Quantum entangled fractional Fourier transform based on the IWOP technique

Ke Zhang(张科)[1], Lan-Lan Li(李兰兰)[1], Pan-Pan Yu(余盼盼)[1], Ying Zhou(周莹)[1], Da-Wei Guo(郭大伟)[1], and Hong-Yi Fan(范洪义)[2,†]

[1] *School of Electronic Engineering, Huainan Normal University, Huainan* 232038, *China*
[2] *Department of Material Science and Engineering, University of Science and Technology of China, Hefei* 230026, *China*

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In our previous papers, the classical fractional Fourier transform theory was incorporated into the quantum theoretical system using the theoretical method of quantum optics, and the calculation produced quantum mechanical operators corresponding to the generation of fractional Fourier transform. The core function of the coordinate–momentum exchange operators in the addition law of fractional Fourier transform was analyzed too. In this paper, the bivariate operator Hermite polynomial theory and the technique of integration within an ordered product of operators (IWOP) are used to establish the entanglement fractional Fourier transform theory to the extent of quantum. A new function generating formula and an operator for generating quantum entangled fractional Fourier transform are obtained using the fractional Fourier transform relationship in a pair of conjugated entangled state representations.

Keywords: fractional Fourier transform, coordinate–momentum exchange operators, bivariate operator Hermite polynomial theory, the technique of integration within an ordered product of operators, quantum entangled fractional Fourier transform

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1. Introduction

Classical optical theory, mainly through the study of the properties of light waves, and classical electromagnetic field theory are used as a foundation to study the transmission laws of light waves in different media, imaging mechanisms and in the application of optical technology, and to explore the different kinds of integral transform of light waves in transmission dimensions. Therefore, it is necessary to develop the theory of spatial distribution function and the corresponding integral transform to describe the characteristics of light waves.[1] Developing the theory of optical transformation, extending the range of imaging and improving the imaging accuracy have become very important research topics in optics. However, most of the operators in quantum optical theory are non-commutative. Therefore, to develop integral transformation theory, we have used an integral within ordered operators (IWOP) technique[2–6] and coherent state representation[7–10] to construct several unitary operators and new representations; thus, the corresponding classical optical transformations have included single-mode fractional Fourier transform,[11,12] fractional Hankel transform,[13,14] and wavelet transform.[15,16] Based on this, using quantum mechanical representation and integral transformation theory, some complex problems in classical optics can be re-described, and the new integral transformation theory can be developed. Therefore, the author believes that it is an effective route from the unitary transformation of quantum optics to the classical transformation. Fractional Fourier transform evolved from the theoretical system of traditional optical Fourier transform, which has a wide range of applications in optical fiber sensing, spectrum analysis, and optical imaging, etc. Wiener's research group was the first to study fractional Fourier transform.[17] In 1980, Namias[18] resummarized fractional Fourier transform in the field of quantum mechanics and analyzed its related characteristics. McBride[19] considered the previous research work on fractional Fourier transform, revised fractional Fourier transform in the quantum field in 1987, and successfully applied it to the field of optical imaging. In 1993, Ozaktas and Mendlovic[20,21] calculated the linear transform kernel, which is contained in fractional Fourier transform, and discussed the relationship between the spatial resolution and spatial bandwidth product of propagation in gradient refractive index media. Furthermore, fractional Fourier transform and numerical examples were also given. Almeida interpreted fractional Fourier transform as the rotation of the time–frequency plane, obtained the relationship between fractional Fourier transform and the time–frequency representation, and analyzed its application in signals in 1994.[22] Zayed[23] and Criolaro[24] studied fractional Fourier transform of various types of signals. Hu's group proposed 3D entangled fractional squeezing transformation and its quantum mechanical correspondence using the technique of integration within an ordered prod-

†Corresponding author. E-mail: fhym@ustc.edu.cn
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uct of operators.[25] The new formulation of complex fractional Fourier transformation was derived from the Einstein–Podolsky–Rosen entangled state representations of continuous variables.[26]

