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Finite-time Mittag–Leffler synchronization of fractional-order complex-valued memristive neural networks with time delay

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Without dividing the complex-valued systems into two real-valued ones, a class of fractional-order complex-valued memristive neural networks (FCVMNNs) with time delay is investigated. Firstly, based on the complex-valued sign function, a novel complex-valued feedback controller is devised to research such systems. Under the framework of Filippov solution, differential inclusion theory and Lyapunov stability theorem, the finite-time Mittag–Leffler synchronization (FTMLS) of FCVMNNs with time delay can be realized. Meanwhile, the upper bound of the synchronization settling time (SST) is less conservative than previous results. In addition, by adjusting controller parameters, the global asymptotic synchronization of FCVMNNs with time delay can also be realized, which improves and enrich some existing results. Lastly, some simulation examples are designed to verify the validity of conclusions.

Keywords: finite-time Mittag–Leffler synchronization, fractional-order complex-valued memristive neural networks, time delay

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1. Introduction

As a crucial component of artificial intelligence, neural networks have been widely applied in optimization, signal processing, associative memory, parallel computation, pattern recognition, artificial intelligence, image processing, control system, and so on.[1–5] Most of the related results concentrate on real-valued neural networks (RVNNs).[6–8] Notably, the research direction of neural networks is changing from the traditional real-valued domain to the complex-valued domain, and some scholars are beginning to study complex-valued neural networks (CVNNs).[9] Compared with RVNNs, CVNNs have more complex characteristics and wider practical applications, which make it possible to solve those problems that RVNNs cannot, for instance, the problem of XOR[10] and the detection problem of symmetry.[11] Nowadays, an increasing number of scholars carry out the study on CVNNs,[12–15] and the potential of CVNNs in electromagnetic, optoelectronics, filtering, remote sensing, signal processing, and other areas are also developed.[16–20]

Fractional-order calculus can describe the systems with genetic and memory properties more accurately due to its nonlocality and infinite memory.[21–25] In recent years, fractional-order complex-valued neural networks (FCVNNs) have drawn extensive attention and some interesting phenomenons have been reported.[26–34] It is noteworthy that most of the research methods are based on splitting the FCVNNs into two real-valued systems and obtaining the relevant conditions by studying these real-valued systems. Since the separation of complex-valued activation functions cannot always be explicitly separated and the increase in system dimensionality may lead to the complexity of system analysis. Therefore, it may be more valuable to analyze the FCVNNs as an entirety. In Ref. [34] we considered the FCVNNs as an entirety and obtained the global dissipativity and quasi-Mittag–Leffler synchronization of the systems by proposing a new fractional-order inequality.

In 1971, Chua first introduced the nonlinear resistive element which is called the memristor.[35] Since memristors have memory properties similar to biological synapses, the memristive neural networks (MNNs) can better simulate the human brain and have been widely used in novel biological models for associative memory, real-time encoding and compression, optimization with massively parallel computing,[36–39] etc. Nowadays, some interesting results on fractional-order complex-valued memristive neural networks (FCVMNNs) have been published.[40–45] However, due to the limited switching speed of the amplifier, time delay is unavoidable in neural network circuits, which could influence the stability of the system.[45,46] Taking these facts into consideration, time delay should be considered in the FCVMNNs.

Synchronization was raised by Pecora and Carroll in 1990,[47] which denotes the phenomenon of system states consistency in time. In recent years, synchronization has been widely applied in secure communication, information processing, biology,[48–50] etc. Currently, related synchronization results of FCVMNNs have been studied in literature.[41,43,44,51,52] In Ref. [41], by designing a simple linear feedback controller, the global asymptotical synchroniza-

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tion of FCVMNNs with parameter uncertainties and multiple time delays was derived according to the comparison principle. By using the Gronwall–Bellman integral inequalities and feedback control technique, the finite-time projective synchronization of FCVMNNs with time delays was achieved in Ref. [43]. The ultimate Mittag–Beffler synchronization of FCVMNNs with time delays was achieved by the fractional-order Leibniz rule, Razumikhin-type method and complex-valued adaptive controller in Ref. [44]. In Ref. [51], the authors dealt with the complex projection synchronization for a class of FCVMNNs with multiple delays by employing the hybrid control strategy. Some sufficient criteria were obtained to ensure the quasi-synchronization of FCVMNNs with switching jumps mismatch by constructing a suitable Lyapunov function in Ref. [52]. Note that most scholars concentrate on asymptotic behavior, which means that synchronization is achieved when the time is infinite. Compared to the infinite-time synchronization, the finite-time synchronization has a faster convergence speed and better robustness,[53] which is more reasonable in realistic applications. It is worth noting that finite-time synchronization has been applied in many practical applications, such as image encryption and satellite positioning. Hence, it is indispensable to investigate the finite-time synchronization of FCVMNNs.

Taking into account the above factors, in this paper we investigate the FTMLS of FCVMNNs with time delay, and the major contributions are summarized as follows:

(i) Unlike the traditional separation technique in Refs. [26,28,29,32,40–45], this paper considers the FCVMNNs as an entirety. Based on the property of complex-variable function and Lyapunov stability theorem, some sufficient conditions are established to realize the FTMLS of FCVMNNs with time delay.

(ii) Based on the complex-valued sign function, a novel complex-valued feedback controller is devised to realize the FTMLS of FCVMNNs with time delay. By adjusting controller parameters, the FTMLS of FCVMNNs with and without time delay can also be realized. In addition, the asymptotic synchronization of FCVMNNs with time delay can be obtained by utilizing the comparison theorem, which extends the existing results in Refs. [33,42,43].

(iii) The upper bound of synchronization settling time (SST) can be estimated by establishing a new fractional differential inequality. Moreover, compared with the previous results in Ref. [43], the upper bound of SST in this paper is more flexible and less conservative.

This article proceeds as follows. In Section 2, some preliminaries are given. The main results and simulation examples are provided in Sections 3 and 4, respectively. Finally, the summary is provided in Section 5.

Notations: In this paper, $\mathbb{N}^+$ represents the set of positive integers. $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and complex numbers. $\mathbb{C}^n$ represent the n-dimensional complex vectors. For any $x\in\mathbb{C}$, $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ represent the real part and imaginary part of x, $\bar{x}$ denotes the conjugate of x, $|x|_1 = |\mathrm{Re}(x)|+|\mathrm{Im}(x)|$, $|x|_2=\sqrt{x\bar{x}}$. For any $X=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}\in\mathbb{C}^n$, $||X||_1=\sum\limits_{i=1}^{n}|x_i|_1$, $||X||_2=\left(\sum\limits_{i=1}^{n}|x_i|_2^2\right)^{1/2}$.

2. Preliminaries and system description

2.1. Preliminaries

Definition 1[54] $g(t):[0,+\infty)\to\mathbb{C}$ is an integrable function and the fractional-order integral of $g(t)$ is

$$ {}_0I_t^{\varepsilon}g(t)=\frac{1}{\Gamma(\varepsilon)}\int_0^t(t-\sigma)^{\varepsilon-1}g(\sigma)\,\mathrm{d}\sigma,\quad \varepsilon>0, $$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2[54] Define $g(t)\in C^n([0,+\infty),\mathbb{C})$, the Caputo fractional-order derivative of $g(t)$ is

$$ {}_0^CD_t^{\varepsilon}g(t)=\frac{1}{\Gamma(n-\varepsilon)}\int_0^t\frac{g^{(n)}(\sigma)}{(t-\sigma)^{\varepsilon-n+1}}\,\mathrm{d}\sigma, $$

where $n\in\mathbb{N}^+$ and $n-1<\varepsilon<n$.

Definition 3[54] The two parameters Mittag–Leffler function is

$$ E_{\lambda,\mu}(\delta)=\sum_{p=0}^{\infty}\frac{\delta^p}{\Gamma(\lambda p+\mu)}, $$

where $\lambda>0$, $\mu>0$ and $\delta\in\mathbb{C}$. Particularly, when $\lambda=\beta=1$, $E_{1,1}(\delta)=\mathrm{e}^{\delta}$.

