

Quantum speed limit for mixed states in a unitary system

Jie-Hui Huang(黄接辉), Li-Guo Qin(秦立国), Guang-Long Chen(陈光龙), Li-Yun Hu(胡利云), and Fu-Yao Liu(刘福窑)

Citation: Chin. Phys. B, 2022, 31 (**11**): 110307. DOI: 10.1088/1674-1056/ac76b4

What follows is a list of articles you may be interested in

Quantum speed limit of the double quantum dot in pure dephasing environment under measurement

Zhenyu Lin(林振宇), Tian Liu(刘天), Zongliang Li(李宗良), Yanhui Zhang(张延惠), and Kang Lan(蓝康)
Chin. Phys. B, 2022, 31 (**7**): 070307. DOI: 10.1088/1674-1056/ac4bd4

Quantum speed limit for the maximum coherent state under the squeezed environment

Kang-Ying Du(杜康英), Ya-Jie Ma(马雅洁), Shao-Xiong Wu(武少雄), and Chang-Shui Yu(于长水)
Chin. Phys. B, 2021, 30 (**9**): 090308. DOI: 10.1088/1674-1056/ac0daf

Influences of spin-orbit interaction on quantum speed limit and entanglement of spin qubits in coupled quantum dots

M Bagheri Harouni
Chin. Phys. B, 2021, 30 (**9**): 090301. DOI: 10.1088/1674-1056/abeef4

Coherent-driving-assisted quantum speedup in Markovian channels

Xiang Lu(鹿翔), Ying-Jie Zhang(张英杰), and Yun-Jie Xia(夏云杰)
Chin. Phys. B, 2021, 30 (**2**): 020301. DOI: 10.1088/1674-1056/abc150

Margolus-Levitin speed limit across quantum to classical regimes based on trace distance

Shao-Xiong Wu(武少雄), Chang-Shui Yu(于长水)
Chin. Phys. B, 2020, 29 (**5**): 050302. DOI: 10.1088/1674-1056/ab7dab

---

Quantum speed limit for mixed states in a unitary system

Jie-Hui Huang(黄接辉)[1,2,†], Li-Guo Qin(秦立国)[1], Guang-Long Chen(陈光龙)[1], Li-Yun Hu(胡利云)[2,‡], and Fu-Yao Liu(刘福窑)[1,§]

[1] *School of Mathematics, Physics and Statistics, Shanghai University of Engineering Science, Shanghai* 201620, *China*
[2] *College of Physics and Communication Electronics, Jiangxi Normal University, Nanchang* 330022, *China*

(Received 23 January 2022; revised manuscript received 17 May 2022; accepted manuscript online 8 June 2022)

Since the evolution of a mixed state in a unitary system is equivalent to the joint evolution of the eigenvectors contained in it, we could use the tool of instantaneous angular velocity for pure states to study the quantum speed limit (QSL) of a mixed state. We derive a lower bound for the evolution time of a mixed state to a target state in a unitary system, which automatically reduces to the quantum speed limit induced by the Fubini–Study metric for pure states. The computation of the QSL of a degenerate mixed state is more complicated than that of a non-degenerate mixed state, where we have to make a singular value decomposition (SVD) on the inner product between the two eigenvector matrices of the initial and target states. By combing these results, a lower bound for the evolution time of a general mixed state is presented. In order to compare the tightness among the lower bound proposed here and lower bounds reported in the references, two examples in a single-qubit system and in a single-qutrit system are studied analytically and numerically, respectively. All conclusions derived in this work are independent of the eigenvalues of the mixed state, which is in accord with the evolution properties of a quantum unitary system.

Keywords: quantum speed limit, instantaneous angular velocity, singular value decomposition

PACS: 03.65.–w, 03.65.Fd, 03.65.Ta **DOI:** 10.1088/1674-1056/ac76b4

1. Introduction

As an important concept in fundamental physics, quantum speed limits (QSLs) impose lower bounds on the evolution time required for a quantum system to evolve between two distinguishable states,[1] and the time–energy uncertainty principle is one of its possible physical interpretations.[2] QSL plays an important role in quantum information processing, which could be used to infer the maximal rate of information transfer,[3,4] analyze the scrambling of information,[5] improve measurement precision in quantum metrology,[6–8] estimate computational capacity,[9,10] explore quantum optimal control,[11–17] investigate entropy production rate in non-equilibrium quantum processes,[18,19] enhance the charging power of quantum batteries,[20–22] and so on. Owing to its significance in both domains of theoretical physics and promising applications in quantum information processing, the issue of QSL is attracting more and more attention in recent years. In a seminal work, Mandelstam and Tamm (MT) derived a lower bound, based on the Heisenberg uncertainty relation between the Hamiltonian and the projection operator to the initial state, on the transition time required for a pure state to naturally evolve to an orthogonal state,[2] $\tau^{(\mathrm{MT})} = \frac{\hbar\pi}{2\Delta E}$, with ΔE the energy spread of the quantum system under consideration. Here we see that energy spread of a quantum system takes effect on its evolution speed. Over half century later, Margolus and Levitin (ML) studied the same problem by using a different algebra method, and another lower bound on the evolution time between two orthogonal pure states is derived in terms of average energy $\bar{E}$,[23] $\tau^{(\mathrm{ML})} = \frac{\hbar\pi}{2\bar{E}}$, where the ground state energy of the quantum system is set zero for simplicity, $E_0 = 0$, and hereafter the average energy $\bar{E}$ should be replaced by $\bar{E} - E_0$ in the case of nonzero ground state energy. The average energy of the system plays another role of affecting the system's evolution speed. Therefore, a unified bound is formed for QSL of unitary dynamics by combining these two independent results as $\tau = \max\{\frac{\hbar\pi}{2\Delta E}, \frac{\hbar\pi}{2\bar{E}}\}$.[24] Based on the changing rate of phase of a quantum system, another unified QSL bound is recently proposed in a mathematical way.[25] The MT bound and ML bound are generalized to $\tau^{(\mathrm{MT})} = \frac{\hbar\Theta}{\Delta E}$ [26] and $\tau^{(\mathrm{ML})} = \frac{\hbar\Theta}{\bar{E}} \tan\frac{\Theta}{2}$,[27] respectively, for the unitary evolution between two nonorthogonal pure states, where the angle Θ is the Fubini–Study angle between them.[28–30]

Based on the fact that the geodesic length between two distinguishable pure states is no longer than the length of any other path connecting the two states, the MT bound could be generalized to a time-dependent quantum system.[31] Generalizations of the MT bound to the mixed state have also been extensively explored in the literatures.[32–40] In the last two decades, the discussion on the "minimum evolution time of a quantum system" has been extended to the damped Jaynes–Cummings model and the Ohimc-like dephasing model,[41–43] the driven

†Corresponding author. E-mail: huangjh@sues.edu.cn
‡Corresponding author. E-mail: hlyun@jxnu.edu.cn
§Corresponding author. E-mail: liu-fuyao@163.com

© 2022 Chinese Physical Society and IOP Publishing Ltd

systems,[44–47] open systems,[48–57] phase space,[58,59] quantum many-body systems,[60,61] thermal states,[62] the generation of quantumness,[63] the regimes across quantum to classical,[64] the non-Hermitian systems,[65] and so on. Some related results will be introduced briefly in the next section.

