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Riemann–Hilbert approach and N double-pole solutions for a nonlinear Schrödinger-type equation

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We investigate the inverse scattering transform for the Schrödinger-type equation under zero boundary conditions with the Riemann–Hilbert (RH) approach. In the direct scattering process, the properties are given, such as Jost solutions, asymptotic behaviors, analyticity, the symmetries of the Jost solutions and the corresponding spectral matrix. In the inverse scattering process, the matrix RH problem is constructed for this integrable equation base on analyzing the spectral problem. Then, the reconstruction formula of potential and trace formula are also derived correspondingly. Thus, N double-pole solutions of the nonlinear Schrödinger-type equation are obtained by solving the RH problems corresponding to the reflectionless cases. Furthermore, we present a single double-pole solution by taking some parameters, and it is analyzed in detail.

Keywords: Schrödinger-type equation, Riemann–Hilbert problem, zero boundary conditions, N double-pole solitons

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1. Introduction

In recent decades, nonlinear partial diferential equations are often used to describe natural phenomena and play a significant role in mathematics and physics, which have attracted much attention in soliton theory and integrable systems.[1–8] The inverse scattering transformation is an important method to study nonlinear partial differential equations. Then, the Riemann–Hilbert (RH) problem is widely adopted to solve nonlinear integrable models, and the solution is an improved version of inverse scattering transformation.[9,10] However, many works focused on studies of nonlinear systems in the case that all discrete spectra are simple. At present, there are some works about the double pole solutions to nonlinear integrable equations.[11–18] Moreover, the higher-order pole solutions have also been studied in some papers.[19–23] In this paper, we study the inverse scattering transform for the Schrödinger-type equation under zero boundary conditions (ZBCs) with the RH approach.

As is well known, the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t + u_{xx} + 2u|u|^2 = 0 \tag{1}$$

comes from a wide range of practical backgrounds, such as deep water waves, plasma physics, quantum field theory, and other fields.[24–27] Based the different concerns of nonlinear phenomena, there are three known modifications of NLS equations including different nonlinear terms by comparing with the NLS equation (1), such as the KN equation, the CLL equation and the GI equation, which are also called the DNSI, DNSII, and DNSIII,[28–35] *etc.*

A new Schrödinger-type equation was proposed in Ref. [36],

$$\mathrm{i}u_t + u_{xx} + 2u|u|^2 - \mathrm{i}u^2u_x^* - 2\mathrm{i}|u|^2u_x = 0, \tag{2}$$

where the asterisk $*$ denotes the complex conjugation. It is an extension of the NLS equation by adding two new terms including two derivates. It is non-trivial to preserve the integrability of the NLS equation by adding proper terms to reach Eq. (2). Then, the Lax pair, a 2×2 matrix spectral problem, is given via the prolongation structure theory,

$$\begin{aligned}\varphi_x &= U\varphi = [\mathrm{i}(z^2-1)\sigma_3 + zU_1]\varphi,\\ \varphi_t &= V\varphi = [-2\mathrm{i}(z^2-1)^2\sigma_3 - 2z^3U_1 + \mathrm{i}z^2U_2 + zU_3]\varphi,\end{aligned} \tag{3}$$

where

$$\begin{aligned}U_1 &= \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} |u|^2 & 0 \\ 0 & -|u|^2 \end{pmatrix},\\ U_3 &= \begin{pmatrix} 0 & \mathrm{i}u_x + 2u + u^2u^* \\ \mathrm{i}u_x^* - 2u^* - u^{*2}u & 0 \end{pmatrix},\end{aligned} \tag{4}$$

$\sigma_3 = \mathrm{diag}(1,-1)$, and $z\in\mathbb{C}$ is a spectral parameter, and N-soliton solutions of the equation with ZBCs are expressed explicitly via the RH approach in Ref. [37]. Furthermore, N-soliton solutions of this equation under NZBCs are expressed explicitly via the RH approach in Ref. [38]. However, for the double-pole soliton solutions of this equation under ZBCs, it is rarely studied. Thus, we will consider N double-pole soliton solutions of the new integrable nonlinear Schrödinger-type (NLST) equation with ZBCs via the RH approach.

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An outline of this paper is as follows. In Section 2, starting from the Lax pair of Eq. (2) and the ZBCs, one can obtain Jost solutions. Then, analyticity, symmetries and asymptotic behaviors associated with eigenfunction and scattering matrix are realized. In Section 3, on the basis of the former analysis, the matrix RH problem is constructed for this integrable equation (2), and the reconstruction formula of potential and trace formula are also derived correspondingly. Furthermore, N double-pole solutions of the nonlinear Schrödinger-type equation are obtained by solving the RH problems corresponding to the reflectionless case, and an example is given, which is discussed in the case of $N=1$. Finally, we present some summary and discussions.

2. The direct scattering with ZBCs

In this section, we analyze the direct scattering with ZBCs,

$$u(x,t)\sim 0,\quad \text{as}\quad x\to\pm\infty, \tag{5}$$

and the case of double poles for the Lax pair of Eq. (2).

2.1. Spectral analysis

Under the boundary condition (5), the Lax pair will be changed into

$$\varphi_x(x,t;z)=U_0\varphi,\quad \varphi_t(x,t;z)=V_0\varphi, \tag{6}$$

where $U_0=\mathrm{i}(z^2-1)\sigma_3$, $V_0=-2\mathrm{i}(z^2-1)^2\sigma_3$, one can find the fundamental matrix solution φ_{bg} of Eq. (2) as follows:

$$\varphi_{\mathrm{bg}}=\mathrm{e}^{\mathrm{i}\theta(x,t;z)\sigma_3},\quad \theta(x,t;z)=(z^2-1)x-2(z^2-1)^2t. \tag{7}$$

Then, the Jost solution $\varphi_\pm(x,t;z)$ can be given as

$$\varphi_\pm(x,t;z)\sim \mathrm{e}^{\mathrm{i}\theta(x,t;z)\sigma_3}\quad \text{as}\quad x\to\pm\infty, \tag{8}$$

where $z\in\Sigma=\mathbb{R}\cup\mathrm{i}\mathbb{R}$. It is convenient to introduce a new matrix spectral function $\mu_\pm(x,t;z)$, which can be defined as

$$\mu_\pm(x,t;z)=\varphi_\pm(x,t;z)\mathrm{e}^{-\mathrm{i}\theta(x,t;z)\sigma_3}, \tag{9}$$

and this results in $\mu_\pm(x,t;z)\sim\mathbb{I}$ as $x\to\pm\infty$. The equivalent Lax pair can be given as follows:

$$\begin{aligned}\mu_{\pm,x}&=\mathrm{i}(z^2-1)[\sigma_3,\mu_\pm]+zU_1\mu_\pm,\\ \mu_{\pm,t}&=-2\mathrm{i}(z^2-1)^2[\sigma_3,\mu_\pm]\\ &\quad+(-2z^3U_1+\mathrm{i}z^2U_2+zU_3)\mu_\pm,\end{aligned} \tag{10}$$

where the Lie bracket $[M,N]=MN-NM$. Moreover, one can obtain the full derivative form and the following Volterra integral equations:

$$\mu_-(x,t;z)=\mathbb{I}+\int_{-\infty}^{x}\mathrm{e}^{\mathrm{i}(z^2-1)(x-y)\hat{\sigma}_3}\left(V_1(y,t)\mu_-(y,t;z)\right)\mathrm{d}y,$$

$$\mu_+(x,t;z)=\mathbb{I}+\int_{+\infty}^{x}\mathrm{e}^{\mathrm{i}(z^2-1)(x-y)\hat{\sigma}_3}\left(V_1(y,t)\mu_+(y,t;z)\right)\mathrm{d}y, \tag{11}$$

where $V_1=zU_1$ and $\mathrm{e}^{\alpha\hat{\sigma}_3}N=\mathrm{e}^{\alpha\sigma_3}N\mathrm{e}^{-\alpha\sigma_3}$ with N being a 2×2 matrix. Next, define $D^\pm=\{z\in\mathbb{C}|\pm\mathrm{Re}(z)\mathrm{Im}(z)>0\}$. Then, one has this result.