Lemma 1[33] Let $0<\varepsilon<1$, $\sigma\in\mathbb{R}$ and $g(t)\in C^1([0,+\infty),\mathbb{C})$, then

$$ {}_0^CD_t^{\varepsilon}g^{\sigma}(t)=\frac{\Gamma(1+\sigma)}{\Gamma(1+\sigma-\varepsilon)}g^{\sigma-\varepsilon}(t){}_0^CD_t^{\varepsilon}g(t). $$

Lemma 2[33] For any $r(t)\in\mathbb{C}$, $0<\varepsilon<1$, the following formulas hold:

(i) ${}_0^CD_t^{\varepsilon}|r(t)|_1\le\frac{1}{2}\left(\overline{[r(t)]}{}_0^CD_t^{\varepsilon}r(t)+[r(t)]{}_0^CD_t^{\varepsilon}\overline{r(t)}\right)$.

(ii) $\overline{[r(t)]}[r(t)]=\mathrm{sign}^2(\mathrm{Re}(r(t)))+\mathrm{sign}^2(\mathrm{Im}(r(t)))=|[r(t)]|_1$, where $[r(t)]=\mathrm{sign}(\mathrm{Re}(r(t)))+\mathrm{i}\,\mathrm{sign}(\mathrm{Im}(r(t)))$ is the sign function of $r(t)$.

Lemma 3[51] Let $L(t):[0,+\infty)\to\mathbb{C}$ satisfy

$$ {}_0^CD_t^{\varepsilon}L(t)\le\rho L(t),\quad 0<\varepsilon<1, $$

where $\rho\in\mathbb{R}$, then the following inequality hold:

$$ L(t)\le L(0)E_{\varepsilon}(\rho t^{\varepsilon}). $$

Lemma 4[55] Let a continuously differentiable function $L(t):[0,+\infty)\to\mathbb{C}$ satisfy

$$ \begin{cases} {}_0^CD_t^{\varepsilon}L(t)=-\sigma L(t)+\sum\limits_{j=1}^{n}\iota_jL(t-\tau), & 0<\varepsilon<1,\\ L(t)=\vartheta(t). \end{cases} \tag{1} $$

If $\sigma>0$ and $\iota_i>0$, then $\lim_{t\to+\infty}L(t)=0$.

2.2. System description

The drive system is denoted as

$$\begin{aligned} {}_0^C D_t^{\varepsilon} s_i(t) \;=\; & -c_i s_i(t) + \sum_{j=1}^{n} a_{ij}(s_j(t)) f_j(s_j(t)) \\ & + \sum_{j=1}^{n} b_{ij}(s_j(t)) g_j(s_j(t-\tau)) + I_i(t), \\ & i, j \in \mathbb{N}^+, \end{aligned} \tag{2}$$

where $\varepsilon \in (0,1)$; $s_i(t) \in \mathbb{C}^n$ represents the state of the i-th neuron at the time t; $c_i > 0$ denotes the self-regulating constant; $f_j(\cdot), g_j(\cdot) \in \mathbb{C}$ are the activation functions of the j-th neuron at the time t and $t-\tau$, respectively; $I_i(t) \in \mathbb{C}$ denotes the external input; $s_i(0) = s_{i0} \in \mathbb{C}$ is the initial condition of Eq. (2); $a_{ij}(s_j(t))$ and $b_{ij}(s_j(t))$ are complex-valued memristive weights, defined by

$$a_{ij}(s_j(t)) = \begin{cases} \dot{a}_{ij} & |s_j(t)| < W_j, \\ \dot{a}_{ij} \quad \text{or} \quad \ddot{a}_{ij} & |s_j(t)| = W_j, \\ \ddot{a}_{ij} & |s_j(t)| > W_j, \end{cases}$$

$$b_{ij}(s_j(t)) = \begin{cases} \dot{b}_{ij} & |s_j(t)| < W_j, \\ \dot{b}_{ij} \quad \text{or} \quad \ddot{b}_{ij} & |s_j(t)| = W_j, \\ \ddot{b}_{ij} & |s_j(t)| > W_j, \end{cases}$$

where the memristive synaptic weights $\dot{a}_{ij}, \ddot{a}_{ij}, \dot{b}_{ij}, \ddot{b}_{ij} \in \mathbb{C}$, and $W_j > 0$ is the switching jump.

The response system is

$$\begin{aligned} {}_0^C D_t^{\varepsilon} \bar{s}_i(t) \;=\; & -c_i \bar{s}_i(t) + \sum_{j=1}^{n} a_{ij}(\bar{s}_j(t)) f_j(\bar{s}_j(t)) \\ & + \sum_{j=1}^{n} b_{ij}(\bar{s}_j(t)) g_j(\bar{s}_j(t-\tau)) + I_i(t) + u_i(t), \end{aligned} \tag{3}$$

where $\varepsilon \in (0,1)$; $\bar{s}_i(t) \in \mathbb{C}$ represents the state of system (3), $u_i(t)$ is the controller; $\bar{s}_i(0) = \bar{s}_{i0} \in \mathbb{C}$ is the initial condition (3); $a_{ij}(\bar{s}_j(t))$ and $b_{ij}(\bar{s}_j(t))$ are complex-valued memristive weights, defined by

$$a_{ij}(\bar{s}_j(t)) = \begin{cases} \dot{a}_{ij} & |\bar{s}_j(t)| < W_j, \\ \dot{a}_{ij} \quad \text{or} \quad \ddot{a}_{ij} & |\bar{s}_j(t)| = W_j, \\ \ddot{a}_{ij} & |\bar{s}_j(t)| > W_j, \end{cases}$$

$$b_{ij}(\bar{s}_j(t)) = \begin{cases} \dot{b}_{ij} & |\bar{s}_j(t)| < W_j, \\ \dot{b}_{ij} \quad \text{or} \quad \ddot{b}_{ij} & |\bar{s}_j(t)| = W_j, \\ \ddot{b}_{ij} & |\bar{s}_j(t)| > W_j, \end{cases}$$

where $\dot{a}_{ij}, \ddot{a}_{ij}, \dot{b}_{ij}, \ddot{b}_{ij} \in \mathbb{C}$ are memristive synaptic weights.

To improve the robustness of FCVMNNs with time delay, we suppose that the activation functions satisfy the following conditions:

(H1) For all $j \in \mathbb{N}^+$, $\iota, \ell \in \mathbb{C}$, $\exists\, F_j, G_j > 0$ such that

$$\begin{aligned} &\left| f_j(\iota) - f_j(\ell) \right| \le F_j |\iota - \ell|, \\ &\left| g_j(\iota) - g_j(\ell) \right| \le G_j |\iota - \ell|. \end{aligned}$$

In FCVMNNs (2) and (3), the memristive connection weights $a_{ij}(\cdot)$ and $b_{ij}(\cdot)$ are discontinuous. Thus, we deal with such systems under the framework of Filippov solution.[56]

Through the differential inclusion theory, system (2) can be transformed to

$$\begin{aligned} {}_0^C D_t^{\varepsilon} s_i(t) \;\in\; & -c_i s_i(t) + \sum_{j=1}^{n} K[a_{ij}(s_j(t))] f_j(s_j(t)) \\ & + \sum_{j=1}^{n} K[b_{ij}(s_j(t))] g_j(s_j(t-\tau)) + I_i(t), \end{aligned} \tag{4}$$

where

$$K[a_{ij}(s_j(t))] = \begin{cases} \dot{a}_{ij} & |s_j(t)| < W_j, \\ \overline{co}\{\dot{a}_{ij}, \ddot{a}_{ij}\} & |s_j(t)| = W_j, \\ \ddot{a}_{ij} & |s_j(t)| > W_j, \end{cases}$$

$$K[b_{ij}(s_j(t))] = \begin{cases} \dot{b}_{ij} & |s_j(t)| < W_j, \\ \overline{co}\{\dot{b}_{ij}, \ddot{b}_{ij}\} & |s_j(t)| = W_j, \\ \ddot{b}_{ij} & |s_j(t)| > W_j, \end{cases}$$

for $i, j \in \mathbb{N}^+$, or there exist $\varphi_{ij}(t) \in K[a_{ij}(s_j(t))]$ and $\beta_{ij}(t) \in K[b_{ij}(s_j(t))]$ such that

$$\begin{aligned} {}_0^C D_t^{\varepsilon} s_i(t) \;=\; & -c_i s_i(t) + \sum_{j=1}^{n} \varphi_{ij}(t) f_j(s_j(t)) \\ & + \sum_{j=1}^{n} \beta_{ij}(t) g_j(s_j(t-\tau)) + I_i(t). \end{aligned} \tag{5}$$