In this paper, we use the tool of instantaneous angular velocity for pure states to study the evolution time of a mixed state, and a lower bound for the evolution time of a mixed state to a target state in a unitary system is derived. This lower bound is tight, and automatically reduces to the quantum speed limit induced by the Fubini–Study metric for pure states. The situation becomes more complicated if a degenerate mixed state is under consideration. In order to study the QSL of a degenerate mixed state with equal nonzero eigenvalues, we have to make a singular value decomposition (SVD) on the inner product between the two eigenvector matrices of the initial and target state. By combing these results, we present a lower bound for the evolution time of a general mixed state. Furthermore, two examples are presented analytically and numerically to compare the tightness among the lower bound presented here and those reported in the references. All conclusions derived in this work are independent of the eigenvalue of the mixed state, which is in accord with the evolution properties of a quantum unitary system.

2. QSL time for a mixed state: a brief introduction

A usual approach for constructing a QSL is based on the geometric property of a quantum state. That is, the length of the geodesic path connecting two states is no longer than the length of any other path. By introducing a distance $\mathcal{D}(\rho_0,\rho_t)$ between the initial and final density matrices, and the "maximum speed of evolution" $Q(t)\equiv(1/t)\int_0^t\sqrt{\mathcal{F}(t')}\,\mathrm{d}t'$ with $\mathcal{F}(t')$ being a metric in the space of density matrices induced by $\mathcal{D}$,[66] the QSL time is then formulated as $\tau=\mathcal{D}(\rho_0,\rho_t)/Q(t)$. This approach is widely used in the issue of QSL in quantum unitary and open systems, and as well in dynamics of classical systems.[67,68] In this section, we briefly introduce three lower bounds for the evolution time of a mixed state in a unitary system, which were reported in more details in Ref. [69].

Combining the MT bound and the ML bound leads to a unified QSL time.[24] It could be extended to the mixed state case by defining the distance between the initial state ρ_0 and the final state ρ_t as their Bures angle $\mathcal{L}(\rho_0,\rho_t)=\arccos(\mathcal{F}(\rho_0,\rho_t))$,[70] with $\mathcal{F}(\rho_0,\rho_t)=\mathrm{tr}[\sqrt{\sqrt{\rho_0}\rho_t\sqrt{\rho_0}}]$ being the Uhlmann root fidelity,[71,72]

$$\tau_{\mathcal{L}}(\rho_0,\rho_t)=\frac{\mathcal{L}(\rho_0,\rho_t)}{\min\{\bar{E},\Delta E\}},\tag{1}$$

where we have set $\hbar=1$ hereafter for simplicity. It bounds the evolution time required to evolve the mixed state ρ_0 to the final state ρ_t by means of a unitary operator U_t, i.e., $\rho_t=U_t\rho_0U_t^\dagger$, where the quantum system is governed by a time-dependent Hamiltonian H_t. Here the average energy is defined as $\bar{E}=(1/t)\int_0^t(\mathrm{tr}[\rho_{t'}H_{t'}]-h_{t'}^{(0)})\,\mathrm{d}t'$, with $h_{t'}^{(0)}$ being the ground state energy of $H_{t'}$, and the average energy spread is $\Delta E=(1/t)\int_0^t\sqrt{\mathrm{tr}[\rho_{t'}H_{t'}^2]-\mathrm{tr}[\rho_{t'}H_{t'}]^2}\,\mathrm{d}t'$.

By defining the distance between two mixed states ρ_0 and ρ_t through the angle between their generalized Bloch vectors,[69] which is equivalent to $\Theta(\rho_0,\rho_t)=\arccos\big((\mathrm{tr}[\rho_0\rho_t]-1/N)/(\mathrm{tr}[\rho_0^2]-1/N)\big)$ with N being the dimensionality, it is proven that the evolution time required to evolve from state ρ_0 to state ρ_t by means of a unitary operation is also bounded from below by

$$\tau_\Theta(\rho_0,\rho_t)=\frac{\Theta(\rho_0,\rho_t)}{Q_\Theta},\tag{2}$$

where the denominator

$$Q_\Theta=\frac{1}{t}\int_0^t\sqrt{\frac{2\,\mathrm{tr}[\rho_{t'}^2H_{t'}^2-(\rho_{t'}H_{t'})^2]}{\mathrm{tr}[\rho_{t'}^2]-1/N}}\,\mathrm{d}t'$$

could be considered as the maximum evolution speed. This bound matches the first one in Eq. (1) for all pure states in qubit systems ($N=2$), where the Bures angle reduces to the Fubini–Study distance.[28–30]

If the distance between two mixed states ρ_0 and ρ_t is defined as

$$\Phi(\rho_0,\rho_t)=\arccos\left(\sqrt{\frac{\mathrm{tr}[\rho_0\rho_t]}{\mathrm{tr}[\rho_0^2]}}\right),$$

and the "average speed of evolution" turns to

$$Q_\Phi=\frac{1}{t}\int_0^t\sqrt{\frac{\mathrm{tr}[\rho_{t'}^2H_{t'}^2-(\rho_{t'}H_{t'})^2]}{\mathrm{tr}[\rho_{t'}^2]}}\,\mathrm{d}t'$$

accordingly, then the third lower bound for the minimal time required to evolve from state ρ_0 to state ρ_t by means of a unitary operation is derived,

$$\tau_\Phi(\rho_0,\rho_t)=\frac{\Phi(\rho_0,\rho_t)}{Q_\Phi}.\tag{3}$$

This bound matches the MT bound for pure states in all unitary systems, because the above distance $\Phi(\rho_0,\rho_t)$ reduces to the Fubini–Study distance in the case of pure states regardless of the dimension of the system, and the above "average speed of evolution" Q_Φ reduces to the energy spread ΔE. It is shown that the two bounds τ_Θ and τ_Φ in Eqs. (2) and (3) are easy to compute and tighter than the bound $\tau_{\mathcal{L}}$ in Eq. (1) in most cases.[69]

3. QSL time for a mixed state

A quantum system governed by a time-dependent Hamiltonian H_t would undergo a unitary evolution, and the state at

time t is connected to the initial one through a unitary transformation U_t, i.e., $\rho_t = U_t\rho_0 U_t^\dagger$, with $U_t = \exp(-\mathrm{i}\int_0^t H_{t'}\,\mathrm{d}t')$. Since the unitary transformation of a matrix does not change its eigenvalues, the time evolution ρ_t has the same spectrum with the initial state ρ_0. Furthermore, the eigenvectors of a density matrix associated with distinct eigenvalues are independent to each other, so the evolution time of a mixed state in a unitary quantum system is just the evolution time of each eigenvector of the mixed state, which provides us a possible way to study the QSL of a mixed state through the evolution time of its eigenvectors. In this section, we present the main results of the QSL for a mixed state in two different circumstances, a non-degenerate mixed state and a degenerate mixed state with equal nonzero eigenvalues. The QSL of a general mixed state could be obtained by combining the two results.

