Transformation relation between coherence and entanglement for two-qubit states

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(Received 30 March 2022; revised manuscript received 23 May 2022; accepted manuscript online 29 May 2022)

Entanglement and coherence are two important resources in quantum information theory. A question naturally arises: Is there some connection between them? We prove that the entanglement of formation and the first-order coherence of two-qubit states satisfy an inequality relation. Two-qubit pure state reaches the upper bound of this inequality. A large number of randomly generated states are used to intuitively verify the complementarity between the entanglement of formation and the first-order coherence. We give the maximum accessible coherence of two-qubit states. Our research results will provide a reliable theoretical basis for conversion of the two quantum resources.

Keywords: entanglement, coherence, first-order coherence, entanglement of formation

PACS: 03.67.Mn, 03.65.Ud, 03.67.–a **DOI:** 10.1088/1674-1056/ac7454

1. Introduction

Entanglement and coherence are two important physical resources in quantum information theory. Schrödinger put forward the concept of entanglement as early as 1935.[1] Quantum entangled states are the most crucial physical resource in quantum information processing and can be widely used in reality, such as super-dense coding,[2] quantum teleportation,[3] and remote state preparation.[4] However, one of the problems in entanglement theory is how to quantify entanglement. In order to solve this problem, a number of entanglement measures were proposed, such as entanglement of formation,[5] concurrence,[6] negativity[7] and relative entropy of entanglement.[8] There is a monotone increasing relationship between entanglement of formation and concurrence of an arbitrary state of two qubits.[5] Because the expression of entanglement of formation is too complicated, it can be expressed by concurrence and entropy. Entanglement of formation quantifies the minimum physical resources required to produce a quantum state.[9] Another question arises: How can entanglement be represented experimentally? Thus, concurrence for two-qubit pure systems was proposed.[10,11] In 2009, different from the above-mentioned experimental methods, only one copy of the state was required for each measurement to determine the concurrence of locally observable values of arbitrary multipartite pure states.[12] Until 2012, an experimental measurement method of entanglement of formation in arbitrary dimensional pure states had not been proposed.[13] Therefore, we acquire a feasible way in reality.

Coherence is one of the most important concepts to describe the characteristics of photon stream,[14] and its concept can be traced back to classical optics as a property of waves. Its surprising role in a range of physical, chemical, and biological applications has made it used more commonly.[15–18] In 2014, Baumgratz *et al.* proposed a framework to quantify coherence and determined that the relative entropy of coherence and the l_1 norm of coherence were the most intuitive and easy measures to calculate coherence.[19] They also put forward the resource-theoretic viewpoint of quantum coherence. This was followed by a number of works that established a rigorous mathematical framework for the resource theory of coherence.[20–24] Svozilik *et al.* proved that the relevant parameter quantifying the maximum hidden coherence was the degree of violation of the CHSH inequality in 2015.[25] In the same year, many studies found the relationship between quantum coherence and other quantum resources, such as entanglement,[26] mixedness,[27] and deficit.[28] Streltsov *et al.* proved that any degree of coherence can be transformed into entanglement through incoherent operation, and provided a clear quantitative and operational connection between coherence and entanglement.[29] In 2016, two basic concepts "coherence distillation" (given by the relative entropy of coherence) and "coherence cost" (by the coherence of formation) were proposed.[30] Quantum coherence also plays an important role in quantum computing.[31,32] Cernoch *et al.* experimentally verified the migration of coherence between the complex quantum systems and their subsystems, and verified the conservation of the maximally accessible first-order coherence while it migrated between classical coherence and quantum correlations.[33]

In recent years, many researchers have studied the intrinsic relationship between coherence and entanglement.[34] These relationships differ markedly due to the variety of resources and measurement methods. The aim of this paper is to establish the relation between entanglement of formation and

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the first-order coherence, i.e., we propose a concrete expression between them.

The rest of this paper is organized as follows. In Section 2, we briefly review entanglement of formation (EOF) and the first-order coherence, and we write the first-order coherence termed in binary entropy. In Section 3, we find the complementary relationship between the two resources, and verify an inequality relation between them for any two-qubit states by using randomly generated states. Then, a specific example is presented to help us understand this relationship. In Section 4, we propose the concept of maximum accessible coherence. In Section 5, we conclude the article with a brief summary.

2. Preliminaries

There are a lot of measures of quantum entanglement, including entanglement of formation (EOF), concurrence, relative entropy of entanglement, negativity, etc. In this section, we will briefly introduce the definitions of concurrence and EOF and the relationship between them. We introduce the measure of coherence: the first-order coherence.