Similarly, system (3) can be denoted as

$$\begin{aligned} {}_0^C D_t^{\varepsilon} \bar{s}_i(t) \;\in\; & -c_i \bar{s}_i(t) + \sum_{j=1}^{n} K[a_{ij}(\bar{s}_j(t))] f_j(\bar{s}_j(t)) \\ & + \sum_{j=1}^{n} K[b_{ij}(\bar{s}_j(t))] g_j(\bar{s}_j(t-\tau)) + I_i(t) + u_i(t), \end{aligned} \tag{6}$$

where

$$K[a_{ij}(\bar{s}_j(t))] = \begin{cases} \dot{a}_{ij} & |\bar{s}_j(t)| < W_j, \\ \overline{co}\{\dot{a}_{ij}, \ddot{a}_{ij}\} & |\bar{s}_j(t)| = W_j, \\ \ddot{a}_{ij} & |\bar{s}_j(t)| > W_j, \end{cases}$$

$$K[b_{ij}(\bar{s}_j(t))] = \begin{cases} \dot{b}_{ij} & |\bar{s}_j(t)| < W_j, \\ \overline{co}\{\dot{b}_{ij}, \ddot{b}_{ij}\} & |\bar{s}_j(t)| = W_j, \\ \ddot{b}_{ij} & |\bar{s}_j(t)| > W_j, \end{cases}$$

for $i, j \in \mathbb{N}^+$, or there exist $\overline{\varphi}_{ij}(t) \in K[a_{ij}(\bar{s}_j(t))]$ and $\overline{\beta}_{ij}(t) \in K[b_{ij}(\bar{s}_j(t))]$ such that

$$\begin{aligned} {}_0^C D_t^{\varepsilon} \bar{s}_i(t) \;=\; & -c_i \bar{s}_i(t) + \sum_{j=1}^{n} \overline{\varphi}_{ij}(t) f_j(\bar{s}_j(t)) \\ & + \sum_{j=1}^{n} \overline{\beta}_{ij}(t) g_j(\bar{s}_j(t-\tau)) + I_i(t) + u_i(t). \end{aligned} \tag{7}$$

Lemma 5[51] Under Assumption (H1), if $f_j(\pm W_j) = g_j(\pm W_j) = 0$, $i, j \in \mathbb{N}^+$, then

$$\begin{aligned} &\left| K[a_{ij}(\bar{s}_j)(t)] f_j(\bar{s}_j) - K[a_{ij}(s_j)(t)] f_j(s_j) \right| \le \varphi_{ij}^* F_j |\bar{s}_j - s_j|, \\ &\left| K[b_{ij}(\bar{s}_j)(t)] g_j(\bar{s}_j) - K[b_{ij}(s_j)(t)] g_j(s_j) \right| \le \beta_{ij}^* G_j |\bar{s}_j - s_j|, \end{aligned}$$

that is, for any $\overline{\varphi}_{ij}(t) \in K[a_{ij}(\bar{s}_j)(t)]$, $\varphi_{ij}(t) \in K[a_{ij}(s_j)(t)]$, $\overline{\beta}_{ij}(t) \in K[a_{ij}(\bar{s}_j)(t)]$, $\beta_{ij}(t) \in K[a_{ij}(s_j)(t)]$,

$$\left|\overline{\varphi}_{ij}(t)f_j(\bar{s}_j) - \varphi_{ij}(t)f_j(s_j)\right| \le \varphi_{ij}^* F_j|\bar{s}_j - s_j|,$$
$$\left|\overline{\beta}_{ij}(t)g_j(\bar{s}_j) - \beta_{ij}(t)g_j(s_j)\right| \le \beta_{ij}^* G_j|\bar{s}_j - s_j|,$$

where $\varphi_{ij}^* = \max\{\dot{a}_{ij}, \ddot{a}_{ij}\}$, $\beta_{ij}^* = \max\{\dot{b}_{ij}, \ddot{b}_{ij}\}$.

Definition 4[57] For any solutions $s(t)$ and $\bar{s}(t)$ of systems (2) and (3) with initial values $s(0)$ and $\bar{s}(0)$, $\exists\ T \in [0, +\infty)$, such that

$$||\bar{s}(t) - s(t)|| \le ||\bar{s}(0) - s(0)||E_\varepsilon(-\lambda t^\varepsilon), \quad t \ge T,$$

then systems (2) and (3) can achieve FTMLS, and

$$T_0 = \inf\{T \ge 0; \bar{s}(t) - s(t) = 0 \ \text{ for } \ t \ge T\}$$

is called the upper bound of SST. Particularly, when $\varepsilon = 1$, the FTMLS are reduced to the finite-time exponential synchronization.

3. Main results

Define the error system as $z_i(t) = \bar{s}_i(t) - s_i(t)$, then

$$\begin{aligned}
{}_0^C D_t^\varepsilon z_i(t) &= {}_0^C D_t^\varepsilon \bar{s}_i(t) - {}_0^C D_t^\varepsilon s_i(t) \\
&= -c_i z_i(t) + \sum_{j=1}^n \left(\overline{\varphi}_{ij}(t)f_j(\bar{s}_j(t)) - \varphi_{ij}(t)f_j(s_j(t))\right) \\
&\quad + \sum_{j=1}^n \left(\overline{\beta}_{ij}(t)g_j(\bar{s}_j(t-\tau)) - \beta_{ij}(t)g_j(s_j(t-\tau))\right) \\
&\quad + u_i(t), \qquad (8)
\end{aligned}$$

where $u_i(t)$ is designed as

$$u_i(t) = -k_i z_i(t) - \xi_i[z_i(t)]|z_i(t)|_1^\varpi - r_i[z_i(t)]|z_i(t-\tau)|_1, \quad (9)$$

where $k_i > 0$, ξ_i and r_i are real numbers.

Then Eq. (8) can be written as

$$\begin{aligned}
&{}_0^C D_t^\varepsilon z_i(t) \\
&= -c_i z_i(t) + \sum_{j=1}^n \left(\overline{\varphi}_{ij}(t)f_j(\bar{s}_j(t)) - \varphi_{ij}(t)f_j(s_j(t))\right) \\
&\quad + \sum_{j=1}^n \left(\overline{\beta}_{ij}(t)g_j(\bar{s}_j(t-\tau)) - \beta_{ij}(t)g_j(s_j(t-\tau))\right) \\
&\quad - k_i z_i(t) - \xi_i[z_i(t)]|z_i(t)|_1^\varpi - r_i[z_i(t)]|z_i(t-\tau)|_1. \qquad (10)
\end{aligned}$$

For convenience, define the following denotations:

$$\partial_1 = \sum_{i=1}^n \left[(\mathrm{Re}(c_i + k_i) - |\mathrm{Im}(c_i + k_i)|) - \sum_{j=1}^n F_i|\varphi_{ij}^*|_1\right],$$
$$\partial_2 = \sum_{i=1}^n \left[r_i - \sum_{j=1}^n G_i|\beta_{ij}^*|_1\right], \quad \partial_3 = \sum_{i=1}^n \xi_i, \quad i, j \in \mathbb{N}^+.$$

Theorem 1 Under controller (9), if Assumption (H1) and the following inequalities hold:

$$\begin{cases} \partial_1 > 0, \quad \partial_2 > 0, \quad \partial_3 > 0, \\ 0 \le \varpi < \varepsilon, \end{cases}$$

then systems (2) and (3) can achieve FTMLS and the SST T_1 can be evaluated by

$$T_1 \le \left(\frac{\varepsilon \sum\limits_{i=1}^n |z_i(0)|_1^{\varepsilon-\varpi} B(\varepsilon, 1-\varpi)}{\sum\limits_{i=1}^n \xi_i}\right)^{1/\varepsilon}.$$