3.1. QSL time for a non-degenerate mixed state

The eigenvectors of a non-degenerate density matrix have a one-to-one correspondence to its eigenvalues. Just as we mentioned above, the evolution of the eigenvectors of a mixed state are independent to each other, so the evolution time of a mixed state in a unitary quantum system is exactly the evolution time of each eigenvector of the mixed state. In a recent work, the concept of instantaneous angular velocity of quantum evolution is proposed for a pure state.[27] It is found that the quantum evolution of a pure state has two "directions" at any moment, one is associated with the fidelity change of the state at this moment, and the other one is related to the state's phase change. The instantaneous angular velocity of the quantum evolution associated with the fidelity change is determined by the time-dependent energy spread $\Delta E_t = \sqrt{\langle H_t^2\rangle - \langle H_t\rangle^2}$ through the simple relation $\omega_t = \Delta E_t/\hbar$. If the time evolution of the state is always inside the plane spanned by the initial vector and the target vector, the quantum system undergoes the fastest evolution, which corresponds to the geodesic path in the geometric description of the quantum evolution. That is to say, the minimum time $\tau_{\min}$ required for a quantum system to evolve from the initial state vector $|\psi_0\rangle$ to the target state vector $|\psi_\tau\rangle$ satisfies the following equation:

$$\int_0^{\tau_{\min}} \Delta E_t\,\mathrm{d}t = \Theta(|\psi_0\rangle, |\psi_\tau\rangle), \tag{4}$$

where $\Theta(|\psi_0\rangle, |\psi_\tau\rangle) = \arccos|\langle\psi_0|\psi_\tau\rangle|$ is the Fubini–Study angle between the two state vectors $|\psi_0\rangle$ and $|\psi_\tau\rangle$, and the real evolution time plays no role here. Although it is hard to work out an analytical solution for $\tau_{\min}$ based on this equation, due to the possibly irregular distribution of the energy spread ΔE_t as a function of time and the integral included, its numerical result is not hard to obtain, and this equation is very useful to estimate the evolution time of a quantum system when the real evolution time is unknown to us. On the contrary, if the real evolution time τ is given, the minimum time $\tau_{\min}$ required to evolve between the two state vectors $|\psi_0\rangle$ and $|\psi_\tau\rangle$ could also be expressed as

$$\tau_{\min}(|\psi_0\rangle, |\psi_\tau\rangle) = \frac{\Theta(|\psi_0\rangle, |\psi_\tau\rangle)}{\frac{1}{\tau}\int_0^\tau \Delta E_t\,\mathrm{d}t}. \tag{5}$$

This definition follows the same idea used in the above lower bounds $\tau_{\mathcal{L}}$, τ_θ and τ_ϕ in Eqs.(1), (2) and (3), respectively. If the unitary system is governed by a time-independent Hamiltonian H, the energy spread ΔE_t becomes a constant ΔE, then the above minimum time $\tau_{\min}$ has a simpler expression

$$\tau_{\min}(|\psi_0\rangle, |\psi_\tau\rangle) = \frac{\Theta(|\psi_0\rangle, |\psi_\tau\rangle)}{\Delta E}. \tag{6}$$

Since the Fubini–Study angle Θ measures the geodesic "distance" between the initial state vector $|\psi_0\rangle$ and its time evolution $|\psi_t\rangle$, the minimum time $\tau_{\min}$ defined in the above Eq. (5) or (6) is a lower bound of the real evolution time between the initial state $|\psi_0\rangle$ and its target state $|\psi_\tau\rangle$.

For a mixed state without degenerate eigenvalue, the time evolution of each eigenvector brings out a minimum time according to the above Eq. (5) or (6). Noting that the evolution time of a mixed state in a unitary quantum system is exactly the evolution time of each eigenvector of the mixed state, every minimum time obtained in this way is a lower bound of the real evolution time, so the real evolution time for a non-degenerate mixed state is bounded from below by the largest one among these minimum time,

$$\tau_{\min}(\rho_0, \rho_\tau) = \max_i\{\tau_{\min}(|\psi_0^{(i)}\rangle, |\psi_\tau^{(i)}\rangle)\}, \tag{7}$$

where $\tau_{\min}(|\psi_0^{(i)}\rangle, |\psi_\tau^{(i)}\rangle)$ is obtained by substituting the Fubini–Study angle $\Theta(|\psi_0^{(i)}\rangle, |\psi_\tau^{(i)}\rangle) = \arccos|\langle\psi_0^{(i)}|\psi_\tau^{(i)}\rangle|$ between the eigenvectors $|\psi_0^{(i)}\rangle$ and $|\psi_\tau^{(i)}\rangle$ of ρ_0 and ρ_τ associated with the same eigenvalue λ_i, into Eq. (5) or (6). In the calculation of $\tau_{\min}(|\psi_0^{(i)}\rangle, |\psi_\tau^{(i)}\rangle)$ in the above equation, the energy spread $\Delta E_t = \sqrt{\langle H_t^2\rangle - \langle H_t\rangle^2}$ involved in Eq. (5) is here imposed on the time evolution $|\psi_t^{(i)}\rangle$. That is to say, the energy spread ΔE_t used in the calculation of $\tau_{\min}(|\psi_0^{(i)}\rangle, |\psi_\tau^{(i)}\rangle)$ stands for the energy band of the time evolution of the i-th eigenvector $|\psi_0^{(i)}\rangle$, rather than the mixed state ρ_0 itself. In fact, the energy spread of the mixed state plays no role in the lower bound proposed here, which is different from some previous results.

3.2. QSL time for a degenerate mixed state with equal nonzero eigenvalues

Suppose an N-dimensional quantum system is initially prepared in a degenerate mixed state whose K ($2 \le K \le N$) nonzero eigenvalues are all equal to $\lambda = 1/K$. The unitary evolution of this system would keep the degeneracy of