Proposition 1 Suppose $u(x,t)\in L^1(\mathbb{R})$ and the Jost solutions $\varphi_\pm(x,t;z)$ ($\varphi_\pm(x,t;z)=(\varphi_{\pm1},\varphi_{\pm2})$), we have the following results:

(1) Eq. (3) has only a solution $\varphi_\pm(x,t;z)$, which can satisfy Eq. (7) on Σ.

(2) $\varphi_{+1}(x,t;z)$ and $\varphi_{-2}(x,t;z)$ can be analytically extended to D^+ and continuously extended to $D^+\cup\Sigma$.

(3) $\varphi_{+2}(x,t;z)$ and $\varphi_{-1}(x,t;z)$ can be analytically extended to D^- and continuously extended to $D^-\cup\Sigma$.

Here, the modified Jost solutions $\mu_\pm(x,t;z)$ possess the same properties as $\varphi_\pm(x,t;z)$.

Proof The proposition was given in, *e.g.*, Refs. [16,37,38]. It can easily be given the analyticity and continuity of $\mu_\pm(x,t;z)$ by the relation (9).

Proposition 2 The Jost solutions $\varphi_\pm(x,t;z)$ can satisfy the Lax pair (3) of the NLST equation (2).

Proof Notice that

$$\begin{aligned}\det(\varphi_\pm(x,t;z))&=\lim_{x\to\pm\infty}\det(\varphi_\pm(x,t;z))\\ &=\lim_{x\to\pm\infty}\det(\mu_\pm(x,t;z))=1,\end{aligned} \tag{12}$$

by Liouville's formula and the zero-curvature condition $U_t-V_x+[U,V]=0$, one can know $\varphi_\pm(x,t;z)-V\varphi_\pm(x,t;z)$ solves the x-part (3). Based on the above analysis, there are two matrices $Q_\pm(t;z)$ satisfying

$$\begin{aligned}&\varphi_{\pm,t}(x,t;z)-V\varphi_\pm(x,t;z)=\varphi_\pm(x,t;z)Q_\pm(t;z),\\ &\text{as}\quad z\in\Sigma.\end{aligned} \tag{13}$$

Using the transformation (9), one can obtain $\mu_\pm(x,t;z)\sim I$ and $Q_\pm(t;z)=0$ as $x\to\pm\infty$, which means that $\varphi_\pm(x,t;z)$ solves the t-part (3).

2.2. Scattering matrix and reflection coefficients

In this subsection, the scattering matrix and reflection coefficients are introduced. Since $\varphi_+(x,t;z)$ and $\varphi_-(x,t;z)$ are two fundamental matrix solutions of the Lax pair (3), there exists a linear relation between $\varphi_+(x,t;z)$ and $\varphi_-(x,t;z)$:

$$\varphi_+(x,t;z)=\varphi_-(x,t;z)S(z),\quad z\in\Sigma, \tag{14}$$

where

$$S(z)=\begin{pmatrix}s_{11}(z) & s_{12}(z)\\ s_{21}(z) & s_{22}(z)\end{pmatrix}$$

is called the scattering matrix and $\det(S(z))=1$, and

$$\varphi_+(x,t;z)=(\varphi_{+1},\varphi_{+2})=\begin{pmatrix}\varphi_{+11} & \varphi_{+12}\\ \varphi_{+21} & \varphi_{+22}\end{pmatrix},$$

$$\varphi_-(x,t;z)=(\varphi_{-1},\varphi_{-2})=\begin{pmatrix}\varphi_{-11} & \varphi_{-12}\\ \varphi_{-21} & \varphi_{-22}\end{pmatrix}.$$

Then, it follows from Eq. (14) that $s_{ij}(z)$ $(i,j=1,2)$ have the Wronskian representations:

$$s_{11}(z)=Wr(\varphi_{+1},\varphi_{-2}),\quad s_{12}(z)=Wr(\varphi_{+2},\varphi_{-2}),$$
$$s_{21}(z)=Wr(\varphi_{-1},\varphi_{+1}),\quad s_{22}(z)=Wr(\varphi_{-1},\varphi_{+2}). \tag{15}$$

As usual, we can introduce the reflection coefficients:

$$\kappa(z)=\frac{s_{21}(z)}{s_{11}(z)},\quad \tilde{\kappa}(z)=\frac{s_{12}(z)}{s_{22}(z)}. \tag{16}$$

Proposition 3 Suppose $u(x,t)\in L^1(\mathbb{R})$. Here, $s_{11}(z)$ is analytic in D^+ and continuous on $D^+\cup\Sigma$, while $s_{22}(z)$ is analytic in D^- and continuous on $D^-\cup\Sigma$. In addition, $s_{12}(z)$ and $s_{21}(z)$ are continuous in Σ.

Proof From Proposition 1 and Eq. (15), it is easy to give the proof of the proposition. Note that we can not exclude the possibility of the existence of the zero point along Σ, in the case of ZBCs, which can lead to the so-called real singularity spectrum. Now we only consider potential functions without spectral singularities.[39]

2.3. Symmetry properties

Proposition 4 $U(x,t;z)$, $V(x,t;z)$, Jost solutions, modified Jost solutions, scattering matrix and reflection coefficients possess two types of symmetry.

(1) The first symmetry reduction:

$$\begin{aligned}
&U(x,t;z)=\sigma_2U(x,t;z^*)^*\sigma_2,\\
&V(x,t;z)=\sigma_2V(x,t;z^*)^*\sigma_2,\\
&\varphi_\pm(x,t;z)=\sigma_2\varphi_\pm(x,t;z^*)^*\sigma_2,\\
&\mu_\pm(x,t;z)=\sigma_2\mu_\pm(x,t;z^*)^*\sigma_2,\\
&S(z)=\sigma_2S(z^*)^*\sigma_2,\quad \kappa(z)=-\tilde{\kappa}(z^*)^*.
\end{aligned} \tag{17}$$

(2) The second symmetry reduction:

$$\begin{aligned}
&U(x,t;z)=\sigma_1U(x,t;-z^*)^*\sigma_1,\\
&V(x,t;z)=\sigma_1V(x,t;-z^*)^*\sigma_1,\\
&\varphi_\pm(x,t;z)=\sigma_1\varphi_\pm(x,t;-z^*)^*\sigma_1,\\
&\mu_\pm(x,t;z)=\sigma_1\mu_\pm(x,t;-z^*)^*\sigma_1,\\
&S(z)=\sigma_1S(-z^*)^*\sigma_1,\quad \kappa(z)=\tilde{\kappa}(-z^*)^*.
\end{aligned} \tag{18}$$

Here, $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\sigma_2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}$.

Proof The similar properties were given in Refs. [37,38], which have been proved by many researchers.[11,16,31,40]

2.4. Asymptotic behaviors

To deal with the RH problem, we must discuss the asymptotic behavior of the modified Jost solutions and scattering matrix as $z\to\infty$ in this subsection.