As we all know, fractional Fourier transform has additivity and satisfies the relation

$$F_\alpha \circ F_\beta = F_{\alpha+\beta}. \tag{1}$$

The final result of two successive fractional Fourier transforms (α- and β-order) can be equivalent to one $(\alpha+\beta)$-order transform. We note that the expression of Fourier transform in quantum mechanics is the transformation from coordinate representation $|q\rangle$ to momentum representation $|p\rangle$,

$$|p\rangle = \int_{-\infty}^{\infty} \mathrm{d}q\,|q\rangle\,\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}q\,|q\rangle\,\mathrm{e}^{\mathrm{i}pq}. \tag{2}$$

Therefore, the theory for generating fractional Fourier transform should also be discussed in the mutually conjugate coordinate representation $|q\rangle$ and momentum representation $|p\rangle$. Our previous research work mainly used Hermitian polynomial function theory based on quantum mechanical operators to obtain the quantum mechanical operators for generating fractional Fourier transform and analyzed the important role of coordinate–momentum exchange quantum operators in the addition rules of fractional Fourier transform. The α-order fractional Fourier transform is defined as

$$F[f](p) = \int_{-\infty}^{\infty} V_\alpha(p,q)\,f(q)\,\mathrm{d}q. \tag{3}$$

Using the matrix element of quantum mechanics, the integral kernel in Eq. (3) can be expressed as

$$\begin{aligned} V_\alpha(p,x) &= \langle p|\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^\dagger a}|q\rangle \\ &= \frac{\mathrm{e}^{\mathrm{i}\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)}}{\sqrt{2\pi\sin\alpha}}\exp\left[\frac{\mathrm{i}\left(p^2+q^2\right)}{2\tan\alpha}-\frac{\mathrm{i}pq}{\sin\alpha}\right], \end{aligned} \tag{4}$$

where $\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)a^\dagger a}$ is the operator for generating fractional Fourier transform, and the production operator $a^\dagger$ and the annihilation operator a satisfy the commutation relation $[a,a^\dagger]=1$.

In view of the increasing attention of quantum entanglement,[27–29] a new problem naturally arises: is it possible to establish the theory of entangled fractional Fourier transform in the quantum sense, and then to explore the fractional Fourier transform between a pair of conjugated entangled state representations? In this paper, we will fully utilize the two-mode operator Hermitian polynomial theory and IWOP technology to analyze the fractional Fourier transform operator, which can be expected to be applied in quantum entanglement theory.

2. A pair of conjugated entangled state representations

The entangled state representation $|\eta\rangle$ in two-mode Fock space can be expressed as

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^* a_2^\dagger+a_1^\dagger a_2^\dagger\right]|00\rangle, \tag{5}$$

where $|00\rangle$ is the two-mode vacuum state, and $\eta=\eta_1+\mathrm{i}\eta_2$ is the complex number and satisfies the eigenstate equation

$$\left(a_1-a_2^\dagger\right)|\eta\rangle = \eta\,|\eta\rangle, \quad \left(a_1^\dagger-a_2\right)|\eta\rangle = \eta^*\,|\eta\rangle. \tag{6}$$

Here, $|\eta\rangle$ is the common eigenstate of the coordinate operator and the momentum operator. Note that

$$\begin{aligned} &X_i = \frac{a_i+a_i^\dagger}{\sqrt{2}}, \quad P_i = \frac{a_i-a_i^\dagger}{\sqrt{2\mathrm{i}}}, \quad i=1,2; \\ &[X_1-X_2, P_1+P_2]=0 \end{aligned} \tag{7}$$

so

$$(X_1-X_2)\,|\eta\rangle = \sqrt{2}\eta_1\,|\eta\rangle, \quad (P_1+P_2)\,|\eta\rangle = \sqrt{2}\eta_2\,|\eta\rangle. \tag{8}$$