the eigenvalue and the structure of the eigenspace invariant. Let $\{|\psi_0^{(i)}\rangle,\ i=1,2,3,\ldots,K\}$ be one set of eigenvectors of the initial mixed state ρ_0, satisfying the normalization condition $\rho_0 = 1/K\sum_i^K |\psi_0^{(i)}\rangle\langle\psi_0^{(i)}|$ and the orthogonality condition $\langle\psi_0^{(i)}|\psi_0^{(j)}\rangle = \delta_{ij}$. Now we align these column vectors to form an $N\times K$ eigenvector matrix $R = (|\psi_0^{(1)}\rangle, |\psi_0^{(2)}\rangle, \ldots, |\psi_0^{(K)}\rangle)$, where only the eigenvectors associated with nonzero eigenvalues are included, and we have $\rho_0 = RR^\dagger/K$. Each eigenvector matrix corresponds to a decomposition of a degenerate mixed state with equal nonzero eigenvalues. Given one eigenvector matrix R, we can obtain another eigenvector matrix, $R' = (|\psi'^{(1)}_0\rangle, |\psi'^{(2)}_0\rangle, \ldots, |\psi'^{(K)}_0\rangle)$, which satisfies $\rho_0 = R'R'^\dagger/K$ too, through the simple matrix product $R' = RS$, with S being an arbitrary $K\times K$ unitary matrix. Accordingly, given an eigenvector matrix $M = (|\psi_\tau^{(1)}\rangle, |\psi_\tau^{(2)}\rangle, \ldots, |\psi_\tau^{(K)}\rangle)$ of the target mixed state ρ_τ, which also has K equal nonzero eigenvalue $\lambda = 1/K$, another eigenvector matrix of ρ_τ, $M' = (|\psi'^{(1)}_\tau\rangle, |\psi'^{(2)}_\tau\rangle, \ldots, |\psi'^{(K)}_\tau\rangle)$, satisfying $\rho_\tau = MM^\dagger/K = M'M'^\dagger/K$, could be expressed as $M' = MT$, with T being a $K\times K$ unitary matrix. In a word, based on an arbitrary eigenvector matrix of the mixed state ρ_0 (or ρ_τ), each $K\times K$ unitary matrix S (or T) corresponds to a decomposition of the state ρ_0 (or ρ_τ).

The inner products between the eigenvectors of the initial mixed state ρ_0 and the corresponding eigenvectors of the target mixed state ρ_τ, are just the diagonal elements of the matrix product $R'^\dagger M' = S^\dagger R^\dagger MT$, with $(R'^\dagger M')_{ii} = \langle\psi'^{(i)}_0|\psi'^{(i)}_\tau\rangle$. Now we choose a particular decomposition of ρ_0 and ρ_τ, so that the matrix product $R'^\dagger M'$ is a non-negative diagonal matrix, which means a SVD is made on the matrix $R^\dagger M$ through the two $K\times K$ unitary matrices S and T. That is to say, for the initial state ρ_0 and the target state ρ_τ sharing the same eigenvalue spectrum, we can always find two particular decompositions, so that the inner products between their pairwise eigenvectors, $\langle\psi'^{(i)}_0|\psi'^{(i)}_\tau\rangle$, are equal to the singular values of $R'^\dagger M'$, i.e., the eigenvalues of $\sqrt{R'^\dagger M'M'^\dagger R'}$. Noting that $\rho_0 = R'R'^\dagger/K$ and $\rho_\tau = M'M'^\dagger/K$, the matrix $\sqrt{R'^\dagger M'M'^\dagger R'}$ and the matrix $K\sqrt{\sqrt{\rho_0}\rho_\tau\sqrt{\rho_0}}$ have identical nonzero eigenvalues, so the average value of the above inner products, $\frac{1}{K}\sum_i^K (R'^\dagger M')_{ii} = \frac{1}{K}\mathrm{tr}(R'^\dagger M')$, is equal to the Uhlmann root fidelity[71,72] between ρ_0 and ρ_τ, $\mathcal{F}(\rho_0,\rho_\tau) = \mathrm{tr}[\sqrt{\sqrt{\rho_0}\rho_\tau\sqrt{\rho_0}}]$, whose arc cosine function, $\arccos(\mathcal{F})$, is a widely-used measure of the "distance" between two mixed states, and we have $\frac{1}{K}\mathrm{tr}(R'^\dagger M') = \mathrm{tr}[\sqrt{\sqrt{\rho_0}\rho_\tau\sqrt{\rho_0}}]$. Thus the two decompositions containing R' and M' are the closest among all decompositions of ρ_0 and ρ_τ.

Based on the above analysis, we now focus on the particular decomposition $R' = (|\psi'^{(1)}_0\rangle, |\psi'^{(2)}_0\rangle, \ldots, |\psi'^{(K)}_0\rangle)$ of the initial degenerate state ρ_0 with equal nonzero eigenvalues, and discuss the evolution of these column vectors of R'. Provided the target state ρ_τ in a unitary system, since the decomposition $M' = (|\psi'^{(1)}_\tau\rangle, |\psi'^{(2)}_\tau\rangle, \ldots, |\psi'^{(K)}_\tau\rangle)$ satisfying $\frac{1}{K}\mathrm{tr}(R'^\dagger M') = \mathrm{tr}[\sqrt{\sqrt{\rho_0}\rho_\tau\sqrt{\rho_0}}]$, is closest to the decomposition R', the potential evolution time between the two decompositions R' and M' is smaller than others. In other words, the minimum time it takes for the state vectors $|\psi'^{(i)}_0\rangle$ $(i=1,2,\ldots,K)$ to simultaneously reach new vectors whose fidelities with the corresponding state vector, $\langle\psi'^{(i)}_0|\psi'^{(i)}_\tau\rangle$ $(i=1,2,\ldots,K)$, has a one-to-one correspondence with the nonzero eigenvalues of $K\sqrt{\sqrt{\rho_0}\rho_\tau\sqrt{\rho_0}}$, bounds the evolution time for the initial state ρ_0 to evolve to the target state ρ_τ. Now we can use the minimum evolution time for a pure state in Eq. (5) or (6) to describe the minimum evolution time of a mixed state ρ_0 to the target state ρ_τ, which is

$$\tau_{\min}(\rho_0,\rho_\tau) = \max_i\{\tau_{\min}(|\psi'^{(i)}_0\rangle, |\psi'^{(i)}_\tau\rangle)\}, \tag{8}$$

where $\tau_{\min}(|\psi'^{(i)}_0\rangle, |\psi'^{(i)}_\tau\rangle)$ is the minimum time required for the column vector $|\psi'^{(i)}_0\rangle$ in R' to evolve to the column vector $|\psi'^{(i)}_\tau\rangle$ in M', obtained by substituting the Fubini–Study angle $\Theta(|\psi'^{(i)}_0\rangle, |\psi'^{(i)}_\tau\rangle) = \arccos|\langle\psi'^{(i)}_0|\psi'^{(i)}_\tau\rangle|$ and the energy spread $\Delta E_t^{(i)} = \sqrt{\langle H_t^2\rangle - \langle H_t\rangle^2}$ imposed on the time evolution of $|\psi'^{(i)}_0\rangle$ into Eq. (5) or (6).

