu\right]}{4\gamma} & -\frac{\zeta k_2\left[2\zeta^2(k_1^2+k_2^2)-\zeta^2+2\mu\right]}{4\gamma} \\ \frac{\zeta k_1\left[2\zeta^2(k_1^2+k_2^2)-\zeta^2+2\mu\right]}{4\gamma} & \Lambda_{22} & -\frac{1}{4}\mathrm{i}\zeta^2k_1k_2 \\ \frac{\zeta k_2\left[2\zeta^2(k_1^2+k_2^2)-\zeta^2+2\mu\right]}{4\gamma} & -\frac{1}{4}\mathrm{i}\zeta^2k_1k_2 & \Lambda_{33} \end{pmatrix},$$

$$\Lambda_{11}=\frac{\mathrm{i}\left[\gamma^2(3\zeta^2-8\mu)(k_1^2+k_2^2)-2\zeta^4+8\zeta^2\mu-8\mu^2\right]}{12\gamma^2},\quad \Lambda_{22}=-\frac{\mathrm{i}\left[\gamma^2(3\zeta^2k_1^2-4\mu k_1^2-4\mu k_2^2)-\zeta^4+4\zeta^2\mu-4\mu^2\right]}{12\gamma^2},$$

$$\Lambda_{33}=-\frac{\mathrm{i}\left[\gamma^2(3\zeta^2k_2^2-4\mu k_1^2-4\mu k_2^2)-\zeta^4+4\zeta^2\mu-4\mu^2\right]}{12\gamma^2}.$$

To solve Eq. (12), the spectral characteristic equation of $\widetilde{U}$ should be

$$\xi^3+\frac{1}{12\gamma^2}\left[3\gamma^2\zeta^2(k_1^2+k_2^2)+(\zeta^2-2\mu)^2\right]\xi$$

$$-\frac{\mathrm{i}}{216\gamma^3}(\zeta^2-2\mu)\left[9\gamma^2\zeta^2(k_1^2+k_2^2)+2(\zeta^2-2\mu)^2\right]=0, \tag{13}$$

where ζ is the spectral parameter and ξ is the unknown number.

As is well known, the rogue waves may be excited when the above-mentioned characteristic equation (13) possesses multiple roots. In this paper, we consider the interactional solution between rogue waves and breathers under the condition that Eq. (13) owns a double-root. By choosing $k_1=-1/2$, $k_2=1$, $\gamma=1$, $\mu=2$ and the spectral parameter $\zeta=\zeta_1=\frac{1}{2}(\sqrt{11}+\mathrm{i}\sqrt{11})$, and substituting them into Eq. (13), we can get a double-root $\xi_1=\xi_2=\frac{1}{24}(\sqrt{55}+5\mathrm{i})$ and a single root $\xi_3=-\frac{1}{2}\sqrt{55}-\frac{5}{12}\mathrm{i}$ for Eq. (13).

By setting $N=1$ in the generalized Darboux transformation (10) and considering the above choice of parameters, one can construct the interaction solution between one breather

and the first-order rogue waves

$$u = -\frac{1}{2}\mathrm{e}^{\frac{5}{2}\mathrm{i}t} + 4\sqrt{55}\,\mathrm{e}^{\frac{5}{2}\mathrm{i}t}\frac{G_1 - 2H_1}{F} + 2\mathrm{i}\sqrt{55}\,\mathrm{e}^{\frac{5}{2}\mathrm{i}t}\left(\frac{G_1 - 2H_1}{F}\right)_x,$$

$$v = \mathrm{e}^{\frac{5}{2}\mathrm{i}t} + 4\sqrt{55}\,\mathrm{e}^{\frac{5}{2}\mathrm{i}t}\frac{H_1 - 2G_1}{F} + 2\mathrm{i}\sqrt{55}\,\mathrm{e}^{\frac{5}{2}\mathrm{i}t}\left(\frac{H_1 - 2G_1}{F}\right)_x,$$

with

$$G_1 = -\frac{a_1^2 - b_1^2}{160}, \quad H_1 = (a_1 + b_1)\,\mathrm{e}^{\frac{\sqrt{55}}{16}(5t-2x) - \frac{5}{16}\mathrm{i}t + \frac{5}{8}\mathrm{i}x},$$