Proposition 5 The asymptotic behaviors of the modified Jost solutions can be expressed as

$$\mu_\pm(x,t;z)=\mathrm{e}^{\mathrm{i}\nu_\pm(x,t)\sigma_3}+O\left(\frac{1}{z}\right),\quad \text{as}\ \ z\to\infty, \tag{19}$$

where $\nu_\pm(x,t)=(1/2)\int_{\pm\infty}^{x}|u(x,t)|^2\mathrm{d}y$.

Proof Taking the asymptotic expansions of $\mu_\pm(x,t;z)$

$$\mu_\pm(x,t;z)=\mu_\pm^{(0)}(x,t)+\frac{\mu_\pm^{(1)}(x,t)}{z}+\frac{\mu_\pm^{(2)}(x,t)}{z^2}+O\Big(\frac{1}{z^3}\Big),$$
$$\text{as}\quad z\to\infty. \tag{20}$$

Substituting Eq. (20) into the Lax pair (10), and comparing the coefficients of different powers of z, we can reach

$$\mathrm{i}[\sigma_3,\mu^{(0)}]=0, \tag{21}$$
$$\mathrm{i}[\sigma_3,\mu^{(1)}]+U_1\mu^{(0)}=0, \tag{22}$$
$$\mu_x^{(0)}=\mathrm{i}[\sigma_3,\mu^{(2)}]-\mathrm{i}[\sigma_3,\mu^{(0)}]-U_1\mu^{(1)}=0. \tag{23}$$

Then, it is easy to obtain

$$(\mu_\pm^{(0)}(x,t))^{\mathrm{off}}=0, \tag{24}$$
$$(\mu_\pm^{(1)}(x,t))^{\mathrm{off}}=\frac{\mathrm{i}}{2}\sigma_3U_1(x,t)\mu_\pm^{(0)}(x,t), \tag{25}$$
$$\mu_\pm^{(0)}(x,t)=C^{\mathrm{diag}}\mathrm{e}^{\mathrm{i}\nu_\pm(x,t)\sigma_3}, \tag{26}$$

where $(\mu^{(n)})^{\mathrm{off}}$ denotes the off-diagonal parts of the matrix. This completes the proof of the proposition.

Proposition 6 The asymptotic behavior for the scattering matrix is given by

$$S(z)=\mathrm{e}^{-\mathrm{i}\nu\sigma_3}+O\Big(\frac{1}{z}\Big),\quad \text{as}\ \ z\to\infty, \tag{27}$$

where ν is a constant, $\nu=(1/2)\int_{+\infty}^{-\infty}|u(x,t)|^2\mathrm{d}y$.

Proof Using Eqs. (15) and (19), we can easily obtain the asymptotic behaviors (27) of the scattering matrix. If we plug the expansion $\mu_\pm$ (20) back into the time part of the Lax pairs and matching the $O(z^i)$ $(i=0,1,2,3,4)$, we can obtain $\nu_t(x,t)=0$ as $x\to\pm\infty$. In other words, ν does not depend on t.

2.5. Discrete spectrum with double zeros

In this subsection, we will focus on the discrete spectrum of the scattering problem with double zeros. We suppose that $s_{11}(z)$ has N double zeros in $\Omega_0=\{z\in\mathbb{C}:\mathrm{Re}\,z>0,\mathrm{Im}\,z>0\}$ denoted by z_n $(n=1,2,\ldots,N)$, that is, $s_{11}(z_n)=s_{11}'(z_n)=0$, and $s_{11}''(z_n)\neq0$. Using the symmetries of the scattering matrix, one has

$$\begin{aligned}
&s_{11}(z_n)=s_{11}(-z_n)=s_{22}(z_n^*)=s_{22}(-z_n^*),\\
&s_{11}'(z_n)=s_{11}'(-z_n)=s_{22}'(z_n^*)=s_{22}'(-z_n^*).
\end{aligned} \tag{28}$$

Then, the set of discrete spectral is defined as

$$\Omega=\{\pm z,\pm z_n^*\}_{n=1}^N. \tag{29}$$

A routine computation gives rise to all residues, which is on the discrete spectrum.

(i) For $s_{11}(z_0)=0$, $s_{11}'(z_0)=0$ and $s_{11}''(z_0)\neq0$, $z_0\in\Omega\bigcap D^+$.

If $s_{11}(z)=0$, $s'_{11}(z)=0$ and $s''_{11}(z)\neq 0$, at $z=z_0$, one can know that $\varphi_{+1}(x,t;z_0)$ and $\varphi_{-2}(x,t;z_0)$ must be linearly dependent by Eq. (15). Then, $\varphi_{+1}(x,t;z_0)$ and $\varphi_{-2}(x,t;z_0)$ follow from

$$\varphi_{+1}(x,t;z_0)=b(z_0)\varphi_{-2}(x,t;z_0), \tag{30}$$

where $b(z_0)$ is a constant. Noticing $s'_{11}(z_0)=0$ and taking the derivative of Eq. (15) with respect to z, one can take $z=z_0$ and obtain

$$s'_{11}(z_0) = \det(\varphi'_{+1}(x,t;z_0) - b(z_0)\varphi'_{-2}(x,t;z_0), \varphi_{-2}(x,t;z_0)), \tag{31}$$

then $\varphi'_{+1}(x,t;z_0)-b(z_0)\varphi'_{-2}(x,t;z_0)$ and $\varphi_{-2}(x,t;z_0)$ are linearly dependent. Thus, we introduce the constant $d(z_0)$ such that

$$\varphi'_{+1}(x,t;z_0)-b(z_0)\varphi'_{-2}(x,t;z_0)=d(z_0)\varphi_{-2}(x,t;z_0). \tag{32}$$

Furthermore, let $\varphi_{+1}(x,t;z_0)$ and $s_{11}(z)$ carry out expansion at $z=z_0$, one has

$$\frac{\varphi_{+1}(x.t;z)}{s_{11}(z)} = \frac{2\varphi_{+1}(x,t;z_0)}{s''_{11}(z_0)}(z-z_0)^{-2}+\left(\frac{2\varphi'_{+1}(x,t;z_0)}{s''_{11}(z_0)}-\frac{2\varphi_{+1}(x,t;z)s'''_{11}(z_0)}{3s''^2_{11}(z_0)}\right)(z-z_0)^{-1}+\cdots. \tag{33}$$

Then, we can obtain

$$\mathop{\mathrm{L}_{-2}}_{z=z_0}\left[\frac{\varphi_{+1}(x.t;z)}{s_{11}(z)}\right] = \frac{2b(z_0)\varphi_{-2}(x,t;z_0)}{s''_{11}(z_0)}, \tag{34}$$

$$\mathop{\mathrm{Res}}_{z=z_0}\left[\frac{\varphi_{+1}(x,t;z)}{s_{11}(z)}\right] = \frac{2b(z_0)\varphi'_{-2}(x,t;z_0)}{s''_{11}(z_0)}+\left(\frac{b(z_0)}{s''_{11}(z_0)}\left(\frac{d(z_0)}{b(z_0)}-\frac{s'''_{11}(z_0)}{3s''_{11}(z_0)}\right)\right)\varphi_{-2}(x,t;z_0), \tag{35}$$

where $\mathop{\mathrm{L}_{-2}}_{z=z_0}[f(x,t;z)]$ denotes the coefficient of $O((z-z_0)^{-2})$ in the Laurent expansion of $f(x,t;z)$ at $z=z_0$, and $\mathop{\mathrm{Res}}_{z=z_0}[f(x,t;z)]$ denotes the coefficient of $O((z-z_0)^{-1})$ in the Laurent expansion of $f(x,t;z)$ at $z=z_0$.