3.3. QSL time for a general mixed state

QSL for a general mixed state in a unitary system could be obtained by combining the above conclusions in subsections A and B. In an N-dimensional unitary system, an arbitrary initial state ρ_0 with rank $K \le N$ and its target state ρ_τ have the same eigenvalue spectrum and eigenspace structure. We rewrite the two mixed states ρ_0 and ρ_τ as their eigenvalue decomposition, $\rho_0 = U\Lambda U^\dagger$ and $\rho_\tau = V\Lambda V^\dagger$, where the two unitary matrices U and V are formed by the eigenvectors of ρ_0 and ρ_τ, and the diagonal matrix Λ is composed of the eigenvalues of ρ_0 (or ρ_τ). Every non-degenerate eigenvalue of ρ_0 or ρ_τ contributes a distinct eigenvector as a column vector to U or V. However, if some of the column vectors of U and V, e.g., $|\psi_0^{(j_1)}\rangle, |\psi_0^{(j_2)}\rangle, \ldots, |\psi_0^{(j_l)}\rangle$ in U and $|\psi_\tau^{(j_1)}\rangle, |\psi_\tau^{(j_2)}\rangle, \ldots, |\psi_\tau^{(j_l)}\rangle$ in V, belong to a degenerate eigenvalue λ_j of multiplicity l, which means the choice of these column vectors is not unique, then we set an extra requirement for these column vectors, the inner product of the two eigenvector matrices $U'^\dagger_{\lambda_j} V'_{\lambda_j}$ with $U'_{\lambda_j} = (|\psi'^{(j_1)}_0\rangle, |\psi'^{(j_2)}_0\rangle, \ldots, |\psi'^{(j_l)}_0\rangle)$ and $V'_{\lambda_j} = (|\psi'^{(j_1)}_\tau\rangle, |\psi'^{(j_2)}_\tau\rangle, \ldots, |\psi'^{(j_l)}_\tau\rangle)$, has to be a non-negative diagonal matrix, which actually implies these eigenvectors satisfy the orthogonality condition, $\langle\psi'^{(j_m)}_0|\psi'^{(j_n)}_\tau\rangle = 0$ for $j_m \ne j_n$.

The two particular eigenvector matrices U'_{λ_j} and V'_{λ_j} could be obtained through the SVD decomposition on the inner product of $U^\dagger_{\lambda_j} V_{\lambda_j}$, with U_{λ_j} (and V_{λ_j}) being an eigenvector matrix of ρ_0 (and ρ_τ) with respect to the eigenvalue λ_j. If the SVD decomposition is expressed as $U^\dagger_{\lambda_j} V_{\lambda_j} = S_{\lambda_j}\Omega_{\lambda_j} T^\dagger_{\lambda_j}$, with the singular values being contained in the diagonal matrix Ω_{λ_j}, then we have $U'_{\lambda_j} = U_{\lambda_j} S_{\lambda_j}$ and $V'_{\lambda_j} = V_{\lambda_j} T_{\lambda_j}$.

Through the two particular eigenvector matrices U' and V', with $U' = (|\psi_0'^{(1)}\rangle, |\psi_0'^{(2)}\rangle, \ldots, |\psi_0'^{(K)}\rangle)$ and $V' = (|\psi_\tau'^{(1)}\rangle, |\psi_\tau'^{(2)}\rangle, \ldots, |\psi_\tau'^{(K)}\rangle)$, the evolution time for the quantum unitary evolution from the initial mixed state ρ_0 to the target state ρ_τ is bounded from below by the largest one among the minimum time for the state vectors $|\psi_0'^{(i)}\rangle$ to evolve to $|\psi_\tau'^{(i)}\rangle$. That is to say, the lower bound for the evolution time in Eq. (8) is also valid for a general mixed state, where the state vectors $|\psi_0'^{(i)}\rangle$ and $|\psi_\tau'^{(i)}\rangle$ are the eigenvectors in the two closest decompositions of ρ_0 and ρ_τ.

The SVD decomposition makes it more complicated to compute the QSL time of a degenerate mixed state than a non-degenerate mixed state. The lower bound for the evolution time of a mixed state proposed here is independent on the eigenvalues of the system, which is in agree with the dynamical properties of a quantum unitary system. Furthermore, it will be shown in the next section that this lower bound is very tight.

4. Results and discussion

4.1. Analytical results in a time-independent single-qubit system

Let us start this section with a simple single-qubit example. In the basis composed of the eigenvectors of the time-independent Hamiltonian $\mathcal{H}$, which has a diagonal form $\mathcal{H} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in this particular basis, an initial state $\rho_0 = \begin{pmatrix} \frac{1}{2}+\lambda & \eta^* \\ \eta & \frac{1}{2}-\lambda \end{pmatrix}$ evolves to a target state $\rho_\tau = \begin{pmatrix} \frac{1}{2}+\lambda & \eta^* e^{i(b-a)\tau} \\ \eta e^{i(a-b)\tau} & \frac{1}{2}-\lambda \end{pmatrix}$ at the real evolution time τ. To ensure the non-negativity of the density matrix ρ_0, the two parameters λ and η should satisfy the conditions $\lambda \in [-0.5, 0.5]$ and $|\eta|^2 \leq \frac{1}{4} - \lambda^2$. Now we could calculate the three lower bounds $\tau_\mathcal{L}$, τ_θ and τ_ϕ in Eqs. (1), (2) and (3) of the evolution time τ,

$$\tau_\mathcal{L} = \frac{1}{\sqrt{\frac{1}{4}-\lambda^2}} \arccos\sqrt{1+\frac{2\lambda^3}{\gamma}-2\lambda^2}, \tag{9a}$$

$$\tau_\theta = \frac{\gamma}{|\eta|} \arccos\left(1-\frac{2|\eta|^2}{\gamma^2}\sin^2\frac{\tau}{2}\right), \tag{9b}$$

$$\tau_\phi = \frac{\sqrt{1+4\gamma^2}}{\sqrt{2}|\eta|} \arccos\sqrt{1-\frac{8|\eta|^2}{1+4\gamma^2}\sin^2\frac{\tau}{2}}, \tag{9c}$$

where the parameter $\gamma = \sqrt{\lambda^2+|\eta|^2} \in [0, 1/2]$ and all involved time $\tau_\mathcal{L}$, τ_θ, τ_ϕ or τ is in units of $1/(a-b)$, or equivalently speaking, the condition $a-b=1$ is assumed for the Hamiltonian $\mathcal{H}$.