Proof Construct the nonnegative function

$$L_1(t) = \sum_{i=1}^n |z_i(t)|_1. \qquad (11)$$

From Lemma 2, one has

$$\begin{aligned}
&{}_0^C D_t^\varepsilon L_1(t) \\
&\le \frac{1}{2}\sum_{i=1}^n \left(\overline{[z_i(t)]}\,{}_0^C D_t^\varepsilon z_i(t) + [z_i(t)]\,{}_0^C D_t^\varepsilon \overline{z_i(t)}\right) \\
&= -\frac{1}{2}\sum_{i=1}^n \left\{(c_i + k_i)z_i(t)\overline{[z_i(t)]} + (\overline{c_i} + \overline{k_i})\overline{z_i(t)}[z_i(t)]\right\} \\
&\quad + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \left\{\overline{[z_i(t)]}\left(\overline{\varphi}_{ij}(t)f_j(\bar{s}_j(t)) - \varphi_{ij}(t)f_j(s_j(t))\right)\right. \\
&\quad \left. + [z_i(t)]\left(\overline{\overline{\varphi}_{ij}(t)f_j(\bar{s}_j(t))} - \overline{\varphi_{ij}(t)f_j(s_j(t))}\right)\right\} \\
&\quad + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \left\{\overline{[z_i(t)]}\left(\overline{\beta}_{ij}(t)g_j(\bar{s}_j(t-\tau)) - \beta_{ij}(t)g_j(s_j(t-\tau))\right)\right. \\
&\quad \left. + [z_i(t)]\left(\overline{\overline{\beta}_{ij}(t)g_j(\bar{s}_j(t-\tau))} - \overline{\beta_{ij}(t)g_j(s_j(t-\tau))}\right)\right\} \\
&\quad - \sum_{i=1}^n \xi_i[z_i(t)]\overline{[z_i(t)]}|z_i(t)|_1^\varpi - \sum_{i=1}^n r_i[z_i(t)]\overline{[z_i(t)]}|z_i(t-\tau)|_1. \qquad (12)
\end{aligned}$$

From Eq. (15), we have

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^n \left[(c_i + k_i)\overline{[z_i(t)]}z_i(t) + (\overline{c_i} + \overline{k_i})[z_i(t)]\overline{z_i(t)}\right] \\
&= -\sum_{i=1}^n \mathrm{Re}\left((c_i + k_i)\overline{[z_i(t)]}z_i(t)\right) \\
&= -\sum_{i=1}^n \left[\mathrm{Re}(c_i + k_i)\left(|\mathrm{Re}(z_i(t))| + |\mathrm{Im}(z_i(t))|\right)\right. \\
&\quad + \mathrm{Im}(c_i + k_i)\left(\mathrm{Re}(z_i(t))\,\mathrm{sign}(\mathrm{Im}(z_i(t)))\right. \\
&\quad \left.\left. - \mathrm{Im}(z_i(t))\,\mathrm{sign}(\mathrm{Re}(z_i(t)))\right)\right] \\
&\le -\sum_{i=1}^n (\mathrm{Re}(c_i + k_i) - |\mathrm{Im}(c_i + k_i)|)|z_i(t)|_1. \qquad (13)
\end{aligned}$$

Under Assumption (H1) and Lemma 5, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \left\{\overline{[z_i(t)]}\left(\overline{\varphi}_{ij}(t)f_j(\bar{s}_j(t)) - \varphi_{ij}(t)f_j(s_j(t))\right)\right. \\
&\left. + [z_i(t)]\left(\overline{\overline{\varphi}_{ij}(t)f_j(\bar{s}_j(t))} - \overline{\varphi_{ij}(t)f_j(s_j(t))}\right)\right\} \\
&\le \sum_{i=1}^n\sum_{j=1}^n \left[\mathrm{Re}\left(f_j(\bar{s}_j(t)) - f_j(s_j(t))\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\times\big(\mathrm{Re}(\varphi_{ij}^*)\,\mathrm{sign}(\mathrm{Re}(z_i(t)))+\mathrm{Im}(\varphi_{ij}^*)\,\mathrm{sign}(\mathrm{Im}(z_i(t)))\big)\big]\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\big[\mathrm{Im}\big(f_j(\bar{s}_j(t))-f_j(s_j(t))\big)\\
&\times\big(\mathrm{Re}(\varphi_{ij}^*)\,\mathrm{sign}(\mathrm{Im}(z_i(t)))-\mathrm{Im}(\varphi_{ij}^*)\,\mathrm{sign}(\mathrm{Re}(z_i(t)))\big)\big]\\
\le&\sum_{i=1}^{n}\sum_{j=1}^{n}\big(|\mathrm{Re}(f_j(\bar{s}_j(t))-f_j(s_j(t)))||\varphi_{ij}^*|_1\\
&+|\mathrm{Im}(f_j(\bar{s}_j(t))-f_j(s_j(t)))||\varphi_{ij}^*|_1\big)\\
\le&\sum_{i=1}^{n}\sum_{j=1}^{n}F_i|\varphi_{ij}^*|_1|z_i(t)|_1. \qquad (14)
\end{aligned}
$$

Similar to Eq. (14),

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\big\{\overline{[z_i(t)]}\big(\overline{\beta}_{ij}(t)g_j(\bar{s}_j(t-\tau))-\beta_{ij}(t)g_j(s_j(t-\tau))\big)\\
&+[z_i(t)]\big(\overline{\overline{\beta}_{ij}(t)g_j(\bar{s}_j(t-\tau))}-\overline{\beta_{ij}(t)g_j(s_j(t-\tau))}\big)\big\}\\
\le&\sum_{i=1}^{n}\sum_{j=1}^{n}G_i|\beta_{ij}^*|_1|z_i(t-\tau)|_1. \qquad (15)
\end{aligned}
$$

Through Lemma 2, one has

$$
\begin{aligned}
&-\sum_{i=1}^{n}\xi_i[z_i(t)]\overline{[z_i(t)]}|z_i(t)|_1^{\varpi}-\sum_{i=1}^{n}r_i[z_i(t)]\overline{[z_i(t)]}|z_i(t-\tau)|_1\\
=&-\sum_{i=1}^{n}\xi_i|z_i(t)|_1^{\varpi}\big(\mathrm{sign}^2(\mathrm{Re}(z_i(t)))+\mathrm{sign}^2(\mathrm{Im}(z_i(t)))\big)\\
&-\sum_{i=1}^{n}r_i|z_i(t-\tau)|_1\big(\mathrm{sign}^2(\mathrm{Re}(z_i(t)))+\mathrm{sign}^2(\mathrm{Im}(z_i(t)))\big)\\
\le&-\sum_{i=1}^{n}\xi_i|z_i(t)|_1^{\varpi}-\sum_{i=1}^{n}r_i|z_i(t-\tau)|_1. \qquad (16)
\end{aligned}
$$

From Eqs. (13)–(16), we have

$$
\begin{aligned}
&{}_0^CD_t^{\varepsilon}L_1(t)\\
\le&-\sum_{i=1}^{n}\Big[(\mathrm{Re}(c_i+k_i)-|\mathrm{Im}(c_i+k_i)|)-\sum_{j=1}^{n}F_i|\varphi_{ij}^*|_1\Big]|z_i(t)|_1\\
&-\sum_{i=1}^{n}\Big[r_i-\sum_{j=1}^{n}G_i|\beta_{ij}^*|_1\Big]|z_i(t-\tau)|_1-\sum_{i=1}^{n}\xi_i|z_i(t)|_1^{\varpi}\\
=&-\partial_1L_1(t)-\partial_2L_1(t-\tau)-\partial_3L_1^{\varpi}(t). \qquad (17)
\end{aligned}
$$

Since $\partial_1>0$, $\partial_2>0$, and $\partial_3>0$, then

$$
\begin{aligned}
{}_0^CD_t^{\varepsilon}L_1(t)&\le-\partial_1L_1(t)-\partial_2L_1(t-\tau)-\partial_3L_1^{\varpi}(t)\\
&\le-\partial_1L_1(t). \qquad (18)
\end{aligned}
$$

Based on Lemma 3, we have

$$
L_1(t)\le L_1(0)E_{\varepsilon}(-\partial_1t^{\varepsilon}), \qquad (19)
$$

that is,

$$
\sum_{i=1}^{n}|z_i(t)|_1\le\sum_{i=1}^{n}|z_i(0)|_1E_{\varepsilon}(-\partial_1t^{\varepsilon}). \qquad (20)
$$

From Eq. (20), we can obtain

$$
||z_i(t)||_1\le||z_i(0)||_1E_{\varepsilon}(-\partial_1t^{\varepsilon}), \qquad (21)
$$

which shows that the error system (10) is Mittag–Leffler stable. In other words, it means that the systems (2) and (3) are Mittag–Leffler synchronization.