Using the normal product form of the two-mode vacuum state ($:\ :$ denotes the normal ordering of the operator)

$$:\exp\left[-a_1^\dagger a_1-a_2^\dagger a_2\right]: = |00\rangle\,\langle 00|. \tag{9}$$

The completeness of entangled states can be obtained using IWOP technology

$$\begin{aligned} &\int\frac{\mathrm{d}^2\eta}{\pi}\,|\eta\rangle\,\langle\eta| \\ &= \int\frac{\mathrm{d}^2\eta}{\pi}:\exp\left\{-\left[\eta-\left(a_1-a_2^\dagger\right)\right]\left[\eta^*-\left(a_1^\dagger-a_2\right)\right]\right\}: \\ &= \int\frac{\mathrm{d}^2\eta}{\pi}:\exp\left[-\left|\eta-\left(a_1-a_2^\dagger\right)\right|^2\right] := 1. \end{aligned} \tag{10}$$

For this aim, we introduce the conjugate state to $|\eta\rangle$

$$|\xi\rangle = \exp\left[-\frac{1}{2}|\xi|^2+\xi a_1^\dagger+\xi^* a_2^\dagger-a_1^\dagger a_2^\dagger\right]|00\rangle, \tag{11}$$

which obeys eigenvector equations

$$\left(a_1+a_2^\dagger\right)|\xi\rangle = \xi\,|\xi\rangle, \quad \left(a_1^\dagger+a_2\right)|\eta\rangle = \xi^*\,|\xi\rangle, \tag{12}$$

so, we can obtain

$$\begin{aligned} &(X_1+X_2)\,|\xi\rangle = \sqrt{2}\xi_1\,|\xi\rangle, \quad (P_1-P_2)\,|\xi\rangle = \sqrt{2}\xi_2\,|\xi\rangle, \\ &\xi = \xi_1+\mathrm{i}\xi_2. \end{aligned} \tag{13}$$

Note that $[X_1+X_2, P_1-P_2]=0$, so $|\xi\rangle$ is also complete

$$\int\frac{\mathrm{d}^2\xi}{\pi}\,|\xi\rangle\,\langle\xi| = 1. \tag{14}$$

Here, $|\xi\rangle$ and $|\eta\rangle$ are mutually conjugate quantum state vectors. Therefore, the inner product between $|\eta\rangle$ and $|\xi\rangle$ can be calculated by

$$\langle\eta|\xi\rangle = \frac{1}{2}\exp\left(\frac{\xi\eta^* - \xi^*\eta}{2}\right). \tag{15}$$

Since $\xi\eta^* - \xi^*\eta$ is a pure imaginary number, equation (15) can be regarded as the kernel of complex fractional Fourier transform.

3. Bivariate Hermite polynomial theory of operator

In this section, the theory of operator bivariate Hermite polynomials will be established.[30–32] On this basis, the quantum entangled fractional Fourier transform is extended. Using the definition of bivariate Hermite polynomials

$$\begin{aligned} &H_{m,n}(\lambda,\lambda^*) \\ &= \sum_{l=0}^{\min(m,n)} \frac{m!n!}{l!(m-l)!(n-l)!}(-1)^l \lambda^{m-l}\lambda^{*n-l}, \end{aligned} \tag{16}$$

and the corresponding generating function is

$$\sum_{m,n=0}^{\infty} \frac{t^m t'^n}{m!n!} H_{m,n}(\lambda,\lambda^*) = \exp\left(-tt' + t\lambda + t'\lambda^*\right). \tag{17}$$

The bivariate Hermite polynomials $H_{m,n}$ have been widely used in quantum mechanics, mathematics and physics. For example, the operator identity in quantum theory can be expressed as

$$:a^{\dagger m}a^n: = a^{\dagger m}a^n = \vdots H_{n,m}\left(a,a^\dagger\right)\vdots, \tag{18}$$

where $\vdots\,\vdots$ denotes the anti-normal ordering, and the proof of Eq. (18) is as follows. Using the formula for transforming the arbitrary operator into anti-normal ordering:

$$\rho = \int \frac{\mathrm{d}^2 z}{\pi} \vdots \langle -z|\rho|z\rangle \exp\left(|z|^2 + z^* a - z a^\dagger + a^\dagger a\right)\vdots, \tag{19}$$

where $|z\rangle$ is the coherent state.

