An incommensurate fractional discrete macroeconomic system: Bifurcation, chaos, and complexity

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This study proposes a novel fractional discrete-time macroeconomic system with incommensurate order. The dynamical behavior of the proposed macroeconomic model is investigated analytically and numerically. In particular, the zero equilibrium point stability is investigated to demonstrate that the discrete macroeconomic system exhibits chaotic behavior. Through using bifurcation diagrams, phase attractors, the maximum Lyapunov exponent and the 0–1 test, we verified that chaos exists in the new model with incommensurate fractional orders. Additionally, a complexity analysis is carried out utilizing the approximation entropy (ApEn) and C_0 complexity to prove that chaos exists. Finally, the main findings of this study are presented using numerical simulations.

Keywords: chaos, macroeconomic system, discrete fractional calculus, complexity

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1. Introduction

Fractional calculus is an old subject that has been evolving for over 300 years. Despite this, it has only been extensively researched in the last few decades, with its applications having been noticed in many domains such as applied physics, engineering, and so on.[1] In comparison to the integer-order derivative, the fractional-order derivative is more exact as it provides useful tools for the expression of memory effects in a wide variety of materials and processing.[2] These considerations have led to an increasing number of researchers being interested in the applications of fractional-order systems.[3] During the same time, interest has also been concentrated on fractional difference operators and discrete fractional calculus within the past decade.[4] Recently, various complex dynamics residing in a fractional order iterated map have been investigated, such as chaos, hyperchaos and coexisting attractors.[5–10] For example, the hyperchaotic dynamics of the fractional generalized Hénon map have been analyzed,[11] while in Ref. [12], Peng *et al.* established the existence of chaos in a fractional discrete memristor system. In Ref. [13], the chaotic dynamics and combined synchronization of three two-dimensional maps were illustrated, whereas in Ref. [14] the authors exhibited the rich behaviors of a new 2D fractional hyperchaotic map with an infinite line of equilibrium. Meanwhile, Ouannas *et al.*[15] presented a three-dimensional fractional iterated map, where the proposed fractional map is observed to have hidden attractors. Thanks to these unique characteristics, fractional order iterated maps have been deeply studied in academic fields.[16–18]

Macroeconomics is a field of economics concerned with the overall performance and behavior of an economy. It aims to achieve gross domestic product growth and price stability. To avoid the effects of inflation and unemployment, governments and businesses use macroeconomic models to assist in the development of economic policies and strategies.[19] Macroeconomic models are a useful tool designed to conduct theoretical investigations of macroeconomic phenomena.[20–22] Their purpose is to examine the dynamics of aggregate quantities and comparative statics, such as revenue earned, services produced, and total products, as well as levels of employment of productive resources and price levels.[23] Currently, fractional calculus is being employed to construct some economic models. For example, in Ref. [24] a discrete fractional order prey–predator model was established using a Caputo-like operator, where the influence of fractional order was examined. In Ref. [25], chaotic phenomena were demonstrated in a discrete-time version of a Cournot duopoly game with fractional order, whereas a fractional order triopoly game with bounded rationality was analyzed.[26] As shown by Refs. [25,26], the olygopoly of the market is increased when the fractional order is changed. Nonetheless, research on fractional order macroeconomic discrete-time systems is scarce in the literature. Recently, Hu *et al.*[27] proposed a new discrete nonlinear fractional-order macroeconomic model with the Jumarie fractional derivative. Later, Chu *et al.*[28] presented a two-dimensional discrete fractional macroeconomic system using the Caputo-like operator and proved that it displayed chaos behavior. In the above literature, only commensurate order is considered, and there is no study on incommensurate-order systems. In actuality, the non-commensurate order is a special case of fractional-order systems in which the orders

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have different values. In recent years, it has been found that models with incommensurate orders have more flexibility. In this context, it is logical to concentrate on examining the complex dynamics behavior of fractional macroeconomic systems with incommensurate order.

The main goal of this essay is to investigate a macroeconomic discrete system with incommensurate fractional order. Stability analysis of the proposed system is carried out. The dynamical behavior of the system is numerically analyzed using bifurcation diagrams, phase attractors, maximum Lyapunov exponents and C_0 complexity. The rest of this article is formulated in the following manner. In Section 2, we give some of the fundamentals connected with discrete fractional calculus, as well as the fractional form of the macroeconomic system and its stability analysis. The behaviors of the suggested fractional system are discussed in Section 3 in detail, including bifurcation diagrams, phase attractors, and the Lyapunov exponent. For the chaos and complexity analysis of the system, Section 4 offers some chaos tests including the 0–1 test method, approximate entropy, and C_0 complexity analysis. Finally, the most important results of the study are summarized in Section 5.

2. Preliminaries and model description

In order to describe our macroeconomic model, some definitions of discrete fractional calculus are discussed and the mathematical model of the fractional macroeconomic discrete system involving the Caputo-left operator is given.

2.1. Discrete fractional calculus

Definition 1[30] The fractional sum of order γ for a function g on a time scale $N_\eta = \{\eta, \eta+1, \eta+2, \ldots\}$ is defined as

$$\Delta_\eta^{-\gamma} g(l) = \frac{1}{\Gamma(\gamma)} \sum_{l=\eta}^{s-\gamma} (s-l-1)^{(\gamma-1)} g(l), \tag{1}$$

with $s \in \mathbb{N}_{\eta+\gamma}$, $\gamma > 0$.

Definition 2[31] The γ-Caputo fractional difference operator for a function $g(s)$ is expressed as

$$\begin{aligned} {}^{\mathrm{C}}\Delta_\eta^{\gamma} g(s) &= \Delta_\eta^{-(m-\gamma)} \Delta^m g(s) \\ &= \frac{1}{\Gamma(m-\gamma)} \sum_{l=\eta}^{s-(m-\gamma)} (s-l-1)^{(m-\gamma-1)} \Delta^m g(l), \end{aligned} \tag{2}$$

where $s \in \mathbb{N}_{\eta+m-\gamma}$, $\gamma \notin \mathbb{N}$ and $m = \lceil \gamma \rceil + 1$. $\Delta^m g(l)$ and $(s-l-1)^{(m-\gamma-1)}$ are the mth integer difference operator and the falling factorial function, respectively, which are given as

$$\begin{aligned} (s-l-1)^{(m-\gamma-1)} &= \frac{\Gamma(s-l)}{\Gamma(s-l-m+\gamma+1)}, \\ \Delta^m g(s) &= \Delta(\Delta^{m-1} g(s)) \\ &= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(s+k), \quad s \in \mathbb{N}_\eta. \end{aligned} \tag{3, 4}$$

2.2. Mathematical model

Recently, Pu[32] modified the Goodwin model by permitting the accelerator function to be non-monotonic. This model takes the following form:

$$X_{k+1} = C_{k+1} + I_{k+1}, \tag{5}$$

$$C_{k+1} = (1-s)X_k + sX_{k-1}, \tag{6}$$

$$I_{k+1} = v(X_k - X_{k-1}) - v(X_k - X_{k-1})^3, \tag{7}$$

where X, C, I are the state variables. Specifically, X denotes income, C denotes consumption, and I denotes investment. The marginal propensity to save is presented by s, while the coefficient of the accelerator function is presented by v.