In the following, the SST of systems (2) and (3) is given. From Eq. (17), we have

$$
{}_0^CD_t^{\varepsilon}L_1(t)\le-\partial_3L_1^{\varpi}(t). \qquad (22)
$$

Since $0<\varpi<\varepsilon$, under Lemma 1, one has

$$
{}_0^CD_t^{\varepsilon}L_1^{\varepsilon-\varpi}(t)\le-\frac{\partial_3\Gamma(1+\varepsilon-\varpi)}{\Gamma(1-\varpi)}. \qquad (23)
$$

Taking the fractional integral of Eq. (23),

$$
\begin{aligned}
L_1^{\varepsilon-\varpi}(t)-L_1^{\varepsilon-\varpi}(0)&\le-{}_0I_t^{\varepsilon}\frac{\partial_3\Gamma(1+\varepsilon-\varpi)}{\Gamma(1-\varpi)}\\
&=-\frac{1}{\Gamma(\varepsilon)}\int_0^t\frac{\partial_3\Gamma(1+\varepsilon-\varpi)}{\Gamma(1-\varpi)(t-\tau)^{1-\varepsilon}}\,\mathrm{d}\tau\\
&=-\frac{\partial_3t^{\varepsilon}}{\varepsilon B(\varepsilon,1-\varpi)}, \qquad (24)
\end{aligned}
$$

where $B(m,n)$ is the Beta function.

Since $L_1^{\varepsilon-\varpi}(t)\ge0$, we have

$$
-L_1^{\varepsilon-\varpi}(0)\le-\frac{\partial_3t^{\varepsilon}}{\varepsilon B(\varepsilon,1-\varpi)}. \qquad (25)
$$

By simplifying Eq. (25), one has

$$
t\le\left(\frac{\varepsilon L_1^{\varepsilon-\varpi}(0)B(\varepsilon,1-\varpi)}{\partial_3}\right)^{1/\varepsilon}, \qquad (26)
$$

where $\partial_3=\sum_{i=1}^{n}\xi_i$. It implies that the SST T_1 can be denoted as

$$
T_1\le\left(\frac{\varepsilon\sum_{i=1}^{n}|z_i(0)|_1^{\varepsilon-\varpi}B(\varepsilon,1-\varpi)}{\sum_{i=1}^{n}\xi_i}\right)^{1/\varepsilon}. \qquad (27)
$$

Therefore, the state trajectories of systems (2) and (3) can be synchronized within T_1 under controller (9).

Remark 1 From Theorem 1, we can realize that the SST T_1 decreases monotonically with respect to the controller gain ξ_i. This means that the SST T_1 can be adjusted by flexibly changing the controller parameters, which improves the reliability and safety of systems.

Remark 2 In Refs. [40–45], researchers divided the FCVMNNs into real part and imaginary part and established the criteria of the two parts. However, this method may cause some problems. First, the complex-valued activation functions cannot always be explicitly separated, e.g., the real and imaginary parts of $g(Z)=(1-\mathrm{e}^{-\overline{Z}})/(1+\mathrm{e}^{-\overline{Z}})$ cannot be explicitly separated. Second, the increase of system dimension will lead to more difficult analysis with the decomposition of complex-valued system. Hence, the separation method cannot be applied in all complex-valued systems. In this article, the system is analyzed as an entirety without the separation method

in the complex-valued domain to avoid these problems. It is worth mentioning that in Ref. [45], the authors investigate the finite-time stabilization of FCVMNNs with proportional delays by using Holder inequality, Gronwall inequality and inequality scaling skills, which is different from the Lyapunov direct method used in this paper.

When $\varpi = 0$, controller (9) can be simplified to

$$u_i(t) = -k_i z_i(t) - \xi_i[z_i(t)] - r_i[z_i(t)]|z_i(t-\tau)|_1. \tag{28}$$

Corollary 1 Under controller (28) and Assumption (H1), if the following inequalities hold:

$$\begin{cases} \overline{\partial}_1 > 0, \\ \overline{\partial}_2 > 0, \\ \overline{\partial}_3 > 0, \end{cases}$$

where $\overline{\partial}_1 = \sum\limits_{i=1}^{n}\Big[(\mathrm{Re}(c_i+k_i) - |\mathrm{Im}(c_i+k_i)|) - \sum\limits_{j=1}^{n} F_i|\varphi_{ij}^*|_1\Big]$, $\overline{\partial}_2 = \sum\limits_{i=1}^{n}\Big[r_i - \sum\limits_{j=1}^{n} G_i|\beta_{ij}^*|_1\Big]$, $\overline{\partial}_3 = \sum\limits_{i=1}^{n}\xi_i$, then systems (2) and (3) can achieve FTMLS. In addition, the SST T_2 is the solution to the following equation:

$$E_\varepsilon(-\overline{\partial}_1 t^\varepsilon) = \frac{\overline{\partial}_3}{\overline{\partial}_1 L_2(0) + \overline{\partial}_3}. \tag{29}$$

Proof Consider the same function:

$$L_2(t) = \sum_{i=1}^{n}|z_i(t)|_1. \tag{30}$$

Similar to Theorem 1,

$${}_0^C D_t^\varepsilon L_2(t) \le -\overline{\partial}_1 L_2(t) - \overline{\partial}_2 L_2(t-\tau) - \overline{\partial}_3. \tag{31}$$

According to Theorem 1, since $\overline{\partial}_1 > 0$, $\overline{\partial}_2 > 0$, and $\overline{\partial}_3 > 0$, systems (2) and (3) are Mittag–Leffler synchronization.

Next, the SST of systems (2) and (3) will be given.

From Eq. (31), one has

$$\begin{aligned} {}_0^C D_t^\varepsilon L_2(t) &\le -\overline{\partial}_1 L_2(t) - \overline{\partial}_2 L_2(t-\tau) - \overline{\partial}_3 \\ &\le -\overline{\partial}_1 L_2(t) - \overline{\partial}_3. \end{aligned} \tag{32}$$

From Eq. (32), $\exists\ \Phi(t) \ge 0$,

$${}_0^C D_t^\varepsilon L_2(t) + \Phi(t) = -\overline{\partial}_1 L_2(t) - \overline{\partial}_3. \tag{33}$$

Taking fractional integral of Eq. (33), one has

$$\begin{aligned} L_2(t) - L_2(0) &= -{}_0I_t^\varepsilon \Phi(t) + {}_0I_t^\varepsilon(-\overline{\partial}_1 L_2(t) - \overline{\partial}_3) \\ &= -{}_0I_t^\varepsilon \Phi(t) - \overline{\partial}_1 {}_0I_t^\varepsilon(L_2(t) + \frac{\overline{\partial}_3}{\overline{\partial}_1}). \end{aligned} \tag{34}$$

From Eq. (34), we have

$$H(t) = L_2(0) + \frac{\overline{\partial}_3}{\overline{\partial}_1} - \overline{\partial}_1 {}_0I_t^\varepsilon H(t) - {}_0I_t^\varepsilon \Phi(t), \tag{35}$$

where $H(t) = L_2(t) + (\overline{\partial}_3/\overline{\partial}_1)$.

By Laplace transform of Eq. (35), we can obtain

$$H(s) = \frac{s^{\varepsilon-1}}{s^\varepsilon + \overline{\partial}_1}(L_2(0) + \frac{\overline{\partial}_3}{\overline{\partial}_1}) - \frac{\Phi(s)}{s^\varepsilon + \overline{\partial}_1}, \tag{36}$$

where $H(s) = \mathcal{L}[H(t)]$ and $\Phi(s) = \mathcal{L}[\Phi(t)]$.