Furthermore, using the integral formula

$$\begin{aligned} (-\mathrm{i})^{m+n}\,\mathrm{e}^{\sigma\tau} &= \int \frac{\mathrm{d}^2 z}{\pi} z^n z^{*m} \exp\left(|z|^2 + \mathrm{i}z^*\sigma + \mathrm{i}z\tau\right) \\ &= H_{m,n}(\sigma,\tau), \end{aligned} \tag{20}$$

we obtain

$$\begin{aligned} a^{\dagger m}a^n &= \int \frac{\mathrm{d}^2 z}{\pi} \vdots \langle -z|a^{\dagger m}a^n|z\rangle \exp\left(|z|^2 + z^* a - z a^\dagger + a^\dagger a\right)\vdots \\ &= \int \frac{\mathrm{d}^2 z}{\pi} \vdots (-z^*)^m z^n \exp\left(-|z|^2 + z^* a - z a^\dagger + a^\dagger a\right)\vdots \\ &= \vdots \sum_{m,n=0}^{\min(m,n)} \frac{n!m!}{l!(n-l)!(m-l)!} a^{n-l} a^{\dagger m-l}(-1)^l \vdots \\ &= \vdots H_{n,m}\left(a,a^\dagger\right)\vdots. \end{aligned} \tag{21}$$

On the other hand, using the completeness of coherent states

$$\begin{aligned} \int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\langle z| &= \int \frac{\mathrm{d}^2 z}{\pi} :\exp\left(-|z|^2 + z a^\dagger + z^* a - a^\dagger a\right): \\ &= 1 \end{aligned} \tag{22}$$

and Eq. (21), we have

$$\begin{aligned} a^m a^{\dagger l} &= \int \frac{\mathrm{d}^2 z}{\pi} a^m |z\rangle\langle z| a^{\dagger l} \\ &= \int \frac{\mathrm{d}^2 z}{\pi} z^m z^{*l} :\exp\left(-|z|^2 + z a^\dagger + z^* a - a^\dagger a\right): \\ &= (-\mathrm{i})^{m+l} : H_{l,m}\left(\mathrm{i}a^\dagger, \mathrm{i}a\right): . \end{aligned} \tag{23}$$

Using $\vdots H_{m,n}\left(a^\dagger,a\right)\vdots = a^{\dagger m}a^n$ and Eq. (17), the generating function of the product between the operator $\vdots H_{m,n}\left(a^\dagger,a\right)\vdots$ and ordinary functions $H_{m,n}(x,y)$ can be obtained as

$$\begin{aligned} &\sum_{m,n=0}^{\infty} \frac{t^n s^m}{m!n!} H_{m,n}(x,y) \vdots H_{m,n}\left(a^\dagger,a\right)\vdots \\ &= \sum_{m,n=0}^{\infty} \frac{:s^m a^{\dagger m} t^n a^n:}{m!n!} H_{m,n}(x,y) \\ &= :\exp\left(-sta^\dagger a + sxa^\dagger + tya\right): . \end{aligned} \tag{24}$$

Changing the normal ordering of Eq. (24) into anti-normal ordering by Eq. (19)

$$\begin{aligned} :\exp\left(sta^\dagger a + sxa^\dagger + tya\right): &= \int \frac{\mathrm{d}^2 z}{\pi} \vdots \langle -z| :\exp\left(-sta^\dagger a + sxa^\dagger + tya\right): |z\rangle \exp\left(|z|^2 + z^* a - z a^\dagger + a^\dagger a\right)\vdots \\ &= \int \frac{\mathrm{d}^2 z}{\pi} \vdots \exp\left(st|z|^2 - sxz^* + tyz\right) \exp\left(|z|^2 + z^* a - z a^\dagger + a^\dagger a\right)\vdots \\ &= \frac{1}{1-st} \vdots \exp\left(\frac{sxa^\dagger + tya - stxy - sta^\dagger a}{1-st}\right)\vdots. \end{aligned} \tag{25}$$