By introducing the variable $y_k = X_{k+1} - X_k$, and substituting Eqs. (6) and (7) into Eq. (5) one may obtain the recurrence relation

$$y_{k+1} = (v-s)y_k - v(y_k)^3. \tag{8}$$

According to Pu in Ref. [32], it is more rational to reformulate Eq. (6) into

$$C_{k+1} = (1-s)X_k + sX_{k-1} + s(1-\varepsilon)X_k. \tag{9}$$

By repeating the previous calculation, the macroeconomic model is derived as

$$\begin{aligned} x_{k+1} &= x_k + y_k, \\ y_{k+1} &= \lambda y_k - (\lambda+1)y_k^3 - \sigma x_k, \end{aligned} \tag{10}$$

where $x = X$ and $\sigma = s(1-\varepsilon)$. By changing the real parameter $\lambda \in [-2.5, 0.5]$, the macroeconomic system (10) can execute various dynamic scenarios. Setting the system parameter $\sigma = 0.2$ and initial condition $x_0 = 2$, $y_0 = 0.1$, figures 1(b) and 1(c) show the bifurcation diagram and the corresponding maximum Lyapunov exponents (MLE), respectively. We can see that the dynamical behavior of the macroeconomic model (10) has evolved from chaotic to periodic as λ increases. particularly, the proposed system is chaotic when $\lambda \in [-2.4, -2]$; where the maximum LE is positive. It serves to explain that, with a proper bifurcation parameter value, model (10) can generate chaotic attractor as illustrated in Fig. 1(a) for $\lambda = -2.3$.

In this work, we expand the integer-order macroeconomic system to the case of the incommensurate fractional order. Herein, the first-order difference of the macroeconomic system is formulated as

$$\begin{aligned} \Delta x(m) &= y(m), \\ \Delta y(m) &= (\lambda-1)y(m) - (\lambda+1)y^3(m) - \sigma x(m). \end{aligned} \tag{11}$$

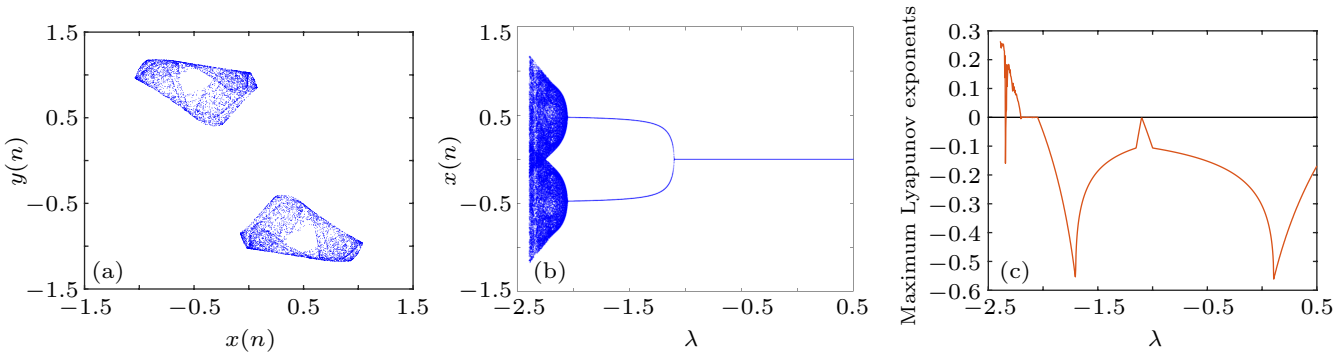


Fig. 1. (a) Chaotic attractor of the macroeconomic system (10) with $\lambda = -2.3$ and initial conditions $(x(0), y(0)) = (2, 0.1)$. (b) Bifurcation diagram versus $\lambda \in (-2.5, 0.5]$ of the macroeconomic model (10), (c) the corresponding MLEs.

According to Eq. (11), we replace the first difference operator Δ with the Caputo difference operator ${}^{\mathrm{C}}\Delta_{\eta}^{\gamma_i}$ and obtain a novel 2D incommensurate fractional discrete-time macroeconomic system. This yields the following discrete-time macroeconomic model with fractional order:

$$\begin{aligned} {}^{\mathrm{C}}\Delta_{\eta}^{\gamma_1} x(s) &= y(s-1+\gamma_1), \\ {}^{\mathrm{C}}\Delta_{\eta}^{\gamma_2} y(s) &= (\lambda-1)y(s-1+\gamma_2) - (\lambda+1)y^3(s-1+\gamma_2) \\ &\quad -\sigma x(s-1+\gamma_2) \end{aligned} \tag{12}$$

for $s \in \mathbb{N}_{\eta+1-\gamma_i}$, where $0 < \gamma_i \leq 1$, $i = 1, 2$ are the fractional orders. When one of the system's orders has different value from the other one $(\gamma_1 \neq \gamma_2)$, we get a discrete-time macroeconomic model with incommensurate fractional order. In general, fractional-order discrete-time models offer more chaos and are more feasible for numerical computing; as a result, the incommensurate discrete-time macroeconomic system simulates economic systems more precisely.

2.3. Stability analysis

In order to determine the equilibrium points of the macroeconomic system with incommensurate order (12), we must solve the system obtained by making the left side of the model (12) equal to zero

$$\begin{aligned} &y = 0, \\ &(\lambda-1)y - (\lambda+1)y^3 - \sigma x = 0. \end{aligned} \tag{13}$$

Obviously, the model has a zero equilibrium point. To analyse the stability of this equilibrium point, we first review the following theorem that is used to examine the stabilization of nonlinear discrete systems with incommensurate fractional orders.

Theorem 1[33] Consider the system

$$\begin{cases} {}^{\mathrm{C}}\Delta_{\eta}^{\gamma_1} y_1(s) = g_1(y(s-1+\gamma_1)), \\ {}^{\mathrm{C}}\Delta_{\eta}^{\gamma_2} y_2(s) = g_2(y(s-1+\gamma_2)), \quad s = 0, 1, \ldots, \\ \qquad\qquad \vdots \\ {}^{\mathrm{C}}\Delta_{\eta}^{\gamma_n} y_n(s) = g_n(y(s-1+\gamma_n)), \end{cases} \tag{14}$$

where $y(s) = (y_1(s), \ldots, y_n(s))^{\mathrm{T}} \in \mathbb{R}^n$ and $g = (g_1, \ldots, g_n) : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that $\gamma_i = w_i/u_i$, $(u_i, w_i) = 1$, $w_i, u_i \in \mathbb{Z}_+$, $i = 1, 2, \ldots, n$ $(0 < \gamma_i < 1)$. Consider that M is the least common multiple (LCM) of $u_1, u_2, \ldots, u_n$. If any root of the Eq. (15)

$$\det(\mathrm{diag}(e^{M\gamma_1}, \ldots, e^{M\gamma_n}) - (1-e^M)J) = 0 \tag{15}$$

is included within a set $\mathbb{C}/K^{\delta}$, then the trivial solution of system (14) corresponding to $x_0 = x(0)$ is locally asymptotically stable, where $\delta = 1/M$, J is the Jacobian matrix of system (14) and

$$\begin{aligned} K^{\sigma} = \Big\{ &\zeta \in \mathbb{C} : |\zeta| \leq \left(2\cos\left(\frac{|\arg\zeta|}{\delta}\right)\right)^{\delta} \\ &\text{and } |\arg\zeta| \leq \frac{\delta\pi}{2} \Big\}. \end{aligned} \tag{16}$$

It is not difficult to find that the Jacobian matrix for a discrete-time macroeconomic system (12) computed at zero point is

$$J = \begin{pmatrix} 0 & 1 \\ -\sigma & \lambda - 1 \end{pmatrix}.$$

To investigate the stability of the macroeconomic model with incommensurate order, we consider the following two cases as an example.