(ii) For $s_{22}(z_0)=0$, $s'_{22}(z_0)=0$ and $s''_{22}(z_0)\neq 0$, $z_0\in\Omega\cap D^-$.

If $s_{22}(z)=0$, $s'_{22}(z)=0$ and $s''_{22}(z)\neq 0$, at $z=z_0$, we follow the same steps and obtain

$$\mathop{\mathrm{L}_{-2}}_{z=z_0}\left[\frac{\varphi_{+2}(x.t;z)}{s_{22}(z)}\right] = \frac{2b(z_0)\varphi_{-1}(x,t;z_0)}{s''_{22}(z_0)}, \tag{36}$$

$$\mathop{\mathrm{Res}}_{z=z_0}\left[\frac{\varphi_{+2}(x.t;z)}{s_{22}(z)}\right] = \frac{2b(z_0)\varphi'_{-1}(x,t;z_0)}{s''_{22}(z_0)}+\left(\frac{b(z_0)}{s''_{22}(z_0)}\left(\frac{d(z_0)}{b(z_0)}-\frac{s'''_{22}(z_0)}{3s''_{22}(z_0)}\right)\right)\varphi_{-1}(x,t;z_0). \tag{37}$$

For simplicity of presentation, we denote $A(z_0)$ and $B(z_0)$ such that

$$A(z_0)=\begin{cases}\dfrac{2b(z_0)}{s''_{11}(z_0)}, & \text{as } z_0\in\Omega\cap D^+,\\ \dfrac{2b(z_0)}{s''_{22}(z_0)}, & \text{as } z_0\in\Omega\cap D^-,\end{cases} \tag{38}$$

$$B(z_0)=\begin{cases}\dfrac{d(z_0)}{b(z_0)}-\dfrac{s'''_{11}(z_0)}{3s''_{11}(z_0)}, & \text{as } z_0\in\Omega\cap D^+,\\ \dfrac{d(z_0)}{b(z_0)}-\dfrac{s'''_{22}(z_0)}{3s''_{22}(z_0)}, & \text{as } z_0\in\Omega\cap D^-.\end{cases} \tag{39}$$

It follows that

$$\begin{aligned}
&\mathop{\mathrm{L}_{-2}}_{z=z_0}\left[\frac{\varphi_{+1}(x.t;z)}{s_{11}(z)}\right]=A(z_0)\varphi_{-2}(x,t;z_0), \quad \text{as } z_0\in\Omega\cap D^+,\\
&\mathop{\mathrm{L}_{-2}}_{z=z_0}\left[\frac{\varphi_{+2}(x.t;z)}{s_{22}(z)}\right]=A(z_0)\varphi_{-1}(x,t;z_0), \quad \text{as } z_0\in\Omega\cap D^-,\\
&\mathop{\mathrm{Res}}_{z=z_0}\left[\frac{\varphi_{+1}(x.t;z)}{s_{11}(z)}\right]=A(z_0)\varphi'_{-2}(x,t;z_0)+B(z_0)\varphi_{-2}(x,t;z_0), \quad \text{as } z_0\in\Omega\cap D^+,\\
&\mathop{\mathrm{Res}}_{z=z_0}\left[\frac{\varphi_{+2}(x.t;z)}{s_{22}(z)}\right]=A(z_0)\varphi'_{-1}(x,t;z_0)+B(z_0)\varphi_{-1}(x,t;z_0), \quad \text{as } z_0\in\Omega\cap D^-.
\end{aligned} \tag{40}$$

Proposition 7 Given $z_0\in\Sigma$, two kinds of symmetries of $A(z_0)$ and $B(z_0)$ can be deduced as follows:
(1) $A(z_0)=-A(z_0^*)^*$, $B(z_0)=B(z_0^*)^*$;
(2) $A(z_0)=A(-z_0^*)^*$, $B(z_0)=-B(-z_0^*)^*$.

Proof Considering Eqs. (38) and (39), from Proposition 4 one can show this proposition.

3. Inverse problem with ZBCs and double poles

In order to study the inverse problem with ZBCs and to obtain the N double-poles solutions of the Eq. (2), we must establish and solve an RH problem based on the above discussions.

3.1. The matrix Riemann–Hilbert problem

For the sake of convenience, we define

$$\eta_n = \begin{cases} z_n, & n = 1,2,\ldots,N, \\ -z_{n-N}, & n = N+1,N+2,\ldots,2N, \end{cases} \tag{41}$$

to establish and solve the RH problem. Then a sectionally meromorphic matrices is proposed:

Proposition 8 Define

$$M(x,t;z) = \begin{cases} M^+(x,t;z) = \left(\dfrac{\mu_{+1}(x,t;z)}{s_{11}(z)}, \mu_{-2}(x,t;z)\right), \\ \qquad\qquad \text{as } z \in D^+, \\ M^-(x,t;z) = \left(\mu_{-1}(x,t;z), \dfrac{\mu_{+2}(x,t;z)}{s_{22}(z)}\right), \\ \qquad\qquad \text{as } z \in D^-, \end{cases} \tag{42}$$

and

$$\lim_{z' \to z, z' \in D^{\pm}} M(x,t;z') = M^{\pm}(x,t;z). \tag{43}$$

Then the following results can be given.

(i) Analyticity: For $M(x,t;z)$, it is not also analytic in $(D^+ \cup D^-) \setminus \Omega$, but only has double-poles in Ω, whose principal parts of the Laurent series at the double pole η_n or η_n^* can be given as follows:

$$\begin{aligned} \operatorname*{Res}_{z=\eta_n} M(x,t;z) &= (A(\eta_n)\mathrm{e}^{-2\mathrm{i}\theta(x,t;\eta_n)}\{\mu'_{-2}(x,t;\eta_n) \\ &\quad + [B(\eta_n) - 2\mathrm{i}\theta'(x,t;\eta_n)]\mu_{-2}(x,t;\eta_n)\},0), \\ \mathrm{L}_{-2}\underset{z=\eta_n}{} M(x,t;z) &= (A(\eta_n)\mathrm{e}^{-2\mathrm{i}\theta(x,t;\eta_n)}\mu_{-2}(x,t;\eta_n),0), \\ \operatorname*{Res}_{z=\eta_n^*} M(x,t;z) &= (0, A(\eta_n^*)\mathrm{e}^{2\mathrm{i}\theta(x,t;\eta_n^*)}\{\mu'_{-1}(x,t;\eta_n^*) \\ &\quad + [B(\eta_n^*) + 2\mathrm{i}\theta'(x,t;\eta_n^*)]\mu_{-2}(x,t;\eta_n^*)\}), \\ \mathrm{L}_{-2}\underset{z=\eta_n^*}{} M(x,t;z) &= (0, A(\eta_n^*)\mathrm{e}^{2\mathrm{i}\theta(x,t;\eta_n^*)}\mu_{-1}(x,t;\eta_n^*)). \end{aligned} \tag{44}$$

(ii) Jump condition:

$$M^-(x,t;z) = M^+(x,t;z)(I - J(x,t;z)), \quad \text{as } z \in \Sigma, \tag{45}$$

where

$$J(x,t;z) = \mathrm{e}^{\mathrm{i}\theta(x,t;z)\hat{\sigma}_3} \begin{pmatrix} 0 & -\tilde{\kappa}(z) \\ \kappa(z) & \kappa(z)\tilde{\kappa}(z) \end{pmatrix}.$$

(iii) Asymptotic behavior:

$$M(x,t;z) = \mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3} + O\left(\frac{1}{z}\right), \quad \text{as } z \to \infty. \tag{46}$$

Proof (i) It follows from the result of Proposition 1 and Proposition 3 that one can obtain the analyticity of $M(x,t;z)$.