Simultaneously, equations (24) and (25) can obtain

$$\sum_{m,n=0}^{\infty} \frac{t^n s^m}{m!n!} H_{m,n}(x,y) \vdots H_{m,n}\left(a^\dagger,a\right)\vdots = \frac{1}{1-st} \vdots \exp\left(\frac{sxa^\dagger + tya - stxy - sta^\dagger a}{1-st}\right)\vdots. \tag{26}$$

Both sides of Eq. (26) are anti-normal ordering. Therefore, another function generating equation corresponding to the bivariate Hermite polynomial is obtained as

$$\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m,n}(\xi,\eta)H_{m,n}(\sigma,\kappa) = \frac{1}{1-st}\exp\left(\frac{s\xi\sigma+t\eta\kappa-st\sigma\kappa-st\xi\eta}{1-st}\right). \quad (27)$$

This method is called the operator bivariate Hermite polynomial method.

4. Quantum entanglement complex fractional Fourier transform

On the basis of Section 3, we can define the entangled complex fractional Fourier transform, which satisfies the following relation:

$$K_\alpha[f] = \int\frac{\mathrm{d}^2\xi}{\pi}K_\alpha(\eta,\xi)f(\xi). \quad (28)$$

The integral kernel $K_\alpha(\eta,\xi)$ in Eq. (28) is the transformation matrix element from the entangled state $|\xi\rangle$ to $\langle\eta|$

$$K_\alpha(\eta,\xi) = \langle\eta|K_\alpha|\xi\rangle. \quad (29)$$

The purpose of this section is to prove that

$$\langle\eta|K_\alpha|\xi\rangle = \frac{\mathrm{e}^{\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}}{2\sin\alpha}\exp\left[\frac{\mathrm{i}\left(|\eta|^2+|\xi|^2\right)}{2\tan\alpha}-\mathrm{i}\frac{\xi\eta^*+\xi^*\eta}{2\sin\alpha}\right]. \quad (30)$$

where

$$K_\alpha(\eta,\xi) = \exp\left[-\mathrm{i}\alpha\left(a_1^\dagger a_1+a_2^\dagger a_2\right)\right]\exp\left[\mathrm{i}\pi a_2^\dagger a_2\right]. \quad (31)$$

Using the definition of the two-mode Fock state $|m,n\rangle = \frac{a_1^{\dagger m}a_2^{\dagger n}}{\sqrt{m!n!}}|00\rangle$, the entangled state $\langle\eta|$ and its corresponding conjugate state $\langle\xi|$ can be expanded in Fock space as

$$\langle\eta| = \langle 00|\sum_{m,n=0}^{\infty}\mathrm{i}^{m+n}\frac{a_1^m a_2^n}{m!n!}H_{m,n}(-\mathrm{i}\eta^*,\mathrm{i}\eta)\,\mathrm{e}^{-|\eta|^2/2}, \quad (32)$$

$$\langle\xi| = \langle 00|\sum_{m,n=0}^{\infty}\frac{a_1^m a_2^n}{m!n!}H_{m,n}(\xi^*,\xi)\,\mathrm{e}^{-|\eta|^2/2}, \quad (33)$$

so

$$\langle\eta|m,n\rangle = \frac{\mathrm{i}^{m+n}H_{m,n}(-\mathrm{i}\eta^*,\mathrm{i}\eta)\,\mathrm{e}^{-|\eta|^2/2}}{\sqrt{m!n!}},$$
$$\langle m,n|\xi\rangle = \frac{H_{m,n}(\xi,\xi^*)\,\mathrm{e}^{-|\xi|^2/2}}{\sqrt{m!n!}}. \quad (34)$$