Case 1 When $(\lambda, \sigma) = (-2.3, 0.2)$ and $(\gamma_1, \gamma_2) = (9/10, 8/10)$. Notice that $M = 10$, and so

$$\begin{aligned} &\det\left(\begin{pmatrix} e^9 & 0 \\ 0 & e^8 \end{pmatrix} - (1-e^10)\begin{pmatrix} 0 & 1 \\ -0.2 & 0.9 \end{pmatrix}\right) = 0 \\ &\Leftrightarrow 0.2e^{20} + 0.9e^{19} + e^{17} - 0.4e^{10} - 0.9e^9 + 0.2 = 0. \end{aligned} \tag{17}$$

The MATLAB platform is used to solve Eq. (17) and the

following results were obtained:

$$\bar{e} = \begin{pmatrix} -4.72405 \\ -0.930571 \\ -0.785502 \pm 0.298862\mathrm{i} \\ -0.732375 \pm 0.609038\mathrm{i} \\ -0.390789 \pm 0.737755\mathrm{i} \\ -0.214189 \pm 1.03814\mathrm{i} \\ 0.182104 \pm 0.812804\mathrm{i} \\ 0.2586 \pm 1.10761\mathrm{i} \\ 0.668233 \pm 0.512763\mathrm{i} \\ 0.710525 \pm 0.641957\mathrm{i} \\ 0.880702 \pm 0.0241438\mathrm{i} \end{pmatrix}. \tag{18}$$

Thus, one can deduce that $e = 0.880702 \pm 0.0241438\mathrm{i} \in K^{1/10}$. Hence, according to Theorem 1, for the given fractional order values and system parameters, the zero equilibrium point of system (12) with incommensurate order is unstable. Therefore, in this case the discrete-time macroeconomic model satisfies the necessary condition to present a chaotic behavior.

Case 2 Let $(\lambda, \sigma) = (1.9, 0.2)$ and $(\gamma_1, \gamma_2) = (0.2, 0.95)$. By the same procedure as the above case, we get

$$\begin{aligned} &\det\left(\begin{pmatrix} e^4 & 0 \\ 0 & e^{19} \end{pmatrix} - (1-e^{20})\begin{pmatrix} 0 & 1 \\ -0.2 & -3.3 \end{pmatrix}\right) = 0 \\ \Leftrightarrow\ & 0.2e^{40} - 3.3e^{24} + e^{23} - 0.4e^{20} + 3.3e^4 + 0.2 = 0. \end{aligned} \tag{19}$$

Thus, by solving system (19), one can deduce that there are solutions $e_j = (0.968528 \pm 0.307566\mathrm{i}, 0.825632 \pm 0.586188\mathrm{i})$ such that $|e_j| \leq \left(2\cos 20 |\arg e_j|\right)^{1/20}$. Then, $e_j \in K^{1/20}$, $j = 1, \ldots, 4$. Hence, in this case the fractional macroeconomic system (12) satisfies the necessary condition for the existence of chaos.

3. Dynamic analysis and numerical simulation

The purpose of this section is to investigate the complex dynamic properties and chaotic behaviors of the macroeconomic model (12) with incommensurate fractional orders via numerical simulations for the same system parameters given in the above section. To demonstrate this, we present the theorem that will enable us to derive the numerical formula for the suggested fractional discrete system.

Theorem 2[34] For the fractional difference equation

$$\begin{aligned} &{}^{\mathrm{C}}\Delta_{\eta}^{\gamma_i} x(s) = g(s+\gamma_i - 1, x(s+\gamma_i - 1)), \\ &\Delta^k x(s) = x_k, \quad m = \lceil \gamma_i \rceil + 1, \quad k = 0, 1, \ldots, m-1, \end{aligned} \tag{20}$$

the unique solution of this initial value problem (20) is given by

$$\begin{aligned} x(s) \ =\ & x_0(s) + \frac{1}{\Gamma(\gamma_i)} \sum_{l=\eta+m-\gamma_i}^{s-\gamma_i} (s - \sigma(l))^{(\gamma_i - 1)} \\ & \times g(l+\gamma_i - 1, x(l+\gamma_i - 1)), \quad s \in N_{\eta+m}, \end{aligned} \tag{21}$$

where

$$x_0(s) = \sum_{k=0}^{m-1} \frac{(s-\eta)^k}{\Gamma(k+1)} \Delta_k x(\eta). \tag{22}$$

The numerical formula of the discrete macroeconomic system (12) is constructed according to Theorem 2 as

$$\begin{aligned} x(n) &= x(0) + \frac{1}{\Gamma(\gamma_1)} \sum_{j=0}^{n-1} \frac{\Gamma(n-1-j+\gamma_1)}{\Gamma(n-1-j+1)} y(j), \\ y(n) &= y(0) + \frac{1}{\Gamma(\gamma_2)} \sum_{j=0}^{n-1} \frac{\Gamma(n-1-j+\gamma_2)}{\Gamma(n-1-j+1)} (\lambda - 1) y(j) \\ &\quad - (\lambda + 1) y^3(j) - \sigma x(j), \end{aligned} \tag{23}$$

where $x(0)$ and $y(0)$ denote initial conditions. This is a novel type of economic system characterized by "memory effects". As seen in Eq. (23), the states $x(n)$, $y(n)$ depends on all preceding variables, namely, $x(0), x(1), \ldots, x(n-1)$ and $y(0), y(1), \ldots, y(n-1)$.

Here, the route to the chaos of the proposed macroeconomic system with non-commensurate fractional orders (12) is extensively examined via a variety of numerical methods, including Lyapunov exponent calculations, bifurcation diagrams, and the display of phase portraits. In particular, the impact of parameter system λ, as well as the fractional values γ_i, $\forall i = 1, 2$, on the model dynamics are discussed in detail. All the numerical simulations are carried out using MATLAB software.

3.1. Bifurcation diagrams and maximum LEs

Here, we explore the fractional macroeconomic system dynamics (12) by varying λ and fractional order γ_i when the control parameter is assigned as $\sigma = 0.2$ and the initial condition is selected as $(x(0), y(0)) = (2, 0.1)$.

Case 3 Setting the system parameter $\lambda = -2.3$ and the fractional order values $(\gamma_1, \gamma_2) = (1, 1)$, the fractional macroeconomic model (12) with commensurate fractional values reveals a strange attractor. The phase portrait of the corresponding attractor is depicted in Fig. 2(f). We can notice that when $\gamma_1 = \gamma_2 = 1$, the trajectories of the proposed fractional model (12) are identical to those of the classical model in Fig. 1(a). Also, as seen in Fig. 2, when the fractional order values vary, the resulting trajectories change. For example, when $(\gamma_1, \gamma_2) = (0.97, 0.94)$ and $(\gamma_1, \gamma_2) = (0.97, 0.99)$, the fractional discrete model shows chaotic attractors while when $(\gamma_1, \gamma_2) = (0.53, 1)$ and $(\gamma_1, \gamma_2) = (0.53, 1)$; the fractional discrete model shows periodic attractors. When the fractional order γ_1 takes small values (close to 0), the trajectories of the proposed fractional map display chaotic attractors as shown in Fig. 2(a). To further investigate the effect of λ on the dynamics of the proposed system (12), we have plotted the bifurcation diagram using λ as a bifurcation parameter. In Fig. 3, corresponding to the three fractional-order values $(\gamma_1, \gamma_2) =$