Here, we only consider double poles $\eta_n \in D^+$, and obtain

$$\begin{aligned} &\operatorname*{Res}_{z=\eta_n} M(x,t;z) \\ &= \Big(A(\eta_n)\mathrm{e}^{-2\mathrm{i}\theta(x,t;\eta_n)}\Big[\mu'_{-2}(x,t;\eta_n) \\ &\quad + \big(B(\eta_n) - 2\mathrm{i}\theta'(x,t;\eta_n)\big)\mu_{-2}(x,t;\eta_n)\Big], 0\Big) \end{aligned} \tag{47}$$

and

$$\mathrm{L}_{-2}\underset{z=\eta_n}{} M(x,t;z) = \Big(A(\eta_n)\mathrm{e}^{-2\mathrm{i}\theta(x,t;\eta_n)}\mu_{-2}(x,t;\eta_n), 0\Big) \tag{48}$$

by combining Eq. (9) with Eq. (40).

(ii) From Eqs. (9) and (14), we have

$$\mu_{-2}(x,t;z) = -\tilde{\kappa}(z)\mathrm{e}^{2\mathrm{i}\theta(x,t;z)}\mu_{-1}(x,t;z) + \frac{\mu_{+2}(x,t;z)}{s_{22}(z)}, \tag{49}$$

$$\begin{aligned} \frac{\mu_{+1}(x,t;z)}{s_{11}(z)} &= \mu_{-1}(x,t;z) + \kappa(z)\mathrm{e}^{-2\mathrm{i}\theta(x,t;z)}\mu_{-2}(x,t;z) \\ &= (1 - \kappa(z)\tilde{\kappa}(z))\mu_{-1}(x,t;z) \\ &\quad + \kappa(z)\mathrm{e}^{-2\mathrm{i}\theta(x,t;z)}\frac{\mu_{+2}(x,t;z)}{s_{22}(z)}, \end{aligned} \tag{50}$$

and then jump condition can be given as follows:

$$\begin{aligned} &\left(\mu_{-1}(x,t;z), \frac{\mu_{+2}(x,t;z)}{s_{22}(z)}\right) \\ &= \left(\frac{\mu_{+1}(x,t;z)}{s_{11}(z)}, \mu_{-2}(x,t;z)\right) \\ &\quad \times \begin{pmatrix} 1 & \mathrm{e}^{2\mathrm{i}\theta(x,t;z)}\tilde{\kappa}(z) \\ -\mathrm{e}^{-2\mathrm{i}\theta(x,t;z)}\kappa(z) & 1 - \kappa(z)\tilde{\kappa}(z) \end{pmatrix}, \end{aligned} \tag{51}$$

$$M^-(x,t;z) = M^+(x,t;z)(I - J(x,t;z)). \tag{52}$$

(iii) According to the asymptotic behaviors of the Jost solution (19) and scattering matrix (27), we have

$$\begin{aligned} M^+(x,t;z) &= \mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3} + O\left(\frac{1}{z}\right), \quad \text{as } z \in D^+, \\ M^-(x,t;z) &= \mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3} + O\left(\frac{1}{z}\right), \quad \text{as } z \in D^-. \end{aligned} \tag{53}$$

Proposition 9 The solution of the matrix RH problem can be given as follows:

$$\begin{aligned} &M(x,t;z) \\ &= \mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3} + \frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{M^+(x,t;\xi)J(x,t;\xi)}{\xi - z}\mathrm{d}\xi \\ &\quad + \sum_{n=1}^{2N}\left(C_n(z)\left[\mu'_{-2}(x,t;\eta_n) + \left(D_n + \frac{1}{z-\eta_n}\right)\mu_{-2}(x,t;\eta_n)\right]\right. \\ &\quad \left. \times \tilde{C}_n(z)\left[\mu'_{-1}(x,t;\eta_n^*) + \left(\tilde{D}_n + \frac{1}{z-\eta_n^*}\right)\mu_{-1}(x,t;\eta_n^*)\right]\right), \end{aligned} \tag{54}$$

where $z \in \Sigma$, and

$$\begin{aligned} C_n(z) &= \frac{A(\eta_n)}{z - \eta_n}\mathrm{e}^{-2\mathrm{i}\theta(x,t;\eta_n)}, \\ \tilde{C}_n(z) &= \frac{A(\eta_n^*)}{z - \eta_n^*}\mathrm{e}^{-2\mathrm{i}\theta(x,t;\eta_n^*)}, \\ D_n(z) &= B(\eta_n) - 2\mathrm{i}\theta'(x,t;\eta_n), \\ \tilde{D}_n(z) &= B(\eta_n^*) - 2\mathrm{i}\theta'(x,t;\eta_n^*), \end{aligned} \tag{55}$$

μ_{-k} and μ'_{-k} $(k = 1,2)$ satisfy

$$
\begin{aligned}
\mu_{-1}(x,t;\eta_n^*) &= \mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3}\begin{pmatrix}1\\0\end{pmatrix}+\sum_{s=1}^{2N}C_s(\eta_n^*)\left[\mu'_{-2}(x,t;\eta_s)+\left(D_s+\frac{1}{\eta_n^*-\eta_s}\right)\cdot\mu_{-2}(x,t;\eta_s)\right]\\
&\quad+\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{(M^+(x,t;\xi)J(x,t;\xi))_1}{\xi-\eta_n^*}\mathrm{d}\xi,\\
\mu_{-2}(x,t;\eta_s) &= \mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3}\begin{pmatrix}0\\1\end{pmatrix}+\sum_{j=1}^{2N}\tilde{C}_j(\eta_s^*)\left[\mu'_{-1}(x,t;\eta_j^*)+\left(\tilde{D}_j+\frac{1}{\eta_s-\eta_j}\right)\cdot\mu_{-1}(x,t;\eta_j)\right]\\
&\quad+\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{(M^+(x,t;\xi)J(x,t;\xi))_2}{\xi-\eta_s^*}\mathrm{d}\xi,\\
\mu'_{-1}(x,t;\eta_n^*) &= -\sum_{s=1}^{2N}\frac{C_s(\eta_n^*)}{\eta_n^*-\eta_s}\left[\mu'_{-2}(x,t;\eta_s)+\left(D_s+\frac{2}{\eta_n^*-\eta_s}\right)\mu_{-2}(x,t;\eta_s)\right]+\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{(M^+(x,t;\xi)J(x,t;\xi))_1}{(\xi-\eta_n^*)^2}\mathrm{d}\xi,\\
\mu'_{-2}(x,t;\eta_s^*) &= -\sum_{j=1}^{2N}\frac{\tilde{C}_s(\eta_n^*)}{\eta_s-\eta_j^*}\left[\mu'_{-1}(x,t;\eta_j^*)+\left(\tilde{D}_j+\frac{2}{\eta_s-\eta_j^*}\right)\mu_{-1}(x,t;\eta_j^*)\right]+\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{(M^+(x,t;\xi)J(x,t;\xi))_2}{(\xi-\eta_s^*)^2}\mathrm{d}\xi. \quad (56)
\end{aligned}
$$