Taking the inverse Laplace transform of Eq. (36), we have

$$\begin{aligned} H(t) &= \left(L_2(0) + \frac{\overline{\partial}_3}{\overline{\partial}_1}\right) E_\varepsilon(-\overline{\partial}_1 t^\varepsilon) \\ &\quad - \Phi(t) * t^{\varepsilon-1} E_{\varepsilon,\varepsilon}(-\overline{\partial}_1 t^\varepsilon). \end{aligned} \tag{37}$$

Due to $\Phi(t)$, $t^{\varepsilon-1}E_{\varepsilon,\varepsilon}(-\overline{\partial}_1 t^\varepsilon)$ are non-negative, we have

$$L_2(t) \le \left(L_2(0) + \frac{\overline{\partial}_3}{\overline{\partial}_1}\right) E_\varepsilon(-\overline{\partial}_1 t^\varepsilon) - \frac{\overline{\partial}_3}{\overline{\partial}_1}. \tag{38}$$

From Eq. (38), we can obtain the SST T_2 which is the solution to the following equation:

$$E_\varepsilon(-\overline{\partial}_1 t^\varepsilon) = \frac{\overline{\partial}_3}{\overline{\partial}_1 L_2(0) + \overline{\partial}_3}. \tag{39}$$

Remark 3 In Theorem 2 of Ref. [43], Zhang and Deng achieved the finite-time projective synchronization of FCVMNNs with time delay by designing a discontinuous feedback controller and obtained the time estimation inequality

$${}_0^C D_t^\varepsilon L_2(t) \le -\overline{\partial}_1 L_2(t) - \overline{\partial}_2 L_2(t-\tau) - \overline{\partial}_3 \le -\overline{\partial}_3. \tag{40}$$

By simplifying Eq. (40), the SST T^* can be denoted as

$$T^* \le \left(\frac{\Gamma(\varepsilon+1)L_2(0)}{\overline{\partial}_3}\right)^{1/\varepsilon}. \tag{41}$$

Obviously, compared to the time estimation inequality (32), Eq. (40) scales the first term $-\overline{\partial}_1 L_2(t)$, which implies that our SST T_2 is less than T^*. Hence, the results of Corollary 1 are less conservation than Theorem 2 in Ref. [43].

When the time delay $\tau = 0$, systems (5) and (7) can be transformed to

$${}_0^C D_t^\varepsilon s_i(t) = -c_i s_i(t) + \sum_{j=1}^{n}\varphi_{ij}(t) f_j(s_j(t)) + I_i(t), \tag{42}$$

and

$${}_0^C D_t^\varepsilon \bar{s}_i(t) = -c_i \bar{s}_i(t) + \sum_{j=1}^{n}\overline{\varphi}_{ij}(t) f_j(\bar{s}_j(t)) + I_i(t) + u_i(t). \tag{43}$$

From Eqs. (42) and (43), the error system is rewritten as

$$\begin{aligned} {}_0^C D_t^\varepsilon z_i(t) &= -c_i z_i(t) + \sum_{j=1}^{n}\Big(\overline{\varphi}_{ij}(t) f_j(\bar{s}_j(t)) \\ &\quad - \varphi_{ij}(t) f_j(s_j(t))\Big) + u_i(t). \end{aligned} \tag{44}$$

Similarly, the controller (9) can be simplified to

$$u_i(t) = -k_i z_i(t) - \xi_i[z_i(t)]|z_i(t)|_1^{\varpi}. \tag{45}$$

Corollary 2 Under controller (45), if Assumption (H1) and the following inequalities hold:

$$\begin{cases} \partial_1 > 0, \\ 0 \le \varpi < \varepsilon, \end{cases}$$

where $\partial_1 = \sum\limits_{i=1}^{n}\Big[(\mathrm{Re}(c_i+k_i) - |\mathrm{Im}(c_i+k_i)|) - \sum\limits_{j=1}^{n} F_i|\varphi_{ij}^*|_1\Big]$, then systems (2) and (3) are FTMLS, and the SST T_3 can be evaluated by

$$T_3 \le \left(\frac{\varepsilon \sum\limits_{i=1}^{n}|z_i(0)|_1^{\varepsilon-\varpi}B(\varepsilon,1-\varpi)}{\sum\limits_{i=1}^{n}\xi_i}\right)^{1/\varepsilon}. \tag{46}$$

Remark 4 The proof of Corollary 2 is similar to Theorem 1. In Theorem 3 of Ref. [33], the finite-time synchronization of FCVNNs was investigated by the property of complex-variable functions. However, time delay and memristive connection weights are not discussed in Ref. [33]. Hence, the result of Corollary 2 can be seen as the generalization of Theorem 3 in Ref. [33].

When neglecting the effect of $|z_i(t-\tau)|_1$ and $|z_i(t)|_1$ in controller (9), the converted controller is as follows:

$$u_i(t) = -k_i z_i(t). \tag{47}$$

Corollary 3 Under controller (47) and Assumption (H1), if the following inequalities hold:

$$\begin{cases} \widetilde{\partial_1^*} > 0, \\ 0 < \widetilde{\partial_2^*} < \widetilde{\partial_1^*}\sin\left(\dfrac{\varepsilon\pi}{2}\right), \end{cases}$$

where $\widetilde{\partial_1^*} = \sum\limits_{i=1}^{n}\Big[(\mathrm{Re}(c_i+k_i) - |\mathrm{Im}(c_i+k_i)|) - \sum\limits_{j=1}^{n} F_i|\varphi_{ij}^*|_1\Big]$, $\widetilde{\partial_2^*} = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} G_i|\beta_{ij}|_1$, then systems (2) and (3) are global asymptotic synchronization.

Proof Consider the same function:

$$L_3(t) = \sum_{i=1}^{n}|z_i(t)|_1. \tag{48}$$

Similar to Theorem 1, one has

$${}_0^C D_t^{\varepsilon} L_3(t) \le -\widetilde{\partial_1^*}L_3(t) + \widetilde{\partial_2^*}L_3(t-\tau). \tag{49}$$

Next, we construct the delayed system as follows:

$${}_0^C D_t^{\varepsilon}\Xi(t) = -\widetilde{\partial_1^*}\Xi(t) + \widetilde{\partial_2^*}\Xi(t-\tau), \tag{50}$$

where $\Xi(t) > 0$, and the initial condition of $\Xi(t)$ is consistent with that of $L_3(t)$.

The Laplace transformation of Eq. (50) can be denoted as

$$s^{\varepsilon}\Xi(s) - s^{\varepsilon-1}\Xi(0) = -\widetilde{\partial_1^*}\Xi(s) + \widetilde{\partial_2^*}\int_0^{+\infty}\mathrm{e}^{-st}\Xi(t-\tau)\,\mathrm{d}t. \tag{51}$$

Then the characteristic equation (51) can be written as

$$s^{\varepsilon} + \widetilde{\partial_1^*} - \widetilde{\partial_2^*}\mathrm{e}^{-s\tau} = 0. \tag{52}$$

Supposed that $s = \hbar i$ ($\hbar \in \mathbb{R}$) is the pure imaginary root of Eq. (52), then

$$\begin{aligned} &|\hbar|^{\varepsilon}\left[\cos\left(\frac{\varepsilon\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\varepsilon\pi}{2}\right)\right] + \widetilde{\partial_1^*} \\ &= \widetilde{\partial_2^*}\Big[\cos(\hbar\tau) - \sin(\hbar\tau)\Big]. \end{aligned} \tag{53}$$

Separating the real part and imaginary part of Eq. (53), one has

$$\begin{cases} |\hbar|^{\varepsilon}\cos\left(\dfrac{\varepsilon\pi}{2}\right) + \widetilde{\partial_1^*} = \overline{\partial_2^*}\cos(\hbar\tau), \\ |\hbar|^{\varepsilon}\sin\left(\dfrac{\varepsilon\pi}{2}\right) = -\widetilde{\partial_2^*}\sin(\hbar\tau). \end{cases} \tag{54}$$

From Eq. (54), we have

$$|\hbar|^2\varepsilon + \widetilde{\partial_1^*}^2 + 2\overline{\partial_1^*}\cos\left(\frac{\varepsilon\pi}{2}\right) = \widetilde{\partial_2^*}^2. \tag{55}$$

Consider $|\hbar|^{\varepsilon}$ as the variable of Eq. (55), we can obtain the discriminant of Eq. (55) as

$$\begin{aligned} \Delta &= \left[2\overline{\partial_1^*}\cos\left(\frac{\varepsilon\pi}{2}\right)\right]^2 - 4\left(\overline{\partial_1^*}^2 - \overline{\partial_2^*}^2\right) \\ &\le 4\left(\overline{\partial_2^*}^2 - \overline{\partial_1^*}^2\sin^2\left(\frac{\varepsilon\pi}{2}\right)\right). \end{aligned} \tag{56}$$

According to $0 < \widetilde{\partial_2^*} < \widetilde{\partial_1^*}\sin\left(\frac{\varepsilon\pi}{2}\right)$ and $0 < \varepsilon < 1$, we can obtain $\Delta < 0$. This means that, for any $\tau > 0$, there have no pure imaginary roots of characteristic equation (50).