$(0.97,0.99)$, $(\gamma_1,\gamma_2)=(0.97,0.94)$, $(\gamma_1,\gamma_2)=(1,0.01)$ for the parameter λ in the range $(-2.5,0.5)$. It can be observed from Fig. 3(a), for $(\gamma_1,\gamma_2)=(0.97,0.99)$, that the fractional discrete macroeconomic system (12) gradually changes from a chaotic state to a periodic state for $\lambda=-1.089$, through a period-doubling bifurcation. However, when γ_2 is decreased from 0.99 to 0.94, the chaotic motion shrinks from $(-2.38,-2.055)$ to $(-2.315,-2.155)$ and a periodic window appears in the interval $(-2.154,-2.037)$. Figure 3(c) shows that when the parameter γ_1 is decreased to the value $\gamma_1=0.01$, the states of the discrete map (12) indicate chaotic motion in the region $\lambda\in(-2.34,-1.912)$, whereas as the parameter λ is increased, the chaos disappears and eight-, four, and two-periodic orbits occur. One can infer that the changes in the fractional-order values have an impact on the bifurcation diagrams and influence the dynamic behaviors of the system.

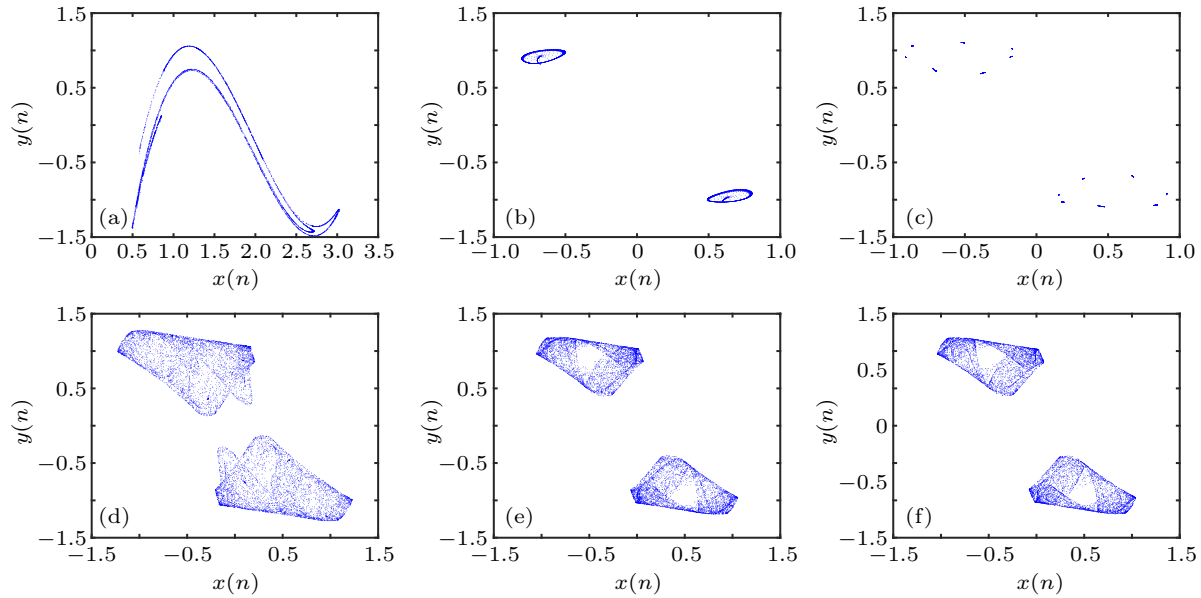


Fig. 2. Different phase portraits of the fractional macroeconomic discrete system (12) with $\lambda=-2.3$ for different incommensurate fractional order values: (a) $(\gamma_1,\gamma_2)=(0.01,1)$, (b) $(\gamma_1,\gamma_2)=(0.53,1)$, (c) $(\gamma_1,\gamma_2)=(0.83,1)$, (d) $(\gamma_1,\gamma_2)=(0.97,0.94)$, (e) $(\gamma_1,\gamma_2)=(0.97,0.99)$, (f) $\gamma_1=\gamma_2=1$.

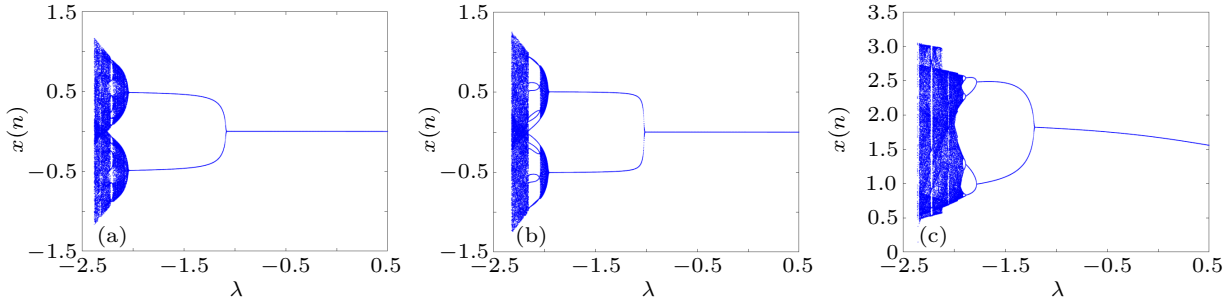


Fig. 3. Bifurcation diagram of the fractional macroeconomic discrete system (12) with fractional order values: (a) $(\gamma_1,\gamma_2)=(0.97,0.94)$, (b) $(\gamma_1,\gamma_2)=(0.97,0.99)$, (c) $\gamma_1=\gamma_2=1$.

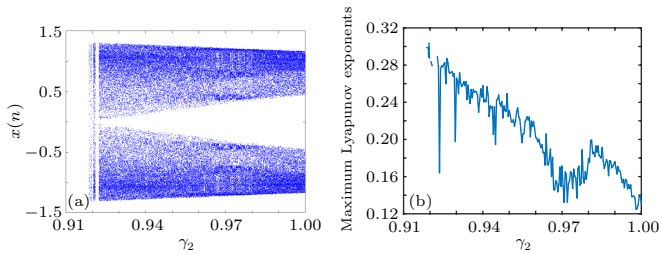


Fig. 4. (a) Bifurcation diagram versus $\gamma_2\in(0,1]$ of the fractional discrete macroeconomic system (12) with $\gamma_1=0.97$, (b) the corresponding MLE.

When the fractional order γ_2 is taken as an adjustable parameter, the bifurcation diagram is seen in Fig. 4(a), for parameter $\lambda=-2.3$ and fractional order $\gamma_1=0.97$. It is clear to see that there is a chaotic region when $\gamma_2\in(0.923,1]$. In addition, when the order is less than 0.915 the system will diverge. Despite the fact that bifurcation plots are helpful tools for identifying the presence of chaos in the fractional system, it is often most practical to compute or estimate the system's Lyapunov exponents. The maximum Lyapunov exponents (MLEs) can be calculated using the Jacobian matrix algorithm.[35] The MLEs are calculated and presented in Fig. 4(b). Obviously, the MLEs are in agreement with the corresponding bifurcation diagrams in Fig. 4(a), where they are positive in the chaotic regions.