Proof To get a normative RH problem, we must remove the asymptotic behaviors and the poles contributions, and rewrite Eq. (42) as

$$
\begin{aligned}
&M^-(x,t;z)-\mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3}-\sum_{n=1}^{2N}\left[\frac{\mathrm{L}_{-2}\underset{z=\eta_n}{M}(x,t;z)}{(z-\eta_n)^2}+\frac{\mathrm{Res}\underset{z=\eta_n}{M}(x,t;z)}{z-\eta_n}+\frac{\mathrm{L}_{-2}\underset{z=\eta_n^*}{M}(x,t;z)}{(z-\eta_n^*)^2}+\frac{\mathrm{Res}\underset{z=\eta_n^*}{M}(x,t;z)}{z-\eta_n^*}\right]\\
&=M^+(x,t;z)-\mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3}-\sum_{n=1}^{2N}\left[\frac{\mathrm{L}_{-2}\underset{z=\eta_n}{M}(x,t;z)}{(z-\eta_n)^2}+\frac{\mathrm{Res}\underset{z=\eta_n}{M}(x,t;z)}{z-\eta_n}+\frac{\mathrm{L}_{-2}\underset{z=\eta_n^*}{M}(x,t;z)}{(z-\eta_n^*)^2}+\frac{\mathrm{Res}\underset{z=\eta_n^*}{M}(x,t;z)}{z-\eta_n^*}\right]-M^+(x,t;z)J(x,t;z). \quad (57)
\end{aligned}
$$

Then, one can obtain the soultion (54) by using Plemelj's formula.

3.2. Reconstruction formula of the potential

In the subsection, our aim is to obtain the reconstruction formula. The first step is to give

$$
\begin{aligned}
M(x,t;z) &= \mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3}+\frac{M^{(1)}(x,t)}{z}+O\left(\frac{1}{z^2}\right),\\
&\quad\text{as } z\to\infty, \quad (58)
\end{aligned}
$$

where

$$
\begin{aligned}
&M^{(1)}(x,t)\\
&=-\frac{1}{2\pi\mathrm{i}}\int_\Sigma M^+(x,t;\xi)J(x,t;\xi)\mathrm{d}\xi+\sum_{s=1}^{2N}\big(A(\eta_n)\mathrm{e}^{-2\mathrm{i}\theta(x,t;\eta_n)}\\
&\quad\times[\mu'_{-2}(x,t;\eta_n)+D_n\mu_{-2}(x,t;\eta_n)],A(\eta_n^*)\mathrm{e}^{2\mathrm{i}\theta(x,t;\eta_n)}\\
&\quad\times[\mu'_{-1}(x,t;\eta_n^*)+\tilde{D}_n\mu_{-1}(x,t;\eta_n^*)]\big). \quad (59)
\end{aligned}
$$

Next, one can obtain

$$
u(x,t)=-2\mathrm{i}\mathrm{e}^{\mathrm{i}\nu_-(x,t)\sigma_3}M_{12}^{(1)}(x,t), \quad (60)
$$

by plugging Eq. (58) back into Eq. (10) and matching $O(z)$ term, where

$$
\begin{aligned}
M_{12}^{(1)}(x,t) &= -\frac{1}{2\pi\mathrm{i}}\int_\Sigma(M^+(x,t;\xi)J(x,t;\xi))_{12}\mathrm{d}\xi\\
&\quad+\sum_{s=1}^{2N}\{A(\eta_n^*)\mathrm{e}^{2\mathrm{i}\theta(x,t;\eta_n)}\\
&\quad\times[\mu'_{-11}(x,t;\eta_n^*)+\tilde{D}_n\mu_{-11}(x,t;\eta_n^*)]\}. \quad (61)
\end{aligned}
$$

For the sake of simplicity, we take $\alpha=(\alpha^{(1)},\alpha^{(2)})$ and $\gamma=(\gamma^{(1)},\gamma^{(2)})$, where

$$
\begin{aligned}
\alpha^{(1)} &= (A(\eta_n^*)\mathrm{e}^{2\mathrm{i}\theta(x,t;\eta_n)}\tilde{D}_n\mu_{-1}(x,t;\eta_n^*))_{1\times2N},\\
\alpha^{(2)} &= (A(\eta_n^*)\mathrm{e}^{2\mathrm{i}\theta(x,t;\eta_n)})_{1\times2N},\\
\gamma^{(1)} &= (\mu_{-11}(x,t;\eta_n^*))_{1\times2N},\\
\gamma^{(2)} &= (\mu'_{-11}(x,t;\eta_n^*))_{1\times2N}. \quad (62)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&u(x,t)\\
&=-2\mathrm{i}\mathrm{e}^{\mathrm{i}\nu_-(x,t)}\left(\alpha\gamma-\frac{1}{2\pi\mathrm{i}}\int_\Sigma\big(M^+(x,t;\xi)J(x,t;\xi)\big)_{12}\mathrm{d}\xi\right). \quad (63)
\end{aligned}
$$

3.3. Trace formulae

Based on the above analysis, $s_{11}(z)$ and $s_{22}(z)$ are analytic respectively on D^+ and D^-, whose discrete zeros are $\pm\eta_n$ and $\pm\eta_n^*$. In order to get the trace formulae, we define

$$
\begin{aligned}
\beta^+(z) &= s_{11}(z)\prod_{n=1}^{2N}\left(\frac{z-\eta_n^*}{z-\eta_n}\right)^2\mathrm{e}^{\mathrm{i}\nu},\\
\beta^-(z) &= s_{22}(z)\prod_{n=1}^{2N}\left(\frac{z-\eta_n}{z-\eta_n^*}\right)^2\mathrm{e}^{-\mathrm{i}\nu}, \quad (64)
\end{aligned}
$$

and it is apparent that $\beta^\pm$ is analytic in $D^\pm$, which has no zero. In addition, there is an relation $\beta^+(z)\beta^-(z)=s_{11}(z)s_{22}(z)$,

and $\beta^{\pm}\to 1$, as $z\to\infty$. Due to $\det(s(z))=1$, one can have

$$\beta^{+}(z)\beta^{-}(z)=\frac{1}{1-\kappa(z)\tilde{\kappa}(z)}. \tag{65}$$

If take logarithm on both sides of Eq. (65) simultaneously and employing Plemelj' formula, we can obtain

$$\log\beta^{\pm}=\mp\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{\log\left(1-\kappa(z)\tilde{\kappa}(z)\right)}{\xi-z}\mathrm{d}\xi,\quad z\in D^{\pm}. \tag{66}$$

From Eqs. (64) and (66), we can obtain

$$\begin{aligned}
s_{11}(z) &= \exp\left(-\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{\log\left(1-\kappa(z)\tilde{\kappa}(z)\right)}{\xi-z}\mathrm{d}\xi\right)\\
&\quad\times\prod_{n=1}^{2N}\left(\frac{z-\eta_n}{z-\eta_n^*}\right)^2\mathrm{e}^{-\mathrm{i}\nu},\\
s_{22}(z) &= \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{\log\left(1-\kappa(z)\tilde{\kappa}(z)\right)}{\xi-z}\mathrm{d}\xi\right)\\
&\quad\times\prod_{n=1}^{2N}\left(\frac{z-\eta_n^*}{z-\eta_n}\right)^2\mathrm{e}^{\mathrm{i}\nu}.
\end{aligned} \tag{67}$$

3.4. Reflectionless potential: double-pole solitons

For reflectionless potential means the reflection coefficients $\kappa(z)=0$ and $\tilde{\kappa}(z)=0$. Next, one can obtain $\varphi_{\pm}(x,t;0)=\mu_{\pm}(x,t;0)=I$ and $s_{11}(0)=1$ from Eq. (11), and then $\nu=8\sum_{n=1}^{N}\arg(\eta_n)+2\pi i$ $(i\in\mathbb{Z})$ is given when combining the trace formula.