The initial mixed state ρ_0 has two eigenvectors

$$|\psi_1\rangle = \frac{1}{\sqrt{2\gamma(\lambda+\gamma)}}\begin{pmatrix} \lambda+\gamma \\ \eta \end{pmatrix},$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2\gamma(\lambda+\gamma)}}\begin{pmatrix} -\eta^* \\ \lambda+\gamma \end{pmatrix},$$

which turn to, at the time τ, the eigenvectors

$$|\psi_1'\rangle = \frac{1}{\sqrt{2\gamma(\lambda+\gamma)}}\begin{pmatrix} \lambda+\gamma \\ \eta e^{i\tau} \end{pmatrix},$$

$$|\psi_2'\rangle = \frac{1}{\sqrt{2\gamma(\lambda+\gamma)}}\begin{pmatrix} -\eta^* e^{-i\tau} \\ \lambda+\gamma \end{pmatrix}$$

of the target state ρ_τ under the action by the above Hamiltonian $\mathcal{H}$. These eigenvectors has the same energy spread $\Delta E = \frac{|\eta|}{2\gamma}$, the Uhlmann root fidelity between $|\psi_j\rangle$ and $|\psi_j'\rangle$ $(j=1,2)$ is $\mathcal{F}(|\psi_j\rangle, |\psi_j'\rangle) = \sqrt{1-\frac{|\eta|^2}{\gamma^2}\sin^2\frac{\tau}{2}}$, then the lower bound of the evolution time τ in Eq. (7) is

$$\tau_{\min} = \frac{2\gamma}{|\eta|} \arccos\sqrt{1-\frac{|\eta|^2}{\gamma^2}\sin^2\frac{\tau}{2}}. \tag{10}$$

It could be proven that the above lower bounds satisfy the following relation:

$$\tau \geq \tau_{\min} = \tau_\theta \geq \tau_\phi \geq \tau_\mathcal{L}. \tag{11}$$

It is very interesting that the lower bound $\tau_{\min}$ proposed here perfectly matches the lower bound τ_θ in a single-qubit system, although they are defined in two completely different ways. In addition, these two lower bounds $\tau_{\min}$ and τ_θ are tighter than the other two lower bounds τ_ϕ and $\tau_\mathcal{L}$ in this system. The above four lower bounds match each other for pure qubit states ($\gamma = 1/2$) or mixed state without quantum coherence[73] ($\eta = 0$). It is found that if a mixed qubit state has no quantum coherence ($\eta = 0$) in the basis composed of the eigenvectors of the time-independent Hamiltonian, it undergoes the fastest evolution, where the three lower bounds $\tau_{\min}$, τ_θ and τ_ϕ are all equal to the real evolution time τ, i.e., $\tau = \tau_{\min} = \tau_\theta = \tau_\phi$. By the way, in the trivial case of maximally mixed state with $\lambda = \eta = 0$, all the above lower bounds $\tau_\mathcal{L}$, τ_θ, τ_ϕ and $\tau_{\min}$ would present a zero result, because a maximally mixed state will not change its density matrix during a unitary evolution.

4.2. Numerical results in a qutrit or higher dimensional systems

Now we turn to a qutrit quantum system with the initial state mixed by three orthogonal pure states $|\psi_1\rangle = \frac{1}{\sqrt{6}}(1 \quad \sqrt{2} \quad \sqrt{3})^\dagger$, $|\psi_2\rangle = \frac{1}{\sqrt{6}}(-2 \quad \sqrt{2} \quad 0)^\dagger$, and $|\psi_3\rangle = \frac{1}{\sqrt{6}}(1 \quad \sqrt{2} \quad -\sqrt{3})^\dagger$ with the mixing coefficients $p_1 = \lambda$, $p_2 = \frac{1}{4}(1-\lambda)$, and $p_3 = \frac{3}{4}(1-\lambda)$, so the density matrix of this initial state could be described as $\rho_0 = \sum_i^3 p_i |\psi_i\rangle\langle\psi_i|$. Under a Hamiltonian $\mathcal{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, this state evolves to a state $\rho_\tau = \sum_i^3 p_i |\psi_i'\rangle\langle\psi_i'|$ at time τ, with $|\psi_i'\rangle = \exp(-i\tau\mathcal{H})|\psi_i\rangle$. In the following, we fix the real evolution time as $\tau = \pi/6$, and numerically investigate the varying behavior of the four lower

bounds $\tau_{\min}$, τ_θ, τ_ϕ, and $\tau_{\mathcal{L}}$ as the parameter λ changes (see Fig. 1). It is shown that the three lower bounds $\tau_{\min}$, τ_θ and τ_ϕ are close to each other in this particular case, and the lower bound $\tau_{\mathcal{L}}$ deviates far from the real evolution time $\tau = \pi/6$ in most cases of the allowed parameter λ. The lower bound $\tau_{\min}$ proposed here (see the top solid lines in Fig. 1), keeps constant and is independent on the parameter λ except for the two special cases of $\lambda = 0.2$ and $\lambda = 3/7$, which actually reflects the fact that the lower bound $\tau_{\min}$ is independent on the spectrum of the mixed state in a time-independent unitary system. In the pure state case of $\lambda = 1$, the three lower bounds $\tau_{\min}$, τ_ϕ, and $\tau_{\mathcal{L}}$ meet with each other, but they are smaller than the real evolution time at this point. Among the three lower bounds $\tau_{\min}$, τ_θ, τ_ϕ (see Fig. 1(b)), the lower bound $\tau_{\min}$ is the largest one in this case, and τ_θ is not always larger than τ_ϕ, which is in agree with the statement presented in Ref. [69].

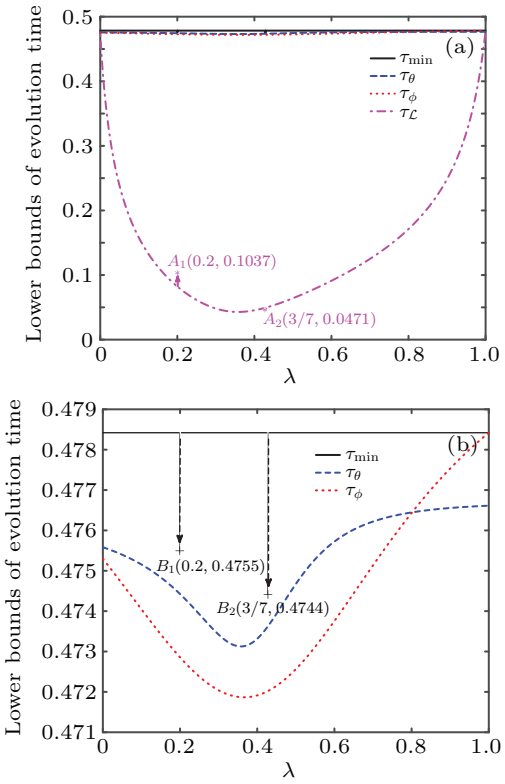


Fig. 1. Numerical results of the four lower bounds $\tau_{\min}$, τ_θ, τ_ϕ, and $\tau_{\mathcal{L}}$ as the parameter λ changes (a). The varying behaviors of the former three lower bounds $\tau_{\min}$, τ_θ and τ_ϕ are redrawn in an enlarged scale to make them clearer (b).

Different from the continuous varying behavior of the two lower bounds τ_θ and τ_ϕ with the parameter λ, the two curves of $\tau_{\min}$ and $\tau_{\mathcal{L}}$ both have two discontinuities at the points $\lambda = 0.2$ and $\lambda = 3/7$. See the points A_1 and A_2 of $\tau_{\mathcal{L}}$ in Fig. 1(a) and B_1 and B_2 of $\tau_{\min}$ in Fig. 1(b), respectively, in which case the condition $p_1 = p_2 = 0.2$ or $p_1 = p_3 = 3/7$ is satisfied for the mixing coefficients, and the initial state ρ_0 is represented by a degenerate density matrix. Here we conjecture that the two lower bounds τ_θ and τ_ϕ proposed in Ref. [69] have a continuous dependence on the initial state ρ_0 and the target state ρ_τ, may not the other two lower bounds $\tau_{\min}$ and $\tau_{\mathcal{L}}$. The discontinuity might occur in the case of quantum degenerate states.