Similarly, setting parameters $\lambda=-2.3$ and fractional order $\gamma_2=1$, figure 5 shows the bifurcation diagram and the MLEs for order $\gamma_1\in(0,1]$. When γ_1 is used as the bifurca-

tion parameter, the fractional macroeconomic system exhibits a relatively complete bifurcation route. For the smallest values of $\gamma_1 \in (0, 0.4248]$, chaotic behavior can be obtained with a positive Lyapunov exponent, while the MLEs change their values between negative and positive when $\gamma_1 \in [0.53, 0.84]$, so the chaos appears and disappears. In addition, when γ_1 becomes larger, chaos occurs again. To verify these findings, we compute the Lyapunov exponents (LE1 and LE2) at some values of the fractional order γ_i. When $(\gamma_1, \gamma_2) = (9.97, 0.99)$, the LE1 $= 0.17$ and LE2 $= -0.03$, indicating the system is chaotic. When $(\gamma_1, \gamma_2) = (0.83, 1)$, the LE1 $= -0.003$ and LE2 $= -0.012$, indicating that the system is periodic, which is in accordance with the phase portraits given in Fig. 2.

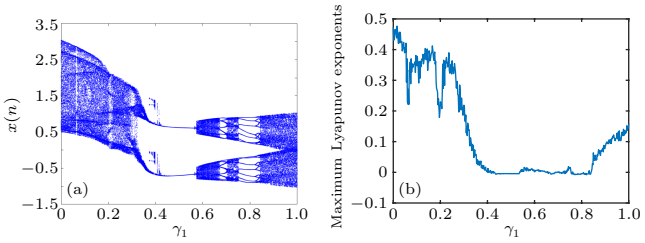


Fig. 5. (a) Bifurcation diagram versus $\gamma_1 \in (0, 1]$ of the fractional discrete macroeconomic system (12) with $\gamma_2 = 1$, (b) the corresponding MLE.

Remark 1 By choosing the same parameters as those in case 3, the resulting bifurcation diagram of the macroeconomic discrete system (12) with commensurate fractional orders is given in Fig. 6. One can see that the dynamic behaviors of the fractional-order system (12) are distinctly different when the fractional orders are incommensurate from those in which the fractional orders are commensurate.

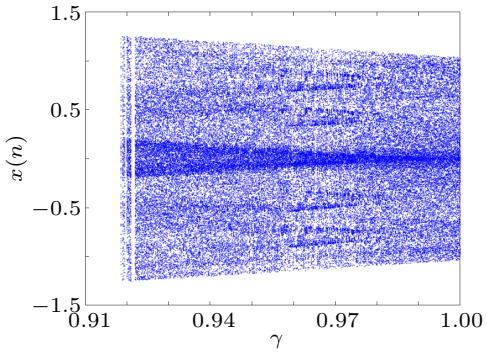


Fig. 6. Bifurcation diagram of the fractional macroeconomic discrete system (12) when $\gamma_1 = \gamma_2 \in (0, 1]$ and $\lambda = -2.3$

Case 4 Similarly, we explore the dynamics of the macroeconomic model with fixed system parameter $\lambda = 1.9$. Firstly, we explore the behaviors of the fractional macroeconomic discrete system when λ ranges from 0 to 2 for fractional order values $\gamma_1 = 0.95$, $\gamma_2 = 0.98$. The bifurcation diagram and the corresponding MLEs of the discrete macroeconomic model are displayed in Fig. 7. As depicted in Fig. 7(a), the stability of the fixed point is lost via Neimark–Sacker bifurcation at $\lambda = 0.85$ which results in chaos, where its corresponding Lyapunov exponent values are LE1 $= 0.0007$ and LE2 $= -0.0026$. Clearly, the bifurcation diagram and the corresponding MLEs are consistent. When $\lambda \in (1.681, 1.968)$ the corresponding MLEs are positive, confirming the presence of chaos after Neimark–Sacker bifurcation has occurred. In addition, Figs. 7(c) and 7(d) display the bifurcation diagrams for $(\gamma_1, \gamma_2) = (0.6, 0.9)$ and $(\gamma_1, \gamma_2) = (0.01, 0.9)$, respectively. One can infer that the bifurcation diagrams are quite different, in which the variations in the fractional orders affect the behaviors of the fractional discrete model (12). When $(\gamma_1, \gamma_2) = (0.01, 0.9)$ the system is stable with period-2 and 4 orbits occurring for $\lambda \in (0, 1.704)$, while for $\lambda > 1.704$, the system loses its stability and becomes chaotic, and the 10-periodic orbit occurs at $\lambda = 1.95$.

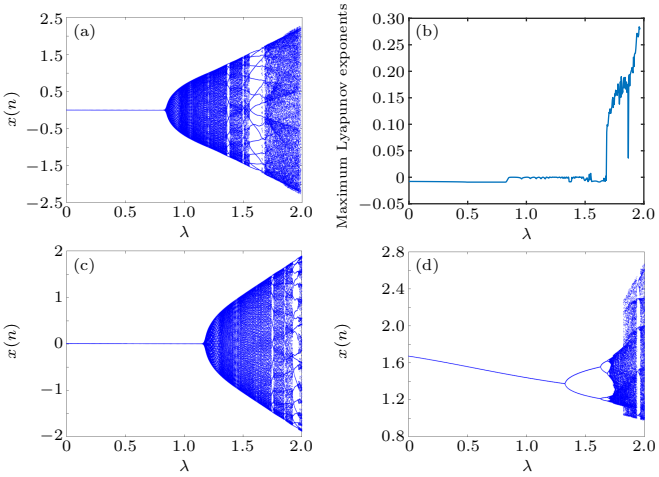


Fig. 7. (a) Bifurcation diagram of the fractional macroeconomic discrete system (12) versus $\lambda \in [0, 2]$ with $(\gamma_1, \gamma_2) = (0.95, 0.98)$, (b) the corresponding MLE. (c) Bifurcation diagram versus $\lambda \in [0, 2]$ with $(\gamma_1, \gamma_2) = (0.6, 0.9)$. (d) Bifurcation diagram versus $\lambda \in [0, 2]$ with $(\gamma_1, \gamma_2) = (0.01, 0.9)$.

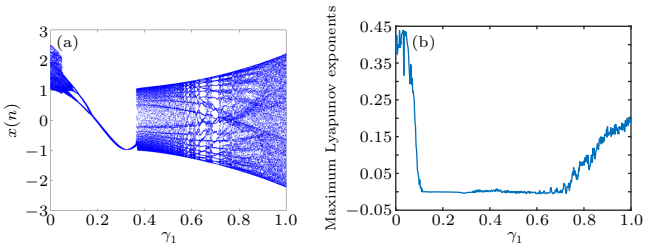


Fig. 8. (a) Bifurcation diagram versus $\gamma_1 \in (0, 1]$ of the fractional macroeconomic discrete system with $\gamma_2 = 0.9$ and $\lambda = 1.9$, (b) the corresponding MLE.

To investigate the bifurcation scenarios with these two fractional order values, a bifurcation diagram is drawn in the region $\gamma_1 \in (0, 1]$ with $\gamma_2 = 0.9$ and $\lambda = 1.9$, as shown in Fig. 8. We can observe that when $\gamma_1 > 0.724$ the MLE is positive, which indicates that the system is chaotic. As γ_1 decreases from 0.328, the system becomes totally periodic. If we continue to decrease γ_1, a small chaotic region can be detected in the range $[0, 0.11]$.