Theorem 1 With zero boundary conditions (ZBCs), the double-pole solution of the NLST equation is given by

$$u(x,t)=2\mathrm{i}\frac{\det(I-S)}{\det(W)}\left(\frac{\det(\hat{W})}{\det(I-\hat{S})}\right)^2, \tag{68}$$

where

$$\begin{aligned}
&W=\begin{pmatrix}0 & \alpha\\ \upsilon & I-S\end{pmatrix},\quad \hat{W}=\begin{pmatrix}0 & \alpha\\ \upsilon & I-\hat{S}\end{pmatrix},\\
&\upsilon=\begin{pmatrix}\upsilon^{(1)}\\ \upsilon^{(2)},\end{pmatrix},\quad \upsilon^{(1)}=(1)_{2N\times 1},\quad \upsilon^{(2)}=(0)_{2N\times 1}.
\end{aligned} \tag{69}$$

The $4N\times 4N$ partitioned matrix $S=\begin{pmatrix}S^{(1,1)} & S^{(1,2)}\\ S^{(2,1)} & S^{(2,2)}\end{pmatrix}$ with $S^{(i,j)}=(s_{n,j}^{(i,j)})_{2N\times 2N}$ $(i,j=1,2)$ is given by

$$\begin{aligned}
s_{n,j}^{(1,1)} &= \sum_{k=1}^{2N}C_k(\eta_n^*)\tilde{C}_j(\eta_k)\left[-\frac{1}{\eta_k-\eta_j^*}(\tilde{D}_j+\frac{2}{\eta_k-\eta_j^*})+\left(D_k+\frac{1}{\eta_n^*-\eta_k}\right)\cdot\left(\tilde{D}_j+\frac{1}{\eta_k-\eta_j^*}\right)\right],\\
s_{n,j}^{(1,2)} &= \sum_{k=1}^{2N}C_k(\eta_n^*)\tilde{C}_j(\eta_k)\left[-\frac{1}{\eta_k-\eta_j^*}+\left(D_k+\frac{1}{\eta_n^*-\eta_k}\right)\right],\\
s_{n,j}^{(2,1)} &= \sum_{k=1}^{2N}\frac{C_k(\eta_n^*)\tilde{C}_j(\eta_k)}{\eta_n^*-\eta_k}\left[\frac{1}{\eta_k-\eta_j^*}\left(\tilde{D}_j+\frac{2}{\eta_k-\eta_j^*}\right)-\left(D_k+\frac{2}{\eta_n^*-\eta_k}\right)\cdot\left(\tilde{D}_j+\frac{1}{\eta_k-\eta_j^*}\right)\right],\\
s_{n,j}^{(2,2)} &= \sum_{k=1}^{2N}\frac{C_k(\eta_n^*)\tilde{C}_j(\eta_k)}{\eta_n^*-\eta_k}\left[\frac{1}{\eta_k-\eta_j^*}-\left(D_k+\frac{2}{\eta_n^*-\eta_k}\right)\right].
\end{aligned} \tag{70}$$

The $4N\times 4N$ partitioned matrix $\hat{S}=\begin{pmatrix}\hat{S}^{(1,1)} & \hat{S}^{(1,2)}\\ \hat{S}^{(2,1)} & \hat{S}^{(2,2)}\end{pmatrix}$ with $\hat{S}^{(i,j)}=(\hat{s}_{n,j}^{(i,j)})_{2N\times 2N}$ $(i,j=1,2)$ is given by

$$\begin{aligned}
\hat{s}_{n,j}^{(1,1)} &= \eta_n^*\sum_{k=1}^{2N}\frac{C_k(\eta_n^*)\tilde{C}_j(\eta_k)}{\eta_k}\left[-\frac{1}{\eta_k-\eta_j^*}\left(\tilde{D}_j+\frac{2}{\eta_k-\eta_j^*}\right)+\frac{\eta_k}{\eta_j^*}\left(D_k+\frac{1}{\eta_n^*-\eta_k}-\frac{1}{\eta_k}\right)\left(\tilde{D}_j+\frac{1}{\eta_k-\eta_j^*}-\frac{1}{\eta_j^*}\right)\right],\\
\hat{s}_{n,j}^{(1,2)} &= \eta_n^*\sum_{k=1}^{2N}\frac{C_k(\eta_n^*)\tilde{C}_j(\eta_k)}{\eta_k}\left[-\frac{1}{\eta_k-\eta_j^*}+\frac{\eta_k}{\eta_j^*}\left(D_k+\frac{1}{\eta_n^*-\eta_k}-\frac{1}{\eta_k}\right)\right],\\
\hat{s}_{n,j}^{(2,1)} &= \sum_{k=1}^{2N}\frac{C_k(\eta_n^*)\tilde{C}_j(\eta_k)}{\eta_n^*-\eta_k}\left[\frac{1}{\eta_k-\eta_j^*}\left(\tilde{D}_j+\frac{2}{\eta_k-\eta_j^*}\right)-\frac{\eta_k}{\eta_j^*}(D_k+\frac{2}{\eta_n^*-\eta_k})\left(\tilde{D}_j+\frac{1}{\eta_k-\eta_j^*}-\frac{1}{\eta_j^*}\right)\right],\\
\hat{s}_{n,j}^{(2,2)} &= \sum_{k=1}^{2N}\frac{C_k(\eta_n^*)\tilde{C}_j(\eta_k)}{\eta_n^*-\eta_k}\left[\frac{1}{\eta_k-\eta_j^*}-\frac{\eta_k}{\eta_j^*}\left(D_k+\frac{2}{\eta_n^*-\eta_k}\right)\right].
\end{aligned} \tag{71}$$

Proof Taking Eqs. (56), (62), and (63) into consideration, the reflectionless potential is expressed as

$$u(x,t)=2\mathrm{i}\frac{\det(W)}{\det(I-S)}\mathrm{e}^{2\mathrm{i}\nu_{-}(x,t)}. \tag{72}$$

In order to obtain an exact form for the reflectionless potential, one can obtain

$$M(x,t;z)=I+z\sum_{n=1}^{2N}\left[\frac{\mathrm{L}_{-2}\underset{z=\eta_n}{(M(x,t;z)/z)}}{(z-\eta_n)^2}+\frac{\mathrm{Res}\underset{z=\eta_n}{(M(x,t;z)/z)}}{z-\eta_n}+\frac{\mathrm{L}_{-2}\underset{z=\eta_n^*}{(M(x,t;z)/z)}}{(z-\eta_n^*)^2}+\frac{\mathrm{Res}\underset{z=\eta_n^*}{(M(x,t;z)/z)}}{z-\eta_n^*}\right], \tag{73}$$

from Eqs. (67) and (11) as $x \to \infty$, and obtain

$$u(x,t) = 2\mathrm{i}\frac{\det(\hat{W})}{\det(I-\hat{S})}\mathrm{e}^{\mathrm{i}\nu_-(x,t)}, \tag{74}$$

by plugging γ back into the formula of the potential, which can be given by using Eq. (73). Then, the exact form of reflectionless potential can be given by combining Eq. (72) with Eq. (74). This proof is completed.