2.1. Entanglement of formation

Consider a two-qubit composite system, its pure state decomposition (not strictly orthogonal decomposition) is

$$\rho = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right|, \tag{1}$$

where $\sum_i p_i = 1$. For each pure state $|\phi_i\rangle$, its entanglement of formation $E(|\phi_i\rangle)$ is defined as the von Neumann entropy of the pure state of subsystem A or B. It can be written as

$$E\left(\left|\phi_i\right\rangle\right) = S\left(\mathrm{Tr}_{\mathrm{A}}\left(\left|\phi_i\right\rangle\left\langle\phi_i\right|\right)\right) = S\left(\mathrm{Tr}_{\mathrm{B}}\left(\left|\phi_i\right\rangle\left\langle\phi_i\right|\right)\right), \tag{2}$$

where $\mathrm{Tr}_{\mathrm{A}}(|\phi_i\rangle\langle\phi_i|)$ and $\mathrm{Tr}_{\mathrm{B}}(|\phi_i\rangle\langle\phi_i|)$ are the reduced density matrix of B and A in pure state $|\phi_i\rangle$, respectively. The EOF of mixed state is defined as the average value of the entanglement of formation of each pure state in the pure states ensemble, and this average value takes the minimum of all possible mean values. Therefore, the entanglement of formation of mixed state is defined as[35]

$$E(\rho) = \min_{\{p_i,|\phi_i\rangle\}} \sum_i p_i E\left(\left|\phi_i\right\rangle\right). \tag{3}$$

Next, we introduce another measure of entanglement: concurrence. For a two-qubit pure state $|\psi\rangle$, its spin inversion state is defined as $|\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y)|\psi^*\rangle$, where $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$ and σ_y is the Pauli matrix. The concurrence of pure state can be defined as

$$E_{\mathrm{c}}\left(\left|\psi\right\rangle\right) = \left|\left\langle\psi \mid \tilde{\psi}\right\rangle\right|. \tag{4}$$

For an arbitrary two-qubit mixed state ρ, the concurrence of mixed state can be expressed as follows:

$$E_{\mathrm{c}}(\rho) = \min_{\{p_i,|\phi_i\rangle\}} \sum_i p_i E_{\mathrm{c}}\left(\left|\phi_i\right\rangle\right). \tag{5}$$

The minimum value can be obtained by taking over all possible pure state decompositions of ρ, and the concurrence can be written as

$$E_{\mathrm{c}}(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{6}$$

where λ_n ($n \in \{1,2,3,4\}$) is the eigenvalue of non-Hermitian matrix $R = \rho\tilde{\rho}$, and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. The spin-flipped density matrix $\tilde{\rho}$ can be expressed as

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y). \tag{7}$$

The relationship between EOF and concurrence can be represent by the function[5]

$$E(\rho) = h\left(\frac{1+\sqrt{1-E_{\mathrm{c}}^2(\rho)}}{2}\right). \tag{8}$$

The binary entropy $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

2.2. The first-order coherence

Let us consider a two-qubit state, $\rho = \sum_{n=1}^4 p_n |\psi_n\rangle\langle\psi_n|$, composed of subsystems A and B, where p_n is the descending eigenvalue of ρ and $\sum_{n=1}^4 p_n = 1$. This quantum state can be obtained by performing operator V on the separable state ρ_Λ. Here, ρ_Λ is a diagonal matrix with corresponding eigenvalues p_n, and the matrix V contains the corresponding eigenvectors $|\psi_n\rangle$. Each subsystem of ρ has the corresponding reduced density matrix $\rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}(\rho)$ and $\rho_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{A}}(\rho)$. The degree of the first-order coherence of each subsystem is[36]

$$D_k = \sqrt{2\mathrm{Tr}\left(\rho_k^2\right) - 1} \tag{9}$$

with $k \in \{\mathrm{A},\mathrm{B}\}$. When the subsystems are considered independently, the first-order coherence for two subsystems has the following form:[25]

$$D = \sqrt{\frac{D_{\mathrm{A}}^2 + D_{\mathrm{B}}^2}{2}}. \tag{10}$$

When both subsystems are coherent, one has $D = 1$, and iff both subsystems show no coherence, $D = 0$.

For convenience, we write the first-order coherence in terms of binary entropy, and use the symbol C to represent it. Its expression of each subsystem can be written as

$$C_k = 1 - h\left(\frac{1+D_k}{2}\right). \tag{11}$$

Moreover, C_k is the monotonically increasing function of D_k. The value range of C_k is the same as D_k, $C_k \in [0,1]$.