Considering the bifurcation diagram and the MLEs of the discrete macroeconomic system (12) with $\gamma_1 = 0.9$ and order $\gamma_2 \in (0, 1]$ (see Fig. 9). As can be observed, the dynamic behavior of the system (12) ranges from periodic state to chaotic

state. Namely, the macroeconomic model is chaotic when $\gamma_2 \in (0, 0.72]$, while it is periodic at the rang $[0.75, 1]$. These different dynamics are expressed by the phase portraits as depicted in Fig. 10. Clearly, the phase portraits in Fig. 10 demonstrate the changes of the fractional macroeconomic model's complex dynamical behaviors, including invariant closed curve, periodic attractor and for chaotic attractors. The above numerical simulation show that the fractional macroeconomic model has rich dynamical properties.

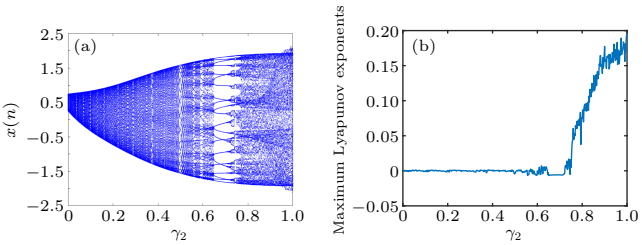


Fig. 9. (a) Bifurcation diagram of the fractional macroeconomic discrete system (12) versus $\gamma_2 \in (0, 1]$ with $\gamma_1 = 0.9$ and $\lambda = 1.9$, (b) the corresponding MLE.

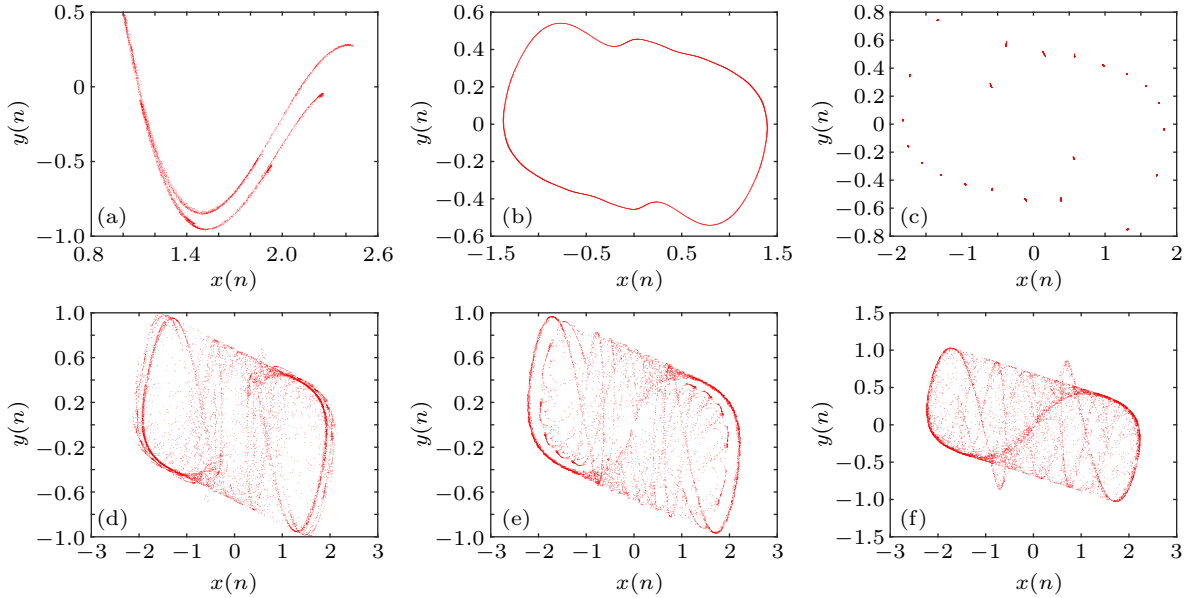


Fig. 10. Different phase portraits of the fractional macroeconomic discrete system (12) with different fractional order values of $\gamma = (\gamma_1, \gamma_2)$, (a) $(\gamma_1, \gamma_2) = (0.01, 0.9)$, (b) $(\gamma_1, \gamma_2) = (0.9, 0.4)$, (c) $(\gamma_1, \gamma_2) = (0.9, 0.7)$, (d) $(\gamma_1, \gamma_2) = (0.9, 1)$, (e) $(\gamma_1, \gamma_2) = (1, 0.9)$, (f) $(\gamma_1, \gamma_2) = (1, 1)$.

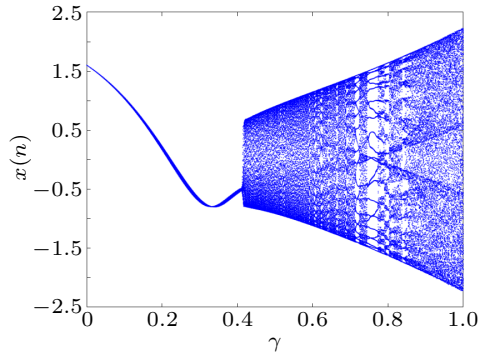


Fig. 11. Bifurcation diagram of the fractional macroeconomic discrete system (12) with commensurate order $\gamma_1 = \gamma_2 \in (0, 1]$ and $\lambda = 1.9$.

Remark 2 Taking the commensurate-order case $\gamma_1 = \gamma_2 \in (0, 1]$, the bifurcation diagram of the fractional macroeconomic discrete-time system is displayed in Fig. 11. One can notice that the bifurcation changes greatly when the order is commensurate. As can be seen from Fig. 11, when $\gamma \in [0.83, 1]$ the system is chaotic. However, when the orders of the macroeconomic system are incommensurate, the system is chaotic when $\gamma_1 = 0.9$ and $\gamma_2 \in [0.72, 1]$, as shown in Fig. 9. Therefore, the dynamic behavior of the fractional macroeconomic discrete-time system with commensurate order is less complex. This further demonstrates that the incommensurate order can describe the economic behavior more accurately.

4. The 0–1 test and complexity analysis of the incommensurate macroeconomic discrete-time system

The complexity of chaotic characteristics may also be used to analyze the dynamic behaviors of chaotic systems. The model becomes increasingly chaotic as its level of complexity rises. The approximate entropy method and the C_0 measure are used to determine the complexity of the discrete macroeconomic model (12). The 0–1 test technique is also used to investigate and assess the chaotic behavior of a discrete-time macroeconomic system with incommensurate order.

4.1. The 0–1 test

The 0–1 test is one technique that may be used to discover chaos.[36] In comparison to the Lyapunov exponents method, the 0–1 test acts on the time series, eliminating the need for phase reconstruction. Specifically, to illustrate the method, we write the translation variables using the time series $x(k)$ for $k = 1, 2, \ldots, N$, as $p_c(n) = \sum_{i=1}^{n} x(i)\cos(ci)$ and $q_c(n) = \sum_{i=1}^{n} x(i)\sin(ci)$, where $c \in (0, \pi)$ is arbitrary. The

mean square displacement $M_c(n)$ is denoted by

$$M_c(n) = \lim_{x\to\infty} \frac{1}{N} \sum_{j=1}^{N} \Big[\big(p_c(i+n) - p_c(i)\big)^2 + \big(q_c(j+n) - q_c(i)\big)^2 \Big], \quad n \le \frac{N}{10}. \tag{24}$$

The asymptotic growth rate K_c is denoted by the expression

$$K_c = \lim_{n\to\infty} \frac{\log M_c(n)}{\log n}. \tag{25}$$

Using the results obtained by plotting p_c and q_c or by calculating the asymptotic growth rate $K = \text{median}(K_c)$, we may determine if there is chaos in the macroeconomic system with incommensurate order. If the trajectories exhibit Brownian-like behaviors and K approaches 1, the dynamics of the incommensurate-order macroeconomic model are chaotic, whereas, if the trajectories' behaviors are bounded and K approaches 0, the dynamics of the incommensurate order discrete-time macroeconomic system are regular.