Using the completeness of particle states

$$\sum_{m,n=0}^{\infty}|m,n\rangle\langle m,n| = 1, \quad (35)$$

we can derive

$$\begin{aligned}&\langle\eta|K_\alpha(\eta,\xi)|\xi\rangle\\ &= \sum_{m,n}^{\infty}\sum_{m',n'}^{\infty}\langle\eta|m,n\rangle\langle m,n|\exp[-\mathrm{i}\alpha(a_1^\dagger a_1\\ &\quad +a_2^\dagger a_2)]\exp[\mathrm{i}\pi a_2^\dagger a_2]|m',n'\rangle\langle m',n'|\xi\rangle\\ &= \sum_{m,n}\sum_{m',n'}\langle\eta|m,n\rangle\langle m',n'|\xi\rangle\,\mathrm{e}^{-\mathrm{i}\alpha m}\,\mathrm{e}^{(\mathrm{i}\pi-\mathrm{i}\alpha)n}\delta_{m,n}\delta_{m',n'}\\ &= \sum_{m,n}\mathrm{e}^{\mathrm{i}\pi n-\mathrm{i}\alpha(m+n)}\frac{H_{m,n}(-\mathrm{i}\eta^*,\mathrm{i}\eta)\,\mathrm{e}^{-|\eta|^2/2}}{m!n!}\\ &\quad\times H_{m,n}(\xi,\xi^*)\mathrm{e}^{-|\xi|^2/2}. \end{aligned} \quad (36)$$

By using the generating function Eq. (27) of the product of bivariate Hermite polynomials, the right side of Eq. (36) can be transformed as

$$\begin{aligned}\langle\eta|K_\alpha(\eta,\xi)|\xi\rangle &= \mathrm{e}^{-\left(|\eta|^2+|\xi|^2\right)/2}\sum_{m,n}\mathrm{i}^{m+n}\frac{\mathrm{e}^{-\mathrm{i}\alpha m}\mathrm{e}^{(\mathrm{i}\pi-\mathrm{i}\alpha)n}}{m!n!}H_{m,n}(-\mathrm{i}\eta^*,\mathrm{i}\eta)H_{m,n}(\xi,\xi^*)\\ &= \mathrm{e}^{-\left(|\eta|^2+|\xi|^2\right)/2}\frac{1}{1-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)}}\exp\left[\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha}(-\mathrm{i}\eta^*)\xi+\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)}\mathrm{i}\eta\xi^*-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)}\left(|\xi|^2+|\eta|^2\right)}{1+\mathrm{e}^{\mathrm{i}(\pi-2\alpha)}}\right]\\ &= \mathrm{e}^{-\left(|\eta|^2+|\xi|^2\right)/2}\frac{1}{1-\mathrm{e}^{-2\mathrm{i}\alpha}}\exp\left[\frac{\mathrm{e}^{-\mathrm{i}\alpha}(\eta^*\xi+\eta\xi^*)-\mathrm{e}^{-2\mathrm{i}\alpha}\left(|\xi|^2+|\eta|^2\right)}{1-\mathrm{e}^{-2\mathrm{i}\alpha}}\right]\\ &= \mathrm{e}^{-\left(|\eta|^2+|\xi|^2\right)/2}\frac{\mathrm{e}^{\mathrm{i}\alpha}}{2\mathrm{i}\sin\alpha}\exp\left[\frac{(\eta^*\xi+\eta\xi^*)-\mathrm{e}^{-\mathrm{i}\alpha}\left(|\xi|^2+|\eta|^2\right)}{2\mathrm{i}\sin\alpha}\right]\\ &= \frac{\mathrm{e}^{\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}}{2\sin\alpha}\exp\left[\frac{\mathrm{i}\left(|\eta|^2+|\xi|^2\right)}{2\tan\alpha}-\mathrm{i}\frac{\eta^*\xi+\eta\xi^*}{2\sin\alpha}\right] = K_\alpha(\eta,\xi). \end{aligned} \quad (37)$$