Therefore, system (50) is global asymptotic stable. Based on Lemma 4, system (8) is global asymptotic stable, which means that systems (2) and (3) are global asymptotic synchronization.

Remark 5 Evidently, by adjusting the controller parameters, the FTMLS of FCVMNNs with and without time delay can be realized, and the asymptotic synchronization of FCVMNNs with time delay can also be obtained by utilizing the comparison theorem, which extends the existing results in Refs. [33,42,43]. Hence, our results are more general and less conservative.

4. Numerical examples

To demonstrate the validity of the results, some numerical simulations are provided as follows:

Example 1 Consider FCVMNNs (2) and (3) as the drive-response systems, where $i = 1,2,3$, $\varepsilon = 0.95$, $\tau = 0.1$, the activation function $f_j(s_j(t)) = \tanh(\mathrm{Re}(s_j(t))) + \mathrm{i}\tanh(\mathrm{Im}(s_j(t)))$

and $g_j(s_j(t-\tau)) = \tanh(\mathrm{Re}(s_j(t-\tau))) + \mathrm{i}\tanh(\mathrm{Im}(s_j(t-\tau)))$; $I(t) = 0$ and

$$C = \mathrm{diag}(c_i) = \begin{pmatrix} 1+\mathrm{i} & 0 & 0 \\ 0 & 1+\mathrm{i} & 0 \\ 0 & 0 & 1+\mathrm{i} \end{pmatrix},$$

$$a_{11}(s_1) = \begin{cases} 1.4+1.4\mathrm{i}, & |s_1(t)| \geq 0.5, \\ 0.8+1.7\mathrm{i}, & |s_1(t)| < 0.5, \end{cases}$$

$$a_{12}(s_2) = \begin{cases} -2.1-1.9\mathrm{i}, & |s_2(t)| \geq 0.5, \\ -1.1-1.1\mathrm{i}, & |s_2(t)| < 0.5, \end{cases}$$

$$a_{13}(s_3) = \begin{cases} 1.1+1.3\mathrm{i}, & |s_3(t)| \geq 0.5, \\ -2.1-2.1\mathrm{i}, & |s_3(t)| < 0.5, \end{cases}$$

$$a_{21}(s_1) = \begin{cases} 1.9+1.9\mathrm{i}, & |s_1(t)| \geq 0.5, \\ 1.6+1.6\mathrm{i}, & |s_1(t)| < 0.5, \end{cases}$$

$$a_{22}(s_2) = \begin{cases} 1.2+1.2\mathrm{i}, & |s_2(t)| \geq 0.5, \\ 1.9+1.7\mathrm{i}, & |s_2(t)| < 0.5, \end{cases}$$

$$a_{23}(s_3) = \begin{cases} -1.1-1.1\mathrm{i}, & |s_3(t)| \geq 0.5, \\ -1.6-3.5\mathrm{i}, & |s_3(t)| < 0.5, \end{cases}$$

$$a_{31}(s_1) = \begin{cases} -0.9-1.1\mathrm{i}, & |s_1(t)| \geq 0.5, \\ 0.9+0.9\mathrm{i}, & |s_1(t)| < 0.5, \end{cases}$$

$$a_{32}(s_2) = \begin{cases} -0.2-1.9\mathrm{i}, & |s_2(t)| \geq 0.5, \\ -0.6-2.5\mathrm{i}, & |s_2(t)| < 0.5, \end{cases}$$

$$a_{33}(s_3) = \begin{cases} 2.1+2.1\mathrm{i}, & |s_3(t)| \geq 0.5, \\ 0.5+0.5\mathrm{i}, & |s_3(t)| < 0.5, \end{cases}$$

$$b_{11}(s_1) = \begin{cases} -1.2+0.5\mathrm{i}, & |s_1(t)| \geq 0.5, \\ -0.2-0.3\mathrm{i}, & |s_1(t)| < 0.5, \end{cases}$$

$$b_{12}(s_2) = \begin{cases} 1.3+0.2\mathrm{i}, & |s_2(t)| \geq 0.5, \\ -1+0.2\mathrm{i}, & |s_2(t)| < 0.5, \end{cases}$$

$$b_{13}(s_3) = \begin{cases} 1.2+0.4\mathrm{i}, & |s_3(t)| \geq 0.5, \\ 2.1-0.3\mathrm{i}, & |s_3(t)| < 0.5, \end{cases}$$

$$b_{21}(s_1) = \begin{cases} 1.9+0.1\mathrm{i}, & |s_1(t)| \geq 0.5, \\ 0.7-0.1\mathrm{i}, & |s_1(t)| < 0.5, \end{cases}$$

$$b_{22}(s_2) = \begin{cases} -1.9+0.3\mathrm{i}, & |s_2(t)| \geq 0.5, \\ -1.6-0.2\mathrm{i}, & |s_2(t)| < 0.5, \end{cases}$$

$$b_{23}(s_3) = \begin{cases} -1.1+0.4\mathrm{i}, & |s_3(t)| \geq 0.5, \\ -0.9+0.8\mathrm{i}, & |s_3(t)| < 0.5, \end{cases}$$

$$b_{31}(s_1) = \begin{cases} -1.2-0.3\mathrm{i}, & |s_1(t)| \geq 0.5, \\ -0.8-0.5\mathrm{i}, & |s_1(t)| < 0.5, \end{cases}$$

$$b_{32}(s_2) = \begin{cases} -1.5-0.4\mathrm{i}, & |s_2(t)| \geq 0.5, \\ -0.3+0.4\mathrm{i}, & |s_2(t)| < 0.5, \end{cases}$$

$$b_{33}(s_3) = \begin{cases} -1.6+0.3\mathrm{i}, & |s_3(t)| \geq 0.5, \\ -1.9-0.2\mathrm{i}, & |s_3(t)| < 0.5. \end{cases}$$

The initial values conditions of systems (2) and (3) are selected as $s_1(0) = -2+0.2\mathrm{i}$, $s_2(0) = 4-0.5\mathrm{i}$, $s_3(0) = 2+0.4\mathrm{i}$ and $\bar{s}_1(0) = -0.5-2\mathrm{i}$, $\bar{s}_2(0) = 0.4+4\mathrm{i}$, $\bar{s}_3(0) = -0.3-2\mathrm{i}$.

Without the controller (9), the phase trajectories of real and imaginary parts of systems (2) and (3) are shown in Fig. 1. Especially, Fig. 2 shows the state trajectories of real and imaginary parts of systems (2) and (3). Obviously, systems (2) and (3) cannot reach the same state in finite time, which means that systems (2) and (3) cannot achieve FTMLS.

By simple computation, we can obtain $F_i = G_i = 1$, $i = 1,2,3$. When choose $\varpi = 0.75$, $K = (10+7\mathrm{i}, 10+7\mathrm{i}, 10+7\mathrm{i})^{\mathrm{T}}$, $\xi = \mathrm{diag}(8,8,8)$, $r_i = \mathrm{diag}(9,9,9)$. Under the controller (9), the state trajectories of systems (2) and (3) are given in Fig. 3. The simulation results show that systems (2) and (3) can realize FTMLS, and the SST $T_1 = 0.8303$ is evaluated according to Theorem 1.

When $\varpi = 0$, $\tau = 0.1$ and the controller parameters remain the same, system (10) can converge to zero in finite time, which is shown in Fig. 4(a). And the SST $T_2 = 1.4766$ is evaluated based on Corollary 1. By choosing 10 random initial conditions, the synchronization errors state can converge to zero in finite time, which is shown in Fig. 4(b).

Remark 6 When we choose the same initial values conditions of systems (2) and (3) and the same controller parameters, in Ref. [43], the SST T^* can be evaluated by

$$T^* = \left(\frac{\Gamma(\varepsilon+1)L_2(0)}{\partial_3}\right)^{1/\varepsilon} = 2.0972.$$

Evidently, in Fig. 4(a), $T_2 = 1.4766 < T^*$, the result given in Corollary 1 is accurate than the result in Ref. [43]. Hence, Corollary 1 provides a more general result.