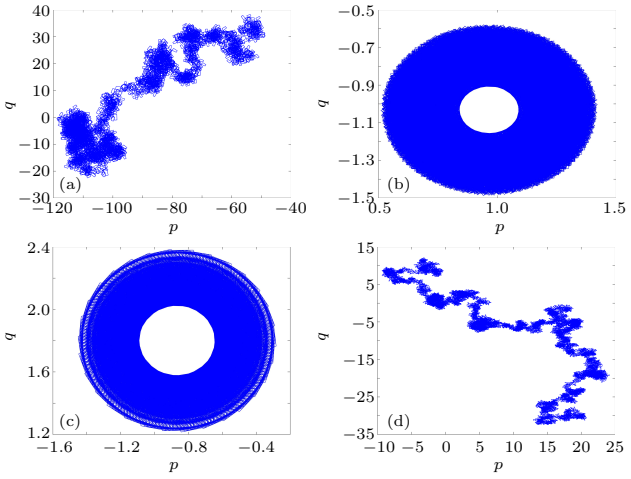


Fig. 12. The p–q trajectories of the fractional macroeconomic discrete system (12) with $\lambda = -2.3$ and for (a) $(\gamma_1, \gamma_2) = (0.01, 1)$, (b) $(\gamma_1, \gamma_2) = (0.53, 1)$, (c) $(\gamma_1, \gamma_2) = (0.83, 1)$, (d) $(\gamma_1, \gamma_2) = (0.97, 0.94)$.

Here, the 0–1 test of the incommensurate-order discrete-time macroeconomic system (12) is performed directly on the states $x(n)$. When $\lambda = -2.3$, and both of the fractional order values γ_1 and γ_2 vary, the p_c–q_c plots of the fractional discrete-time macroeconomic system (12) are shown in Fig. 12. In light of Fig. 12, it is easily noticed that the trajectories exhibit Brownian-like behavior when $(\gamma_1, \gamma_2) = (0.01, 1)$ and $(\gamma_1, \gamma_2) = (0.97, 094)$ which indicates that the fractional macroeconomic system with incommensurate order is chaotic. Otherwise, when $(\gamma_1, \gamma_2) = (0.53, 1)$ and $(\gamma_1, \gamma_2) = (0.83, 1)$ the fractional macroeconomic system provides a regular behavior in which it is evident that the translation components p and q in the p–q plan show bounded behavior. Furthermore, figure 13 shows the p–q trajectories corresponding with various fractional orders, in which $\lambda = 1.9$. When $(\gamma_1, \gamma_2) = (0.9, 0.4)$ and $(\gamma_1, \gamma_2) = (0.9, 0.7)$ the trajectories in the p–q plane are bounded which means that the system is regular, and for the commensurate order, i.e. when $(\gamma_1, \gamma_2) = (0.92, 0.92)$ and integer order $(\gamma_1, \gamma_2) = (1, 1)$, figure 13 shows Brownian-like trajectories in the p–q plane which indicates that the macroeconomic system is chaotic. Thus, it can be seen that the 0–1 test confirms very well the results of the bifurcation diagram and the MLEs shown above.

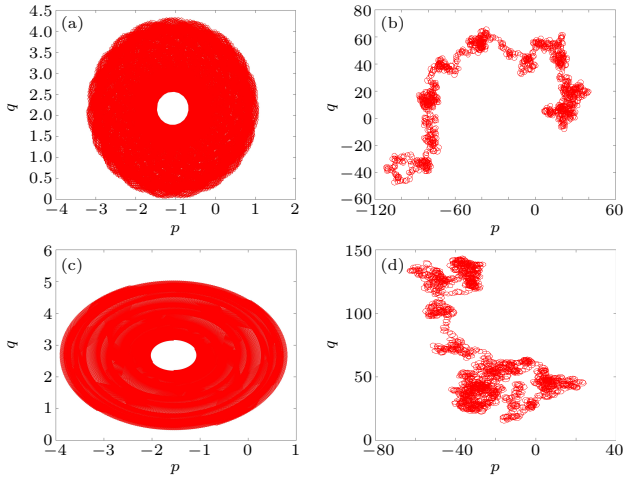


Fig. 13. The p–q trajectories of the fractional macroeconomic discrete system (12) with $\lambda = 1.9$ and for (a) $(\gamma_1, \gamma_2) = (0.9, 0.4)$, (b) $(\gamma_1, \gamma_2) = (0.92, 0.92)$, (c) $(\gamma_1, \gamma_2) = (0.9, 0.7)$, (d) $(\gamma_1, \gamma_2) = (1, 1)$.

4.2. Approximate entropy

The approximate entropy (ApEn) is a regularity measure that may be used to quantify the degree of complexity in the system that is created by a time sequence. In general, sequences with larger values of ApEn are regarded as being the more complex ones.[37] Note that the value of ApEn depends on two important parameters: the embedding dimension m and the similar tolerance r. In this paper we put $r = 0.2\text{std}(x)$ and $m = 2$; $\text{std}(x)$ represents the standard deviation of the data x. ApEn is computed as follows:

$$\text{ApEn} = \phi^m(r) - \phi^{m+1}(r), \tag{26}$$

where $\phi^m(r)$ is given by

$$\phi^m(r) = \frac{1}{n-m-1} \sum_{i=1}^{n-m+1} \log C_i^m(r). \tag{27}$$

The ApEn complexity of the macroeconomic discrete-time system with incommensurate order γ_1, γ_2 is studied and the findings are shown in Table 1. As can be seen, the ApEn complexity of the macroeconomic system (12) varies when γ_i $(i = 1, 2)$ varies, and the highest ApEn are obtained when the discrete-time macroeconomic system is chaotic, which is consistent with the findings of the maximum LE. As a consequence, to obtain a high structural complexity, we need to be conscious in determining the values of the fractional order γ_i in the system (12).

Table 1. Approximate entropy test for different incommensurate fractional order.

λ	γ_1	γ_2	ApEn
−2.3	0.01	1	0.5495
−2.3	0.53	1	0.0460
−2.3	0.97	0.99	0.4647
1.9	0.9	0.7	0.1126
1.9	1	0.9	0.4705
1.9	1	1	0.4961

4.3. The C_0 complexity

According to the inverse Fourier transform, we use the C_0 algorithm to compute the complexity of chaotic models. The algorithm is described as follows:[38] First, the Fourier transform of the series $[x(0),x(1),\ldots,x(N-1)]$ is figured out by

$$Y_N(j)=\frac{1}{N}\sum_{k=0}^{N-1}x(k)\exp\left(-2\pi\mathrm{i}\left(\frac{kj}{N}\right)\right),$$
$$j=0,1,\ldots,N-1. \tag{28}$$

We calculate the mean square as $G_N=(1/N)\sum_{j=0}^{N-1}|Y_N(j)|^2$ and we let

$$\overline{Y}_N(j)=\begin{cases}Y_N(j) & \text{if } \|Y_N(j)\|^2>rG_N,\\ 0 & \text{if } \|Y_N(j)\|^2\le rG_N.\end{cases} \tag{29}$$