Based on Eq. (11), we can rewrite Eq. (10) as

$$C = \frac{C_{\mathrm{A}} + C_{\mathrm{B}}}{2}. \tag{12}$$

3. The complementary relation between first-order coherence and entanglement of formation

In this section, we put forward two theorems, which are the complementary relationship between EOF and C for two-qubit pure state and mixed state. Sequentially, we will prove our theorems.

3.1. Pure states

Theorem 1 For arbitrary two-qubit pure state $|\psi\rangle$, the entanglement of formation $E(|\psi\rangle)$ and the first-order coherence $C(|\psi\rangle)$ in binary entropy satisfy the following complementary relation

$$E(|\psi\rangle)+C(|\psi\rangle)=1. \tag{13}$$

Proof The relationship between the first-order coherence (D) and concurrence for two-qubit pure states $|\psi\rangle$ can be written as[37]

$$E_{\mathrm{c}}^{2}(|\psi\rangle)+D^{2}(|\psi\rangle)=1. \tag{14}$$

Combining with Eq. (8), we can obtain the relation between EOF and the first-order coherence for two-qubit pure state, it can be expressed as

$$E(|\psi\rangle)=h\left(\frac{1+D(|\psi\rangle)}{2}\right). \tag{15}$$

Due to $C=C_{\mathrm{A}}=C_{\mathrm{B}}$ for two-qubit pure state, combining the definition of C, we obtain

$$E(|\psi\rangle)=1-C(|\psi\rangle), \tag{16}$$

which recovers Eq. (13) as shown in Theorem 1.

3.2. Mixed states

We have proved the relationship between EOF and C in two-qubit pure states before. Naturally, there is a question: What is the relationship between them in two-qubit mixed states? Now we are going to write out the derivation process and to verify the results with random states.

Theorem 2 For arbitrary two-qubit mixed state ρ, the EOF and the first-order coherence (C) termed in binary entropy satisfy the following complementary relation as

$$E(\rho)+C(\rho)\leq 1. \tag{17}$$

Proof A quantum system is made up of many different subsystems which can be described by the vector $(|\psi_i\rangle, i=1,2,\ldots,N)$. If it is a pure state, its projection operator will be $\rho_i=|\psi_i\rangle\langle\psi_i|$. If the system is mixed ensembles, we can obtain its density operator

$$\rho=\sum_i p_i\rho_i. \tag{18}$$

Meanwhile, its subsystems can be written as

$$\rho_k=\sum_i p_i\rho_i^k, \tag{19}$$

with $k\in\{\mathrm{A},\mathrm{B}\}$. In Bloch space, Eq. (19) is equivalent to the following equation:

$$\boldsymbol{a}=\sum_i p_i\boldsymbol{a}_i, \tag{20}$$

where $\boldsymbol{a}_i$ represent the Bloch vector of ρ_i. Since the modulus of a vector satisfies convexity, we have the following inequality for the modulus of $\boldsymbol{a}$:

$$|\boldsymbol{a}|=\left|\sum_i p_i\boldsymbol{a}_i\right|\leq\sum_i p_i|\boldsymbol{a}_i|. \tag{21}$$

The first-order coherence of the subsystem is equal to its modulus $D(\rho_k)=|\boldsymbol{a}|$.[37] Therefore, the first-order coherence of the subsystem is also convex,

$$D(\rho_k)\leq\sum_i p_i D\left(\rho_i^k\right). \tag{22}$$

The first-order coherence (C) termed in binary entropy is the monotonically increasing function of the first-order coherence (D), so C also satisfies convexity

$$C(\rho)\leq\sum_i p_i C(\rho_i). \tag{23}$$

For an arbitrary two-qubit state we can clearly know

$$\min_{\{p_i,|\phi_i\rangle\}}\sum_i p_i E(|\phi_i\rangle)\leq\sum_i p_i E(|\phi_i\rangle). \tag{24}$$

According to the definition of entanglement of formation (see Eq. (3)), it is natural to find that EOF satisfies convexity. Thus, we obtain

$$E(\rho)\leq\sum_i p_i E(\rho_i). \tag{25}$$

We add the two convexity inequalities for coherence (Eq. (23)) and entanglement (Eq. (25)), then we can reach

$$E(\rho)+C(\rho)\leq\sum_i p_i[E(\rho_i)+C(\rho_i)]. \tag{26}$$

Because ρ_i is pure state, the right-hand side of Eq. (26) is going to be 1, which proves Eq. (17) as shown in Theorem 2.