$$F = (40\mathrm{i}x - 150\mathrm{i}t + 16 - 400tx + 775t^2 + 80x^2)\frac{\sqrt{55}}{40} + (\sqrt{11} - \mathrm{i}\sqrt{5})\frac{5\alpha^2}{2}\mathrm{e}^{\frac{\sqrt{55}}{8}(5t-2x)} + \frac{\alpha}{40}\left[(8c_1 + d_1)\sqrt{4\sqrt{5}+5} + (8c_1 - d_1)\mathrm{i}\sqrt{4\sqrt{5}-5}\right] \times \mathrm{e}^{\frac{\sqrt{55}}{48}(5t-2x) + \frac{5}{16}\mathrm{i}t - \frac{5}{8}\mathrm{i}x},$$

$$a_1 = \frac{\sqrt{5}}{640}\left[(15\sqrt{55}\mathrm{i}t - 4\sqrt{55}\mathrm{i}x)\sqrt{4\sqrt{5}+5} + (5\mathrm{i}t + 20\mathrm{i}x + 16)\sqrt{4\sqrt{5}-5}\right],$$

$$b_1 = \frac{\sqrt{5}}{640}\left[(4\sqrt{55}x - 15\sqrt{55}t)\sqrt{4\sqrt{5}-5} + (5t + 20x - 16\mathrm{i})\sqrt{4\sqrt{5}+5}\right],$$

$$c_1 = 4\sqrt{5}\mathrm{i}x - 10\sqrt{5}\mathrm{i}t - \sqrt{11}\mathrm{i} - 5\sqrt{11}t - \sqrt{5},$$

$$d_1 = 85\sqrt{5}\mathrm{i}t - 12\sqrt{5}\mathrm{i}x + 8\sqrt{11}\mathrm{i} + 35\sqrt{11}t - 20\sqrt{11}x + 8\sqrt{5},$$

where α is the constant. The second-order interaction solution can be derived according to the generalized Darboux transformation (10) by setting $N = 2$, but we do not demonstrate them here due to the complicated and lengthy expression, which can be verified with the aid of Maple software.

When choosing the proper value of parameters, one can observe the interaction between one breather and first-order rogue wave in Fig. 3. In Figs. 3(c) and 3(d), the parameter α is smaller than that in Figs. 3(a) and 3(b). Through comparison among the figures, one can find that the distance between the breather and rogue wave becomes larger, which shows that the parameter α will influence the distance between the breather and rogue wave. When increasing the value of α, the breather merges with the rogue wave, while α decreases, the breather and the rogue wave will separate. To better understand the interaction between the breather and first-order rogue wave, we display the profile of temporal evolution corresponding to Figs. 3(a) and 3(b), as those presented in Fig. 4. At $t = 0$, one can observe that the first-order rogue wave coexists with one general breather, as the dashed line shown in Fig. 4. The left envelopes of the dashed line represent the general breather in two components, and the right ones of the dashed line indicate the first-order rogue wave. At $t = 0$, the breather has a higher amplitude than the first-order rogue wave in component u, while the breather has a smaller amplitude than the first-order rogue wave in component v. At $t = 12$ in component u and $t = 16$ in component v, one can see the profile of the single breather as the solid line shown in Fig. 4. Here, for the sake of obtaining complete amplitudes of the breathers in two components, we choose the diverse choice of time at $t = 12$ and $t = 16$ respectively. Meanwhile, one may notice that the breather has a higher amplitude in component u than that in component v, as the left contour of the dashed lines displayed in Fig. 4, while the rogue wave in component v has a higher amplitude than that in component u, which can be seen in the right contour of the dashed line in Fig. 4.