We make a numerical comparison on the tightness of the lower bound $\tau_{\min}$ proposed here and the maximum one among the other three lower bounds $\tau'_{\min} = \max[\tau_\theta, \tau_\phi, \tau_{\mathcal{L}}]$. In a time-independent qutrit system, we generate 10^6 random positive matrixes as the initial mixed states through $\rho_0 = U\Lambda U^\dagger$ in a numerical program, where each unitary matrix U containing the eigenvectors of the mixed state is generated through a QR decomposition of a random square matrix, and the eigenvalues, contained in the diagonal matrix Λ, are also generated randomly, but the conditions of non-negativity and normalization are guaranteed. The corresponding target states ρ_τ is generated by imposing a random Hamiltonian $\mathcal{H}$ on the initial state with a randomly-chosen evolution time τ, i.e., $\rho_\tau = \mathrm{e}^{-\mathrm{i}\tau\mathcal{H}}\rho_0\,\mathrm{e}^{\mathrm{i}\tau\mathcal{H}}$. It is found that only about 4% of these sampled states have a larger lower bound $\tau'_{\min}$ than $\tau_{\min}$ for their evolution time to evolve from ρ_0 to the corresponding target states ρ_τ. Our numerical results show that this ratio goes down to about 2%, 0.9%, 0.3%, and 0.1% as the dimension of the quantum system enhances from 4 to 7, where each numerical result is derived based on a sample size of 10^6.

Degeneracy of a density matrix may reduce the lower bound $\tau_{\min}$ proposed here, according to the definition of the lower bound $\tau_{\min}$ introduced in Subsection 3.2, because the decomposition of a quantum state represented by a degenerate matrix is not unique, and we have to figure out two closest decompositions in the initial and target states to estimate the evolution time. In a time-independent qutrit system, we generate 10^6 random positive matrixes as the initial states ρ with the eigenvalues $\lambda_1 = \lambda_2 = 1/2$ and $\lambda_3 = 0$, and the eigenvectors are from the column vectors of the unitary matrix generated through the QR decomposition of a random 3×3 matrix. The corresponding target state ρ_τ is also constructed through $\rho_\tau = \mathrm{e}^{-\mathrm{i}\tau\mathcal{H}}\rho_0\,\mathrm{e}^{\mathrm{i}\tau\mathcal{H}}$ with randomly-chosen Hamiltonian $\mathcal{H}$ and evolution time τ. It is found that nearly half of these sampled states have a larger lower bound $\tau'_{\min}$ than $\tau_{\min}$ for their evolution time to evolve from ρ_0 to the corresponding target states ρ_τ, and this ratio is much larger than the ratio for the nondegenerate states discussed above. Similar to the above case, this ratio becomes smaller as the dimension of the quantum system enhances. For example, in a 10-dimensional system, about 20% of the mixed states represented by a rank-deficient density matrix with equal nonzero eigenvalues have larger lower bound of $\tau'_{\min}$ than $\tau_{\min}$.

5. Conclusions

In this paper, we have studied the quantum speed limits of a mixed state in a unitary system. A lower bound of the evolution time for the mixed state in a quantum unitary system

is proposed based on the instantaneous angular velocity of its eigenvectors. This lower bound is independent on the eigenvalues of the mixed state, which is in accord with the properties of a quantum evolution, because unitary evolution does not change the spectrum of a quantum state, and no evidence shows that the eigenvalues of a quantum state would accelerate or decelerate the unitary evolution of a quantum state. In a single-qubit system, the lower bound proposed here is equivalent to lower bound τ_θ proposed in Ref. [69], and they are tighter than the other two lower bounds τ_ϕ and $\tau_{\mathcal{L}}$ introduced in the same reference. In a qutrit or higher quantum system, our numerical results show that the lower bound proposed here is very tight. Even for degenerate quantum states, in which case the lower bound proposed here is smaller or equal to that of the non-degenerate states containing the same set of eigenvectors, more than half of these states have larger lower bound than the maximum one among τ_θ, τ_ϕ and $\tau_{\mathcal{L}}$.

Since the lower bound proposed here is based on the evolution time of the eigenvectors contained in the mixed state, it naturally reduces to the Mandelstam–Tamm bound in Ref. [2] for pure states of arbitrary dimension. As a price to obtain a tighter bound of the evolution time for a mixed state, the computation of the current lower bound need to make an eigenvalue decomposition of the mixed state under consideration, and even a SVD is required if a degenerate mixed state is considered, which is more complicated than the computation of the lower bounds τ_θ, τ_ϕ, and $\tau_{\mathcal{L}}$ introduced in Ref. [69]. It is another example to demonstrate the trade-off relation between the tightness of a QSL and its computational complexity. As indicated in Eq. (8), the evolution time of a mixed state is bounded from below by the maximum evolution time of the eigenvectors, which implies that reducing the maximum evolution time of the eigenvectors may be a possible way to speed up quantum computation.

The approach developed here could not be directly used in a quantum non-unitary system, because a non-unitary dynamics would affect the spectrum of a mixed state, and the evolution of a mixed state is not equivalent to the joint evolution of the eigenvectors in this case. The generalization of the instantaneous angular velocity of a pure state to the mixed state case remains open. Furthermore, the derivation of the current lower bounds of the evolution time is mainly based on mathematical analysis on the density matrix and its unitary evolution, rather than the operational meanings of quantum evolution, and a physically meaningful QSL for open dynamics would be very interesting and remains open too at present.

Acknowledgement

Project supported by the National Natural Science Foundation of China (Grant Nos. 11664018, 12174247, and U2031145).