Equation (17) reveals that EOF and C satisfy the complementary relation. Then we take 20000 randomly generated states to verify our relation, and we plot the EOF and C of these states in Fig. 1. We find that all coordinate points are in the region bounded by Eq. (17), C-axis, and E-axis, so Eq. (17) is correct. The upper bound is indeed Eq. (13). According to the above results, we conclude that EOF is anti-correlated with C. The greater the C, the smaller the EOF, and vice versa.

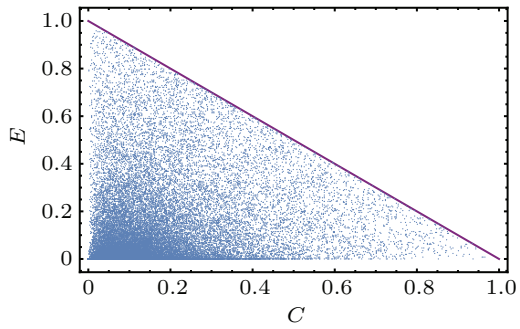


Fig. 1. The regional map of entanglement of formation (E) versus the first-order coherence (C) in binary entropy for arbitrary two-qubit states. The upper bounds (purple line) are denoted by $E+C=1$. The figure plots the entanglement of formation (E) along the y-axis, and the C along the x-axis, for 50000 randomly generated two-qubit states.

3.3. Example

In this section, we will use an example to confirm the above theorems.

The Werner state ($\rho_{\rm W}$) is a combination of the Bell state $|\psi_{\rm B}\rangle$ (maximum entangled pure state) and the maximum mixed state ($\frac{I\otimes I}{4}$). I is the identity matrix with second order. For convenience, we choose the Bell state $|\psi_{\rm B}\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$. We take the unitary transformation U for the Werner state, and we obtain the Werner-type ρ that can be written as

$$\rho = p\,|\psi\rangle\langle\psi| + (1-p)\frac{I\otimes I}{4}. \tag{27}$$

It is obvious that $|\psi\rangle = U\,|\psi_{\rm B}\rangle$. Here, p $(0\le p\le 1)$ is the probability of Werner state becoming Bell state. If $p>1/3$, the Werner state is an entangled state.

In this paper, we find a U operator that makes Werner state convert to Werner-type state

$$U(\theta) = \begin{pmatrix} -\sin(\theta-\frac{\pi}{4}) & 0 & 0 & \sin(\theta+\frac{\pi}{4}) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin(\theta+\frac{\pi}{4}) & 0 & 0 & \sin(\theta-\frac{\pi}{4}) \end{pmatrix}. \tag{28}$$

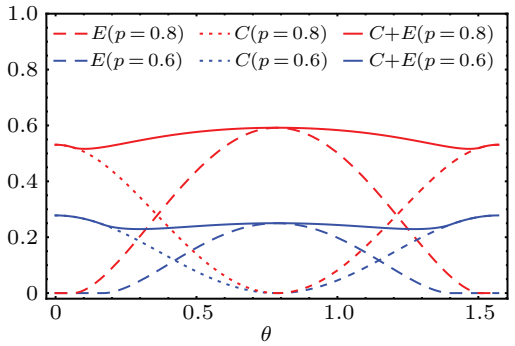


Fig. 2. The entanglement of formation E and the first-order coherence C and their sum $C+E$ of Werner-type states with respect to the state's parameter θ. The blue lines represent $p=0.6$ and the red lines represent $p=0.8$.

Since the numerical results are too complicated, we will use the form of pictures to show our results more intuitively. We provide the EOF and C as a function of the parameter θ with different p in Fig. 2. It can be seen from Fig. 2 that when $\theta=\pi/4$, the entanglement of formation reaches its maximum and the first-order coherence (C) is minimum. The C of Werner-type state ρ will decrease with the increase of its entanglement of formation.

4. Maximum accessible coherence

The maximum accessible coherence is defined as[25]

$$C_{\max} = \max_{U} C\left(U\rho U^{\dagger}\right). \tag{29}$$

For two-qubit pure state, we have obtained the equation between EOF and the first-order coherence (C) termed in binary entropy: $C+E=1$. It is not difficult to find that, when the EOF is minimum ($E=0$), the first-order coherence (C) reaches the maximum, and the maximum accessible coherence is 1.