Now, we compute the inverse Fourier transform of $\overline{Y}_N$ as

$$\bar{y}(m)=\frac{1}{N}\sum_{j=0}^{N-1}\overline{Y}_N(j)\exp\left(2\pi\mathrm{i}\left(\frac{kj}{N}\right)\right),$$
$$m=0,1,\ldots,N-1. \tag{30}$$

Then, we calculate the C_0 complexity as

$$C_0=\frac{\sum_{m=0}^{N-1}\|\bar{y}(m)-x(m)\|}{\sum_{m=0}^{N-1}\|x(m)\|^2}. \tag{31}$$

The C_0 complexity of the fractional macroeconomic discrete-time system (12) with varying fractional order values γ_i are calculated and the results are presented in Fig. 14. From Fig. 14, as with the MLEs and ApEn, the C_0 complexity value of the fractional macroeconomic system increases rapidly when the value of the fractional-order γ_i approaches 1. As a consequence, we can conclude that the C_0 method is an effective tool for measuring complexity accurately. When $\gamma_2=0.9$, figure 14(a) illustrates that the fractional macroeconomic system (12) has higher complexity when $\gamma_1\in(0.724,1]$. When $\gamma_1=0.9$, the C_0 complexity increases with an increase in fractional order γ_2 as shown in Fig. 14(b). These results are consistent with previous results of the MLE shown in Figs. 8(b) and 9(b).

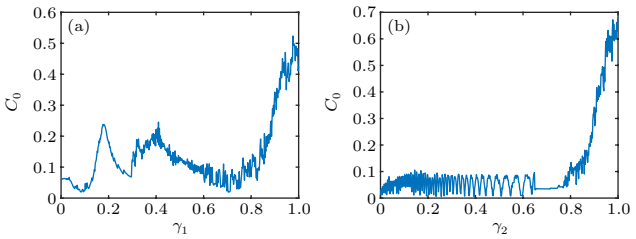


Fig. 14. The C_0 complexity analysis of the system (12) with (a) $\gamma_2=0.9$, (b) $\gamma_1=0.9$.

5. Conclusion

This paper has considered a new fractional macroeconomic discrete system. Through using bifurcation diagrams, phase attractors and maximum LE, the complex dynamics of this proposed system, with some changes in the incommensurate fractional order, are discussed. The results show that chaos exists in this fractional macroeconomic system and that the type and range of chaotic behavior depend on the fractional order values. Furthermore, we have also computed and presented the 0–1 test, C_0 complexity and approximation entropy of the macroeconomic discrete system for various values of the fractional order to illustrate and measure the complex dynamics of the system. In addition, the results show that the fractional macroeconomic system with incommensurate order has greater complexity compared to that of the fractional system with commensurate order. Owing to the diverse complex dynamical behavior, this study may offer a theoretical foundation and assist in the advancement of research in economic fields.

References

[1] Podlubny I 1999 *Fractional Differential Equations* vol. 198 (Elsevier)
[2] Ostalczyk P 2015 *Discrete Fractional Calculus: Applications in Control and Image Processing* (Singapore: World Scientific Publishing)
[3] Yan B, He S and Wang S 2020 *Math. Probl. Eng.* **2020** 2468134
[4] Edelman M, Macau E and Sanjuan M 2018 *Chaotic, Fractional, and Complex Dynamics: New Insights and Perspectives* (Berlin: Springer Cham) pp. 147–71
[5] Gao F, Li W Q, Tong H Q and Li X L 2019 *Chin. Phys. B* **28** 090501
[6] Liu C R, Yu P, Chen X Z, Xu H Y, Huang L and Lai Y C 2019 *Chin. Phys. B* **28** 100501
[7] Ouannas A, Khennaoui A A, Momani S, Pham V T and El-Khazali R 2020 *Chin. Phys. B* **29** 050504
[8] Yu Y J and Wang Z H 2013 *Chin. Phys. Lett.* **30** 110201
[9] Wu J, Zhan X S, Zhang X H and Gao H L 2012 *Chin. Phys. Lett.* **29** 050203
[10] Zhang D, Shi J Q, Sun Y, Yang X H and Ye L 2019 *Acta. Phys. Sin.* **68** 240502 (in Chinese)
[11] Shukla M K and Sharma B B 2017 *AEU Int. J. Electron. Commun.* **78** 265
[12] Peng Y, He S and Sun K 2021 *Results Phys.* **24** 104106
[13] Khennaoui A A, Ouannas A, Bendoukha S, Grassi G, Lozi R P and Pham V T 2019 *Chaos Solitons Fractals* **119** 150
[14] Almatroud A O, Khennaoui A A, Ouannas A and Pham V T 2021 *Int. J. Nonlinear Sci. Numer. Simul.* **2021**
[15] Ouannas A, Khennaoui A A, Momani S, Grassi G and Pham V T 2020 *AIP Adv.* **10** 045310
[16] Yan S L 2019 *Acta. Phys. Sin.* **68** 170502 (in Chinese)

[17] Abbes A, Ouannas A, Shawagfeh N and Khennaoui A A 2022 *Eur. Phys. J. Plus* **137** 1
[18] He Z Y, Abbes A, Jahanshahi H, Alotaibi N D and Wang Y 2022 *Mathematics* **10** 165
[19] Wieland V and Wolters M H 2010 *Econ. Theory* **47** 247
[20] Morgenstern O 1963 *Limits to the Uses of Mathematics in Economics* (Philadelphia: American Academy of Political and Social Science)
[21] Allen R G D 1970 *Macro-Economic Theory: A Mathematical Treatment* (London: Macmillan)
[22] Blanchard O 2018 *Oxford Rev. Econ. Policy.* **34** 43
[23] Masson P 1999 *J. Int. Money Finance* **18** 587
[24] Aldurayhim A, Elsadany A A and Elsonbaty A 2021 *Fractals* **29**
[25] Xin B, Peng W and Kwon Y 2020 *Physica A* **558** 124993
[26] Khennaoui A A, Almatroud A O, Ouannas A, Al-sawalha M M, Grassi G and Pham V T 2021 *Discrete Contin. Dyn. Syst. B* **26** 4549
[27] Hu Z and Chen W 2013 *Discrete Dyn. Nat. Soc.* **2013** 275134
[28] Chu Y M, Bekiros S, Zambrano-Serrano E, Orozco-López O, Lahmiri S, Jahanshahi H and Aly A A 2021 *Chaos Solitons Fractals* **145** 110776
[29] Puu T 1986 *Reg. Sci. Urban Econ.* **16** 81
[30] Atici F M and Eloe P 2009 *Electron. J. Qual. Theory Differ. Equ.* **3** 1
[31] Abdeljawad T 2011 *Comput. Math. Appl.* **62** 1602
[32] Puu T 1986 *Reg. Sci. Urban Econ.* **16** 81
[33] Djenina N, Ouannas A, Batiha I M, Grassi G and Pham V T 2020 *Mathematics* **8** 1754
[34] Wu G C and Baleanu D 2013 *Nonlinear Dyn.* **75** 283
[35] Wu G C and Baleanu D 2015 *Commun. Nonlinear Sci. Numer. Simul.* **22** 95
[36] Gottwald G A and Melbourne I 2016 *Chaos Detection and Predictability* (Berlin-Springer) pp. 221–247
[37] Pincus S M 1991 *Proc. Natl. Acad. Sci. USA* **88** 2297
[38] Ran J 2018 *Adv. Differ. Equ.* **2018** 1