From the above analysis, it is easy to obtain N double-pole soliton solution of the NLST equation (2) under ZBCs via Eq. (68). For example, the single double-pole solution is given in Fig. 1. Here, Figs. 1(a)–1(c) show the dynamical characteristics of double-pole soliton of the new Schrödinger-type equation for $t = -3, 0, 3$, and the nonlinear phenomena is caused by two bright-solitons bumping into each other approximatively.

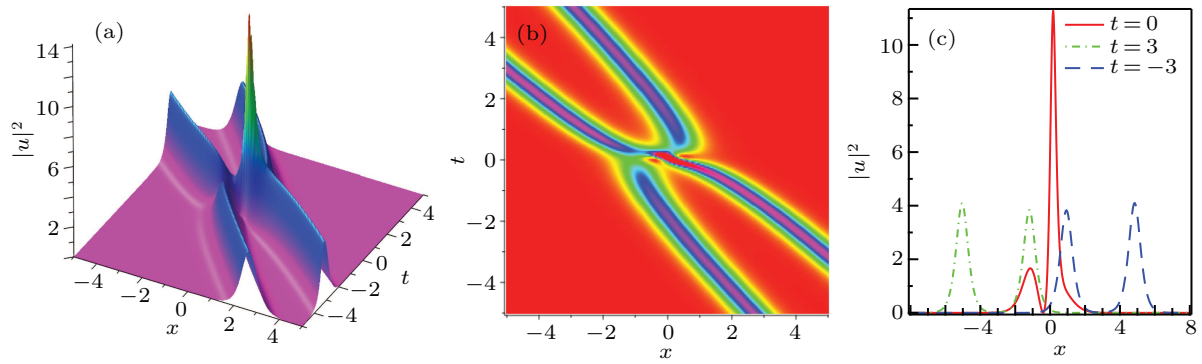


Fig. 1. The first-order soliton solution u of Eq. (2) under ZBCs, and $A(\eta_1)=1$, $B(\eta_1)=1$, $\eta_1 = 1+(1/2)\mathrm{i}$. (a) The three-dimensional plot. (b) The density plot. (c) The sectional drawings at $t=-3$ (dashed line), $t=0$ (solid line), and $t=3$ (dot-dashed line).

4. Summary and discussion

In this work, we mainly investigate a new Schrödinger-type equation (NLST) with double zeros of analytical scattering coefficients, which is regarded as a new extension of the classical NLS equation, under ZBCs via the RH approach. For the direct scattering problem and the inverse scattering problem, we not only discuss the key properties including the analyticity, symmetries, asymptotic behaviors and discrete spectrum, but also construct the matrix RH problem via analyzing the spectral problem of the lax pair, and the reconstruction formula of potential, and trace formula are also derived correspondingly. Therefore, the expression of N double-pole soliton solution for the NLST equation (2) is given. Furthermore, an example is given when taking parameters $A(\eta_1)=1$, $B(\eta_1)=1$, $\eta_1 = 1+(1/2)\mathrm{i}$, and the solution is also analyzed in detail. In our future work, we will consider this equation under nonzero boundary conditions via the RH approach, and study the long-time asymptotic behavior of the solution for the equation via the Deift–Zhou nonlinear steepest descent method. Moreover, the mathematical structure and physical properties of this equation are also necessarily studied.

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References

[1] Lax P D 1968 *Commun. Pure Appl. Math.* **21** 467
[2] Ablowitz M J and Fokas A S 2003 *Complex Variables: Introduction and Applications*, (Cambridge: Cambridge University Press)
[3] Hasegawa A and Tappert F 1973 *Appl. Phys. Lett.* **23** 142
[4] Hasegawa A and Kodama Y 1981 *Proc. IEEE* **69** 1145
[5] Hasegawa A and Kodama Y 1995 *Solitons in Optical Communication* (Oxford: Oxford University Press)
[6] Turitsyn S K, Prilepsky J E, Le S T, Wahls S, Frumin L L, Kamalian M and Derevyanko S A 2017 *Optica* **4** 307
[7] Biswas A 2005 *Opt. Quantum Electron.* **37** 649
[8] Agalarov A, Zhulego V and Gadzhimuradov T 2015 *Phys. Rev. E* **91** 042909
[9] Novikov S, Manakov S V, Pitaeskii L P and Zakharov V E 1984 *Theory of soliton: the inverse scattering method* (Berlin: Springer Science and Business Media)
[10] Coifman R R and Beals R 1984 *Commun. Pure Appl. Math.* **37** 39
[11] Zhang G Q and Yan Z Y 2020 *Physica D* **410** 132521
[12] Peng W Q and Chen Y 2022 *J. Math. Phys.* **63** 033502
[13] Zhang G Q, Chen S Y and Yan Z Y 2020 *Commun. Nonlinear Sci. Numer. Simul.* **80** 104927
[14] Pu J C and Chen Y 2021 arXiv:2105.06098 [nlin.SI]
[15] Zhu J Y and Chen Y 2022 *J. Math. Phys.* **62** 123501
[16] Zhang G Q and Yan Z Y 2020 *J. Nonlinear Sci.* **30** 3089
[17] Zhang X F, Tian S F and Yang J J 2021 *Anal. Math. Phys.* **11** 1
[18] Wang X B and Han B 2020 *J. Math. Anal. Appl.* **487** 123968
[19] Yang B and Chen Y 2019 *Nonlinear Anal.-Real World Appl.* **45** 918
[20] Weng W F and Yan Z Y 2021 *Mod. Phys. Lett. B* **35** 2150483
[21] Zhang Z Z anf Fan E G 2021 *Z. Angew. Math. Phys.* **72** 153
[22] Mao J J, Tian S F, Xu T Z and Shi L F 2021 *Nonlinear Dyn.* **104** 2639
[23] Zhang Y S, Tao X X, Yao T T and He J S 2020 *Stud. Appl. Math.* **145** 812
[24] Zakharov V E 1968 *J. Appl. Mech. Tech. Phys.* **9** 190
[25] Johnson R S and Stewartson K 1977 *Proc. R. Soc. Lond. A* **357** 131
[26] Subhadarshan S and Santanu S R 2017 *J. Appl. Anal. Comput.* **7** 824
[27] Clarkson P A and Tuszynski J A 1990 *J. Phys. A: Math. Gen.* **23** 4269
[28] Mjølhus E 1976 *J. Plasma Phys.* **16** 321
[29] Chen H H, Lee Y C and Liu C S 1979 *Phys. Scr.* **20** 490
[30] Gerdjikov V S and Ivanov I 1983 *Bulg. J. Phys.* **2** 130
[31] Kaup D J and Newell A C 1978 *J. Math. Phys.* **19** 798
[32] Hou Y, Fan E G and Zhao P 2013 *J. Math. Phys.* **54** 073505
[33] Zhao P and Fan E G 2020 *Physica D* **402** 132213
[34] Nie H, Zhu J Y and Geng X G 2018 *Anal. Math. Phys.* **8** 415
[35] Takaoka H 2021 *J. Differ. Equ.* **291** 90
[36] Wadati M, Konno K and Ichikawa Y 1979 *J. Phys. Soc. Jpn.* **46** 1965
[37] Lin Y, Fang Y and Dong H 2019 *Math. Probl. Eng.* **2019** 4058041
[38] Zhang B and Fan E G 2021 *Mod. Phys. Lett. B* **35** 2150208
[39] Zhou X 1989 *Commun. Pure Appl. Math.* **42** 895
[40] Biondini G and Kovačič G 2014 *J. Math. Phys.* **55** 031506