We focus on an arbitrary 2×2 dimensional mixed state, which can be expressed as a probabilistic mixture of its eigenvector, as follows:

$$\rho_{\rm AB} = \sum_{k=1}^{4} x_k\,|\psi_k\rangle\langle\psi_k|, \tag{30}$$

where $\langle\psi_m|\psi_n\rangle = \delta_{mn}$. Its subsystem $\rho_{\rm A} = {\rm Tr}_{\rm B}(|\psi_k\rangle\langle\psi_k|)$ and $\rho_{\rm B} = {\rm Tr}_{\rm A}(|\psi_k\rangle\langle\psi_k|)$ can be expressed in terms of Bloch vectors as follows:

$$\begin{aligned} |a\rangle &= \sum_{k=1}^{4} x_k\,|a_k\rangle, \\ |b\rangle &= \sum_{k=1}^{4} x_k\,|b_k\rangle. \end{aligned} \tag{31}$$

Thus the first-order coherence termed in binary entropy of the state $\rho_{\rm AB}$ can be written as

$$\begin{aligned} C(\rho_{\rm AB}) \;=\; & 1-\frac{1}{2}h\left[\frac{1}{2}\left(1+\sum_{k=1}^{4} x_k\,|a_k\rangle\right)\right] \\ & -\frac{1}{2}h\left[\frac{1}{2}\left(1+\sum_{k=1}^{4} x_k\,|b_k\rangle\right)\right]. \end{aligned} \tag{32}$$

Then, we apply a global unitary operation U to the state $\rho_{\rm AB}$ and we obtain a post-operation state

$$\rho'_{\rm AB} = U\rho_{\rm AB}U^{\dagger} = \sum_{k=1}^{4} x_k\,|\psi'_k\rangle\langle\psi'_k|, \tag{33}$$

where $|\psi'_k\rangle = U\,|\psi_k\rangle$. The pure state $|\psi'_k\rangle$ is the eigenvector of the final state $\rho'_{\rm AB}$, since $\langle\psi'_m|\psi'_n\rangle = \langle\psi_m|U^{\dagger}U\,|\psi_n\rangle = \langle\psi_m|\psi_n\rangle = \delta_{mn}$ for all m and n. The state $\rho'_{\rm AB}$ can reach the maximum accessible coherence if and only if the pure states $\{|\psi'_k\rangle\}_{k=1}^{4}$ are separable. When the unitary operation satisfy $U=V^{\dagger}$, these pure states are separable, and they can be expressed as $|\psi'_k\rangle = |i_{\rm A}\rangle\otimes|j_{\rm B}\rangle\,(i,j=1,2)$. The subsystems of the state $\rho'_{\rm AB}$ can be written as

$$\rho'_{\rm A} = \sum_{i=1}^{2}\sum_{j=1}^{2} x_{j+2(i-1)}\,|i_{\rm A}\rangle\langle i_{\rm B}|,$$

$$\rho'_{\rm B} = \sum_{j=1}^{2}\sum_{i=1}^{2} x_{j+2(i-1)} \left|j_{\rm A}\right\rangle \left\langle j_{\rm B}\right|. \tag{34}$$

The global unitary operator $V^{\dagger}$ can make two-qubit state $\rho_{\rm AB}$ obtain the maximum accessible coherence ($C_{\rm max}$). Thus, the maximal accessible coherence can be obtained

$$\begin{aligned} C_{\rm max}\left(\rho_{\rm AB}\right) &= \max_{U} C\left(U\rho_{\rm AB}U^{\dagger}\right) \\ &= C\left(V^{\dagger}\rho_{\rm AB}V\right) \\ &= C\left(\rho'_{\rm AB}\right) \\ &= \frac{C\left(\rho'_{\rm A}\right)+C\left(\rho'_{\rm B}\right)}{2} \\ &= 1-\frac{1}{2}h\left[\sum_{i=1}^{2}\left(\sum_{j=1}^{2}x_{j+2(i-1)}\right)^{2}\right] \\ &\quad -\frac{1}{2}h\left[\sum_{j=1}^{2}\left(\sum_{i=1}^{2}x_{j+2(i-1)}\right)^{2}\right]. \end{aligned} \tag{35}$$

5. Conclusion

In summary, we have investigated the intrinsic relationship between EOF and the first-order coherence termed in binary entropy. First, we write the first-order coherence termed in binary entropy. Next, we derive an inequality relation between EOF and C for any two-qubit states. The relation in pure states is the upper bound of the inequality. We reveal a complementary relationship between EOF and C. The first-order coherence (C) termed in binary entropy is largest when EOF is minimal. Then we give an example to more intuitively reflect the inverse correlation between EOF and C. Finally, based on the above investigation, we propose the maximal accessible coherence $C_{\rm max}$. We indirectly provide the relationship between EOF and the first-order coherence. We hope that our results can provide some help in the mutual transformation of entanglement and coherence.

Acknowledgements

This work was supported by the National Science Foundation of China (Grant Nos. 12175001 and 12075001) and the Natural Science Foundation of Education Department of Anhui Province, China (Grant No. KJ2016SD49).

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