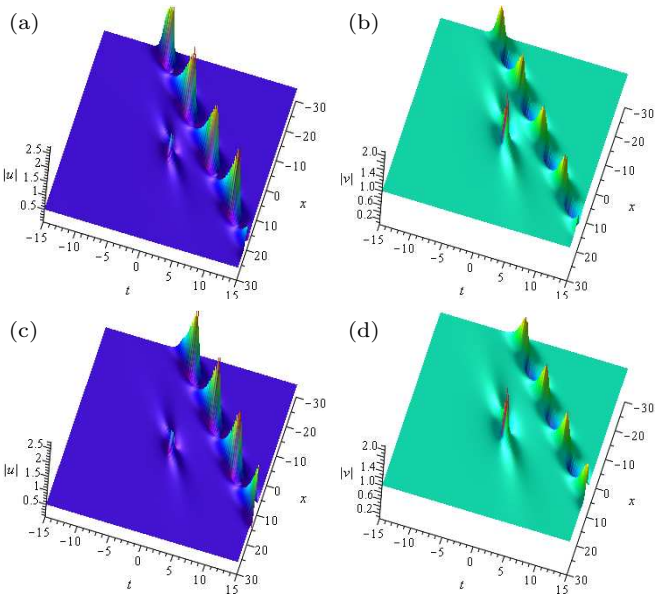


Fig. 3. The evolution of the first-order interactional solutions between one general breather and the first-order rogue wave with parameters $\mu = 2$, $\gamma = 1$, $\zeta = \frac{1}{2}\sqrt{11} + \frac{1}{2}\mathrm{i}\sqrt{5}$, $k_1 = 1$, $k_2 = -1/2$. $\alpha = 10^{-5}$ in (a) and (b), $\alpha = 10^{-7}$ in (c) and (d).

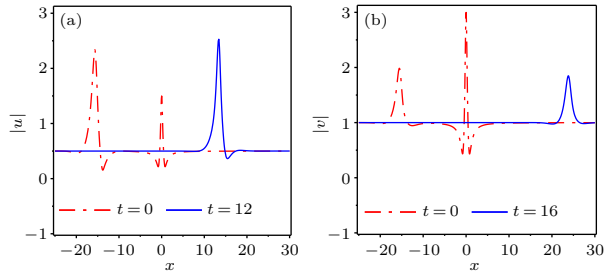


Fig. 4. The evolution of the first-order interactional solutions between one general breather and the first-order rogue wave at different times with parameters $\mu = 2$, $\gamma = 1$, $\zeta = \frac{1}{2}\sqrt{11} + \frac{1}{2}\mathrm{i}\sqrt{5}$, $\alpha = 10^{-5}$, $k_1 = 1$, and $k_2 = -1/2$.

The second-order rogue wave usually includes two types:

the fundamental case and the triangular pattern case. For this reason, we will discuss two kinds of the second-order interactional solutions. Firstly, we consider the interaction between two breathers and the fundamental second-order rogue waves, as displayed in Fig. 5. Here, the breathers show different profiles in two components. The amplitudes above the plane wave background of the two breathers are bigger than that at the bottom in component u, while the opposite case can be found in component v, where the amplitudes below the plane wave background of the two breathers are larger than that above. The breather existing in component v is usually called a quasi-breather. To better observe the difference between the two components, we also present the profile of the two breathers and the fundamental second-order rogue waves at different times, as shown in Fig. 6. At $t = 0$, the two breathers exist with the second-order rogue wave, and the two left envelopes of the dashed line correspond to the breathers, and the right one of the dashed line denotes the second-order rogue wave. Simultaneously, one can see that the two breathers have a similar amplitude as the rogue wave in component u, as the dashed line shown in Fig. 6(a), while the rogue wave has a very large amplitude in component v, which is characteristic of a rogue wave and is associated with the right envelope of the dashed line in Fig. 6(b). At $t = 16$ in component u and $t = 20$ in component v, only the contours of the two breathers can be observed, and the two breathers appear inapparent in component v as a result of the large amplitude of the rogue wave. Here, we choose the different times to display the obvious amplitudes of the two breathers in the two components due to the different period of breathers.

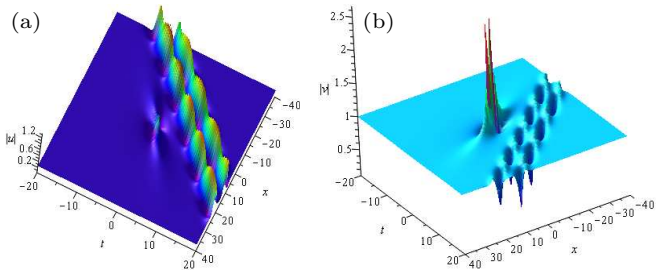


Fig. 5. The evolution of the second-order interactional solutions between two breathers and the fundamental second-order rogue waves with parameters $\mu = 2$, $\gamma = 1$, $\zeta = \frac{1}{4}\sqrt{47} + \frac{1}{4}\mathrm{i}\sqrt{17}$, $\alpha = 10^{-7}$, $k_1 = 1$, and $k_2 = 1/4$.