It can be seen that the integral transformation kernel $\langle\eta|K_\alpha|\xi\rangle$ is generated by the operator $\exp\left[-\mathrm{i}\alpha\left(a_1^\dagger a_1+a_2^\dagger a_2\right)\right]\exp\left[\mathrm{i}\pi a_2^\dagger a_2\right]$

when changing the entangled state $|\xi\rangle$ to $\langle\eta|$.

5. The addition rule of entangled fractional Fourier transforms

The criterion for defining an entangled fractional Fourier transform is that it must satisfy the rule of addition: $F_\alpha \circ F_\beta[f] = K_{\alpha+\beta}[f]$. That is, two consecutive transformations are equivalent to one transformation:

$$\begin{aligned} F_\alpha \circ F_\beta[f] &= \int \frac{\mathrm{d}^2\xi}{\pi} \int \frac{\mathrm{d}^2\xi'}{\pi} K_\alpha(\eta,\xi') \left[K_\beta(\eta',\xi)|_{\eta'=\xi'}\right] f(\xi) = \int \frac{\mathrm{d}^2\xi}{\pi} K_{\alpha+\beta}(\eta,\xi) f(\xi) \\ &= \frac{\mathrm{e}^{\mathrm{i}\left(\alpha+\beta-\frac{\pi}{2}\right)}}{2\sin(\alpha+\beta)} \int \frac{\mathrm{d}^2\xi}{\pi} \exp\left[\frac{\mathrm{i}\left(|\eta|^2+|\xi|^2\right)}{2\tan(\alpha+\beta)} - \mathrm{i}\frac{\eta^*\xi+\eta\xi^*}{2\sin(\alpha+\beta)}\right] f(\xi) = K_{\alpha+\beta}[f]. \end{aligned} \tag{38}$$

In fact, with the help of Dirac symbols, equation (38) can be rewritten as

$$\int \frac{\mathrm{d}^2\xi}{\pi} \int \frac{\mathrm{d}^2\xi'}{\pi} \langle\eta|K_\alpha|\xi'\rangle_{\eta'=\xi'} \langle\eta'|K_\beta|\xi\rangle\langle\xi|f\rangle = \int \frac{\mathrm{d}^2\xi}{\pi} \langle\eta|K_\alpha W K_\beta|\xi\rangle\langle\xi|f\rangle = \int \frac{\mathrm{d}^2\xi}{\pi} \langle\eta|K_{\alpha+\beta}|\xi\rangle\langle\xi|f\rangle, \tag{39}$$

$$\begin{aligned} W &= \int \frac{\mathrm{d}^2\xi'}{\pi} |\xi'\rangle_{\eta'=\xi'} \langle\eta'| \\ &= \int \frac{\mathrm{d}^2\xi'}{\pi} \exp\left[-\frac{1}{2}|\xi'|^2 + \xi' a_1^\dagger + \xi'^* a_2^\dagger - a_1^\dagger a_2^\dagger\right] |00\rangle\langle 00| \times \exp\left[-\frac{1}{2}|\eta'|^2 + \eta'^* a_1 - \eta' a_2 + a_1 a_2\right]_{\eta'=\xi'} \\ &= \int \frac{\mathrm{d}^2\xi'}{\pi} : \exp\left[-|\xi'|^2 + \xi'\left(a_1^\dagger - a_2\right) + \xi'^*\left(a_2^\dagger + a_1\right) - a_1^\dagger a_2^\dagger + a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\right] : \\ &=: \exp\left[(a_1^\dagger - a_2)(a_2^\dagger + a_1) - a_1^\dagger a_2^\dagger + a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\right] : =: \exp[-2a_2^\dagger a_2] : = \exp[-\mathrm{i}\pi a_2^\dagger a_2], \end{aligned} \tag{40}$$

where W is called the core operator of the entangled fractional Fourier transform, and the function of W is to transform $|\eta\rangle$ into $|\xi\rangle_{\xi=\eta}$

$$\begin{aligned} W|\eta\rangle &= \exp\left[-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger \mathrm{e}^{-\mathrm{i}\pi} + \mathrm{e}^{-\mathrm{i}\pi} a_1^\dagger a_2^\dagger\right] |00\rangle \\ &= |\xi\rangle_{\xi=\eta}. \end{aligned} \tag{41}$$