References

[1] Vaidman L 1992 *Am. J. Phys.* **60** 182
[2] Mandelstam L and Tamm I 1945 *J. Phys. (USSR)* **9** 249
[3] Bekenstein J D 1981 *Phys. Rev. Lett.* **46** 623
[4] Murphy M, Montangero S, Giovannetti V and Calarco T 2010 *Phys. Rev. A* **82** 022318
[5] del Campo A, Molina-Vilaplana J and Sonner J 2017 *Phys. Rev. D* **95** 126008
[6] Giovannetti V, Lloyd S and Maccone L 2006 *Phys. Rev. Lett.* **96** 010401
[7] Giovannetti V, Lloyd S and Maccone L 2011 *Nat. Photon.* **5** 222
[8] Chin A W, Huelga S F and Plenio M B 2012 *Phys. Rev. Lett.* **109** 233601
[9] Lloyd S 2000 *Nature* **406** 1047
[10] Lloyd S 2002 *Phys. Rev. Lett.* **88** 237901
[11] Carlini A, Hosoya A, Koike T and Okudaira Y 2006 *Phys. Rev. Lett.* **96** 060503
[12] Caneva T, Murphy M, Calarco T, Fazio R, Montangero S, Giovannetti V and Santoro G E 2009 *Phys. Rev. Lett.* **103** 240501
[13] Reich D M, Ndong M and Koch C P 2012 *J. Chem. Phys.* **136** 104103
[14] Brody D C and Meier D M 2015 *Phys. Rev. Lett.* **114** 100502
[15] Wang X T, Allegra M, Jacobs K, Lloyd S, Lupo C and Mohseni M 2015 *Phys. Rev. Lett.* **114** 170501
[16] Mukherjee V, Giovannetti V, Fazio R, Huelga S F, Calarco T and Montangero S 2015 *New J. Phys.* **17** 063031
[17] Deffner S and Campbell S 2017 *J. Phys. A-Math. Theor.* **50** 453001
[18] Deffner S and Lutz E 2010 *Phys. Rev. Lett.* **105** 170402
[19] del Campo A, Goold J and Paternostro M 2014 *Sci. Rep.* **4** 6208
[20] Binder F C, Vinjanampathy S, Modi K and Goold J 2015 *New J. Phys.* **17** 075015
[21] Campaioli F, Pollock F A, Binder F C, Céleri L, Goold J, Vinjanampathy S and Modi K 2017 *Phys. Rev. Lett.* **118** 150601
[22] Dou F Q, Wang Y J and Sun J A 2022 *Front. Phys.* **17** 31503
[23] Margolus N and Levitin L B 1998 *Physica D* **120** 188
[24] Levitin L B and Toffoli T 2009 *Phys. Rev. Lett.* **103** 160502
[25] Sun S N, Peng Y G, Hu X H and Zheng Y J 2021 *Phys. Rev. Lett.* **127** 100404
[26] Bhattacharyya K 1983 *J. Phys. A* **16** 2993
[27] Huang J H, Hu L Y and Liu F Y 2020 *Phys. Rev. A* **102** 062221
[28] Fubini G 1904 *Atti Istit. Veneto* **63** 502
[29] Study E 1905 *Mathematische Annalen* **60** 321
[30] Page D N 1987 *Phys. Rev. A* **36** 3479
[31] Anandan J and Aharonov Y 1990 *Phys. Rev. Lett.* **65** 1697
[32] Pfeifer P 1993 *Phys. Rev. Lett.* **70** 3365
[33] Pfeifer P and Fröhlich J 1995 *Rev. Mod. Phys.* **67** 759
[34] Giovannetti V, Lloyd S and Maccone L 2003 *Phys. Rev. A* **67** 052109
[35] Jones P J and Kok P 2010 *Phys. Rev. A* **82** 022107
[36] Mondal D, Datta C and Sazim S 2016 *Phys. Lett. A* **380** 689
[37] Mondal D and Pati A K 2016 *Phys. Lett. A* **380** 1395
[38] Pires D P, Cianciaruso M, Celeri L C, Adesso G and Soares-Pinto D O 2016 *Phys. Rev. X* **6** 021031
[39] Wu S X and Yu C S 2018 *Phys. Rev. A* **98** 042132
[40] Wu S X and Yu C S 2020 *Sci. Rep.* **10** 5500
[41] Zhang Y J, Han W, Xia Y J, Cao J P and Fan H 2014 *Sci. Rep.* **4** 4890
[42] Xu K, Han W, Zhang Y J and Fan H 2018 *Chin. Phys. B* **27** 010302
[43] Du K Y, Ma, Y J, Wu S X and Yu C S 2021 *Chin. Phys. B* **30** 090308
[44] Deffner S and Lutz E 2013 *J. Phys. A-Math Theor* **46** 335302
[45] Poggi P M, Lombardo F C and Wisniacki D A 2013 *Europhys. Lett.* **104** 40005
[46] Poggi P M 2019 *Phys. Rev. A* **99** 042116
[47] Lu X, Zhang Y J and Xia Y J 2021 *Chin. Phys. B* **30** 020301
[48] Taddei M M, Escher B M, Davidovich L and de Matos Filho R L 2013 *Phys. Rev. Lett.* **110** 050402
[49] del Campo A, Egusquiza I L, Plenio M B and Huelga S F 2013 *Phys. Rev. Lett.* **110** 050403
[50] Sun Z, Liu J, Ma J and Wang X 2015 *Sci. Rep.* **5** 8444
[51] Deffner S and Lutz E 2013 *Phys. Rev. Lett.* **111** 010402
[52] Ektesabi A, Behzadi N and Faizi E 2017 *Phys. Rev. A* **95** 022115
[53] Mirkin N, Toscano F, Wisniacki D A 2016 *Phys. Rev. A* **94** 052125
[54] Villamizar D V, Duzzioni E I Leal A C S and Auccaise R 2018 *Phys. Rev. A* **97** 052125

[55] Funo K, Shiraishi N and Saito K 2019 *New J. Phys.* **21** 013006
[56] Sun S N and Zheng Y J 2019 *Phys. Rev. Lett.* **123** 180403
[57] Das A, Bera A, Chakraborty S and Chruściński D 2021 *Phys. Rev. A* **104** 042202
[58] Deffner S 2017 *New J. Phys.* **19** 103018
[59] Hu X H, Sun S N and Zheng Y J 2020 *Phys. Rev. A* **101** 042107
[60] Bukov M, Sels D and Polkovnikov A 2019 *Phys. Rev. X* **9** 011034
[61] Fogarty T, Deffner S, Busch T and Campbell S 2020 *Phys. Rev. Lett.* **124** 110601
[62] Il'in N and Lychkovskiy O 2021 *Phys. Rev. A* **103** 062204
[63] Jing J, Wu L A and del Campo A 2016 *Sci. Rep.* **6** 38149
[64] Wu S X and Yu C S 2020 *Chin. Phys. B* **29** 050302
[65] Wang Y Y and Fang M F 2020 *Chin. Phys. B* **29** 030304
[66] Provost J P and Vallee G 1980 *Commun. Math. Phys.* **76** 289
[67] Shanahan B, Chenu A, Margolus N and del Campo A 2018 *Phys. Rev. Lett.* **120** 070401
[68] Okuyama M and Ohzeki M 2018 *Phys. Rev. Lett.* **120** 070402
[69] Campaioli F, Pollock F A, Binder F C and Modi K 2018 *Phys. Rev. Lett.* **120** 060409
[70] Bures D 1969 *Trans. Am. Math. Soc.* **135** 199
[71] Wootters W K 1981 *Phys. Rev. D* **23** 357
[72] Uhlmann A 1992 *The Metric of Bures and the Geometric Phase*, In: *Groups and Related Topics* (New York, Kluwer Academic) pp. 267–274
[73] Baumgratz T, Cramer M and Plenio M B 2014 *Phys. Rev. Lett.* **113** 140401