By changing the parameters, one can obtain another view of the interaction between two breathers and the triangular pattern rogue wave, which is presented in Figs. 7(a) and 7(b). Comparing the result with Fig. 5, one can see that the fundamental second-order rogue wave splits into three first-order ones, which are usually called the triangular pattern rogue wave. The two breathers in two components still have different profiles, namely, breather and quasi-breather, similar to those shown in Fig. 5. Furthermore, one can ascertain that the amplitude of the rogue wave in component u is far less than that in component v. By changing the value of parameter α and giving Figs. 7(c) and 7(d), where α is smaller than that in Figs. 7(a) and 7(b), we find a similar phenomenon that when α decreases, the two breathers and the triangular pattern rogue wave separate, which is consistent with the previous analysis.

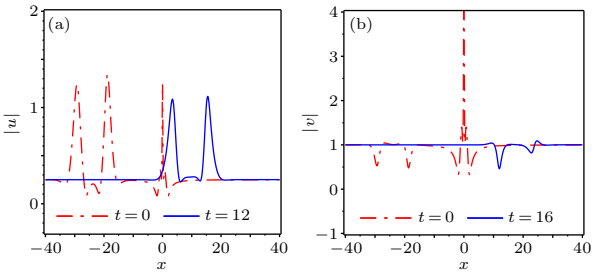


Fig. 6. The evolution of the second-order interactional solutions between two breathers and the fundamental second-order rogue waves at different times with parameters $\mu = 2$, $\gamma = 1$, $\zeta = \frac{1}{4}\sqrt{47} + \frac{1}{4}\mathrm{i}\sqrt{17}$, $\alpha = 10^{-7}$, $k_1 = 1$, and $k_2 = 1/4$.

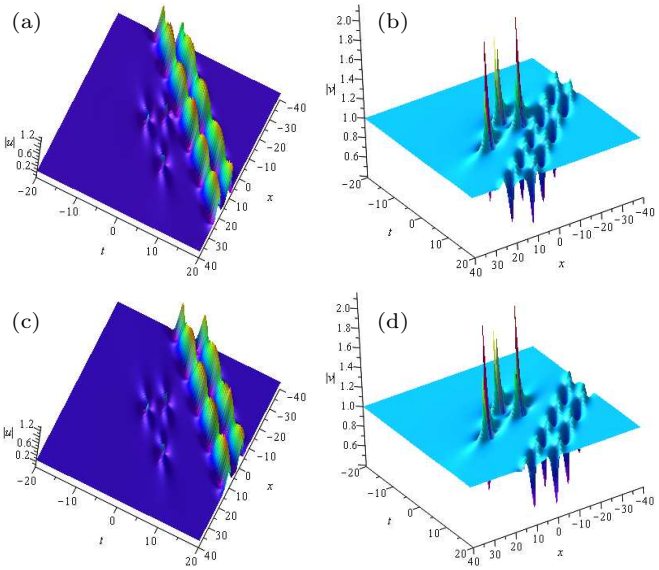


Fig. 7. The evolution of the second-order interactional solutions between two breathers and the triangular pattern rogue wave with parameters $\mu = 2$, $\gamma = 1$, $\zeta = \frac{1}{4}\sqrt{47} + \frac{1}{4}\mathrm{i}\sqrt{17}$, $k_1 = 1$, and $k_2 = 1/4$. $\alpha = 10^{-7}$ in (a) and (b), $\alpha = 10^{-9}$ in (c) and (d).

4. Conclusion

In this paper, we investigate the coupled modified NLS system by the Darboux transformation technique. The multi-breather solutions are generated by the traditional Darboux transformation and the dynamics are discussed in detail. In addition, the concrete expression of one breather is given and the case of a general breather is discussed. Furthermore, we give the expression of the N-fold generalized Darboux transformation and the condition whereby the characteristic equation has a double-root to derive the interactional solution. We exhibit the first-order interactional solution and omit the second-order one due to the length expression here. Under certain parameters, we graphically analyze the dynamics of one general breather and first-order rogue wave, and two breathers and second-order rogue wave based on the interactional solutions.

For the coupled modified NLS equation, due to the simultaneous existence of nonlinear and self-steepening terms, different profiles in two components for the breathers and rogue wave are presented, as shown in Figs. 3–7. We hope the result presented here can be useful to demonstrate the dynamic of a localized wave in the coupled modified NLS equation.