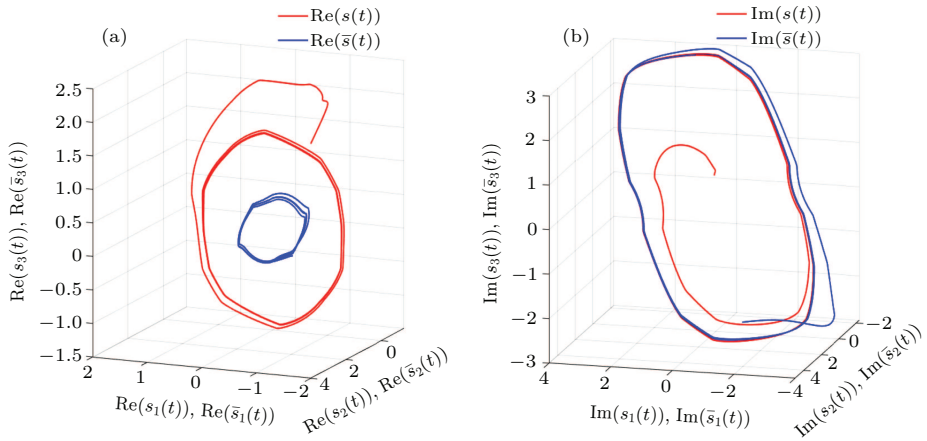


Fig. 1. (a) Phase plot of the real part of drive-response systems (2) and (3) without the controller (9). (b) Phase plot of the imaginary part of drive-response systems (2) and (3) without the controller (9).

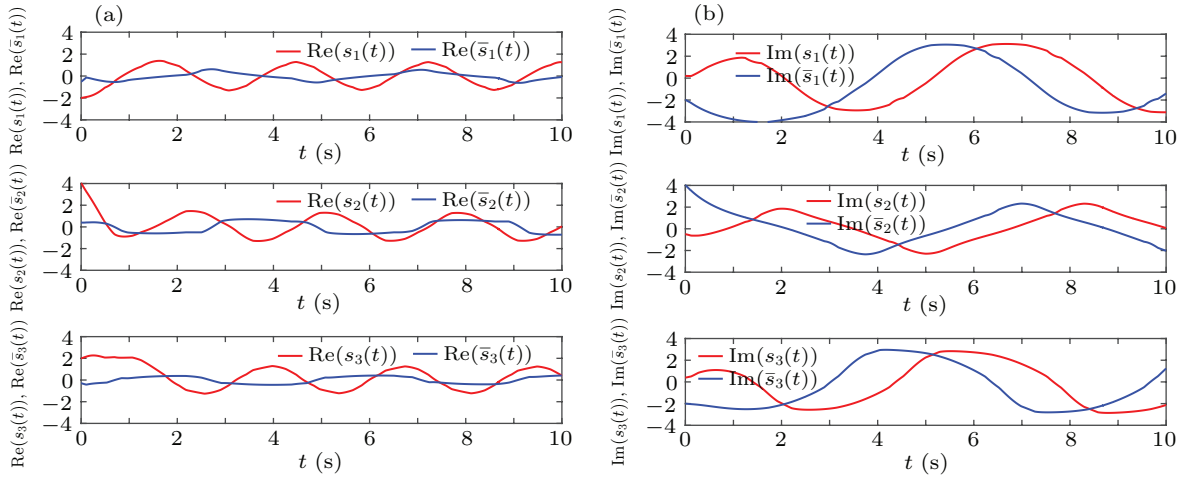


Fig. 2. Under $\tau = 0.1$, $\varpi = 0.75$: (a) the state trajectories of the real parts of drive-response systems (2) and (3) without controller (9), (b) the state trajectories of the imaginary parts of drive-response systems (2) and (3) without controller (9).

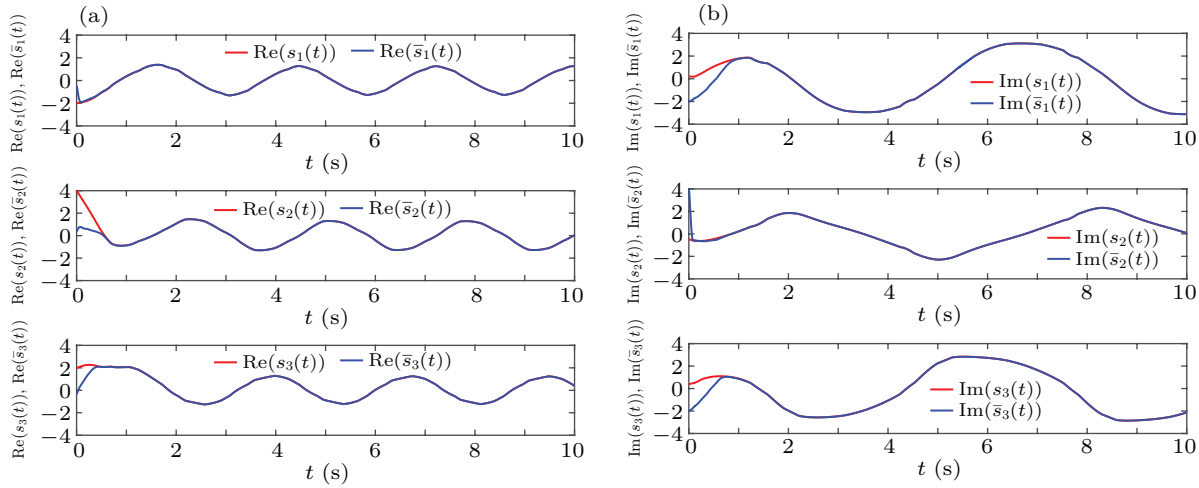


Fig. 3. Under $\tau = 0.1$, $\varpi = 0.75$: (a) the state trajectories of the real parts of drive-response systems (2) and (3) under controller (9), (b) the state trajectories of the imaginary parts of drive-response systems (2) and (3) under controller (9).

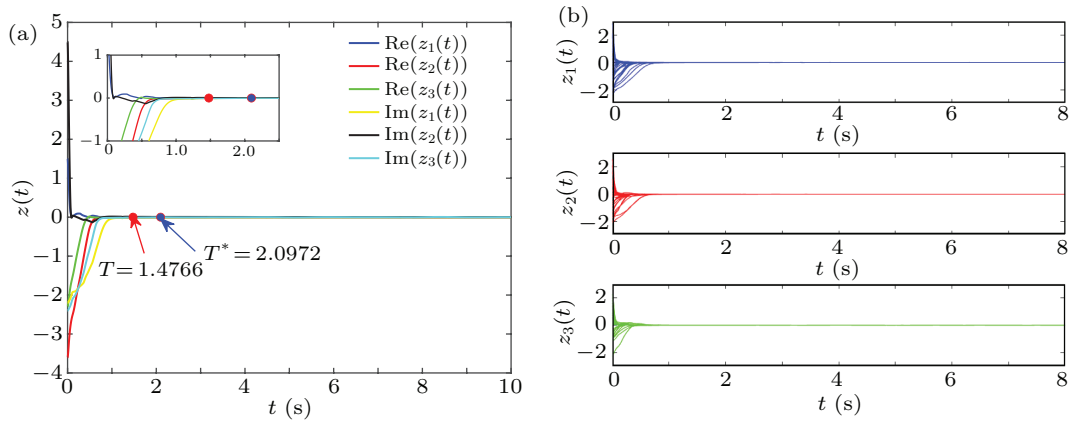


Fig. 4. Under $\tau = 0.1$, $\varpi = 0$: (a) the error state trajectories under controller (28), (b) state trajectories of error system $z(t)$ with 10 random initial conditions.

5. Conclusions

In summary, we have considered the FTMLS of a class of FCVMNNs with time delay. Different from the separation technique, we take the FCVMNNs as an entirety to analyze the system. Firstly, under the framework of Filippov solution, the original systems are transformed into the right-hand continuous systems. Moreover, based on the property of complex-variable function and Lyapunov stability theorem, some sufficient conditions of FTMLS are obtained by devising a complex-valued feedback controller. Meanwhile, the upper bound of the SST can be estimated by fractional dif-

ferential inequality. Compared with the results in Ref. [43], our SST is less conservative. Furthermore, by adjusting controller parameters, the global asymptotic synchronization of FCVMNNs with time delay can also be realized, which improves and enrich the results in Refs. [33,42,43]. Note that in Ref. [36], the authors focused on the application of memristive neural networks in image encryption. In the future, we will consider applications of FCVMNNs in image encryption.

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