It can be obtained from Eqs. (39) and (40) that

$$\begin{aligned} &K_{\alpha+\beta} = K_\alpha W K_\beta \\ &= \exp\left[-\mathrm{i}(\alpha+\beta)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] \exp\left[\mathrm{i}\pi a_2^\dagger a_2\right]. \end{aligned} \tag{42}$$

6. Application

Enlightened by the core operator Eq. (40), we create a new combinatorial operator

$$S_\alpha = \exp\left[-\mathrm{i}\alpha\left(a_1^\dagger a_2^\dagger + a_1 a_2\right)\right] \mathrm{e}^{-\mathrm{i}\pi a_2^\dagger a_2}, \tag{43}$$

where $\exp[-\mathrm{i}\alpha(a_1^\dagger a_2^\dagger + a_1 a_2)]$ is a two-mode squeezing operator. Using the IWOP technique and following the same procedures as in Section 4, we prove that

$$\begin{aligned} &\langle\eta|S_\alpha|\xi\rangle \\ &= \frac{1}{2\mathrm{i}\sinh\alpha} \exp\left[\frac{\mathrm{i}\left(|\eta|^2+|\xi|^2\right)}{2\tanh\alpha} - \mathrm{i}\frac{\xi\eta^*+\xi^*\eta}{2\sinh\alpha}\right]. \end{aligned} \tag{44}$$

Comparing Eq. (44) with Eq. (30), one can see that

$$\sin\alpha \to \sinh\alpha, \quad \tan\alpha \to \tanh\alpha, \tag{45}$$

which indicates that the core operator is indispensable to fractional transformation in the two-mode case.

7. Conclusion

To establish the theory of fractional Fourier transform in the sense of quantum entanglement, a pair of mutually conjugate entangled state representations are introduced, and their mutual transformation is complex Fourier transform. On this basis, we developed the new content of the quantum entangled fractional Fourier transform. A new function generating formula,

$$\begin{aligned} &\sum_{m,n=0}^{\infty} \frac{t^n s^m}{m!n!} H_{m,n}(\xi,\eta) H_{m,n}(\sigma,\kappa) \\ &= \frac{1}{1-st} \exp\left(\frac{s\xi\eta + t\eta\kappa - st\sigma\kappa - st\xi\eta}{1-st}\right), \end{aligned}$$

is obtained using the bivariate Hermite polynomial theory of the quantum mechanics operator and IWOP technology. The core operators, $\exp[-\mathrm{i}\alpha(a_1^\dagger a_1 + a_2^\dagger a_2)]\exp[\mathrm{i}\pi a_2^\dagger a_2]$ for generating entangled fractional Fourier transforms are obtained, and the addition law of its satisfaction is analyzed. The quantum mechanical operator we used in this paper to generate the entangled fractional Fourier transform is different from the previous two papers; we emphasize the role of the core operator,

which the two references had not noted. Moreover, to prove the result, we used two-variable Hermite polynomials, which is also a new method. It is worth noting that fractional Fourier transform is different from the direct product of the fractional Fourier transforms of two single-mode real functions in the sense of entanglement; the fractional Fourier transform operator of the latter is $e^{i\left(\frac{\pi}{2}-\alpha\right)\left(a^{\dagger}a+b^{\dagger}b\right)}$. The method in this paper lays a theoretical foundation for the discovery of new optical transformations.

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