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References

[1] Pethick C J and Smith H 2008 *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge)
[2] Hasegawa A and Matsumoto M 2003 *Optical Solitons in Fibers* (Springer, Berlin)
[3] Lonngren K E 1983 *Plasma Phys.* **25** 943
[4] Li L, Li Z D, Malomed B A, Mihalache D and Liu W M 2005 *Phys. Rev. A* **72** 033611
[5] Feng Y Y and Bilige S D 2021 *Nonlinear Dyn.* **106** 879
[6] Li L X 2022 *Nonlinear Dyn.* **107** 3779
[7] Wang L, Zhang J H, Wang Z Q, Liu C, Li M, Qi F H and Guo R 2016 *Phys. Rev. E* **93** 012214
[8] Wang L L and Liu W J 2020 *Chin. Phys. B* **29** 070502
[9] Liu Y K and Li B 2017 *Chin. Phys. Lett.* **34** 010202
[10] Liu S Z, Wang J and Zhang D J 2022 *Stud. Appl. Math.* **148** 651
[11] Akhmediev N, Ankiewicz A and Soto-Crespo J M 2009 *Phys. Rev. E* **80** 026601
[12] Xue B, Shen J and Geng X G 2020 *Phys. Scr.* **95** 055216
[13] Wu Q L and Zhang H Q 2022 *Wave Motion* **110** 102890
[14] Xu S W, He J S and Wang L H 2011 *J. Phys. A: Math. Theor.* **44** 305203
[15] Du Z, Tian B, Qu Q X and Zhao X H 2020 *Chin. Phys. B* **29** 030202
[16] Li M, Wang B T, Xu T and Shui J J 2020 *Acta Phys. Sin.* **69** 010502 (in Chinese)
[17] Chen S H and Song L Y 2013 *Phys. Rev. E* **87** 032910
[18] Liu C and Akhmediev N 2019 *Phys. Rev. E* **100** 062201
[19] Akhmediev N and Korneev V I 1986 *Theor. Math. Phys.* **69** 1089
[20] Liu C, Wu Y H, Chen S C, Yao X K and Akhmediev N 2021 *Phys. Rev. Lett.* **127** 094102
[21] Agrawal G P 2007 *Nonlinear Fiber Optics* (San Diego: Academic)
[22] Zhang H Q, Tian B, Lü X, Li H and Meng X H 2009 *Phys. Lett. A* **373** 4315
[23] Zhang H Q 2012 *Z. Naturforsch. A* **67** 711
[24] Li M, Tian B, Liu W J, Jiang Y and Sun K 2010 *Eur. Phys. J. D* **59** 279
[25] Hang C, Wu Q L and Zhang H Q 2021 *Mod. Phys. Lett. B* **35** 2150373
[26] Yan X W 2020 *Appl. Math. Lett.* **107** 106414
[27] Morris H C and Dodd R K 1979 *Phys. Scr.* **20** 505
[28] Ling L L and Liu Q P 2010 *J. Phys. A: Math. Theor.* **43** 434023
[29] Xu T and He G L 2020 *Nonlinear Dyn.* **100** 2823
[30] Baronio F, Degasperis A, Conforti M and Wabnitz S 2012 *Phys. Rev. Lett.* **109** 044102
[31] Guo B L and Ling L L 2011 *Chin. Phys. Lett.* **28** 110202
[32] Wang X, Yang B, Chen Y and Yang Y Q 2014 *Chin. Phys. Lett.* **31** 090201
[33] Zhao L C and Liu J 2012 *J. Opt. Soc. Am. B* **29** 3119
[34] Priya N V, Senthilvelan M and Lakshmanan M 2013 *Phys. Rev. E* **88** 022918
[35] Chen S C, Liu C, Yao X K, Zhao L C and Akhmediev N 2021 *Phys. Rev. E* **104** 024215
[36] Liu C, Chen S C, Yao X K and Akhmediev N 2022 *Physica D* **433** 133192
[37] Che W J, Chen S C, Liu C, Zhao L C and Akhmediev N 2022 *Phys. Rev. A* **105** 043526
[38] Zhang G Q, Yan Z Y, Wen X Y and Chen Y 2017 *Phys. Rev. E* **95** 042201