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Hongwei Zhang(张红伟), Ran Cheng(程然), and Dawei Ding(丁大为)†

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Finite-time synchronization of uncertain fractional-order multi-weighted complex networks with external disturbances via adaptive quantized control

Hongwei Zhang(张红伟), Ran Cheng(程然), and Dawei Ding(丁大为)†

School of Electronics and Information Engineering, Anhui University, Hefei 230601, *China*

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The finite-time synchronization of fractional-order multi-weighted complex networks (FMCNs) with uncertain parameters and external disturbances is studied. Firstly, based on fractional calculus characteristics and Lyapunov stability theory, quantized controllers are designed to guarantee that FMCNs can achieve synchronization in a limited time with and without coupling delay, respectively. Then, appropriate parameter update laws are obtained to identify the uncertain parameters in FMCNs. Finally, numerical simulation examples are given to validate the correctness of the theoretical results.

Keywords: fractional-order complex networks, uncertain parameter, finite-time synchronization, quantized control

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1. Introduction

In the last decade, as a new discipline of complexity science, complex networks (CNs) have attracted extensive attention and discussion in the academic field. The theories of CNs have been applied in biology,[1] social science,[2] computer science[3] and power system,[4,5] and some excellent achievements have been obtained. In real life, the connection among network nodes is often multi-weighted.[6,7] For example, in the information exchange between persons, in addition to the use of mobile phones to send messages, people also use e-mail, network chat software, and other party for information communication. It can be seen that the CNs with multi-weighted node connection have more practical significance and application value. However, at present, most of the studies only consider the dynamical behavior of CNs with a single weight and lack the in-depth study of multi-weighted CNs. Given the importance of this issue, some researchers have focused on it and achieved some results.[8–10]

Synchronization, as a crucial behavioral feature in the dynamic system, is a hot research field in CNs. In general, network synchronization refers to the process in which the states of all coupled dynamical systems tend to behave together over time.[11] Over the last several years, researchers have explored a variety of synchronization in CNs, such as complete synchronization,[12] projective synchronization,[13,14] quasi synchronization,[15,16] cluster synchronization,[17–19] and exponential synchronization.[20] In engineering applications, due to the requirements of security and efficiency, it is important to know the maximum synchronization time of the network.[21] Therefore, compared with the general synchronization mode without the upper limit of synchronization time, the finite-time synchronization is more meaningful and can be used in a wider range of scenarios.[22–25] In Ref. [23], Ying *et al.* articulated a new method to solve the problem of CNs synchronization within a limited time with Markovian switching topologies. In Ref. [24], He *et al.* established a synchronization criterion for input delay CNs in limited time via non-fragile controller. In Ref. [25], Zhang researched the synchronization problem of CNs with disturbances in finite time when the node states are uncertain.

It is worth mentioning that in the above literature, some parameters of CNs are assumed to be known. But in practical applications, these parameters are often uncertain because of the constraints of some actual conditions. Furthermore, these parameters often have a significant effect on the whole network synchronization process.[26] Therefore, considering the synchronization of CNs with uncertain parameters has stronger practical significance.[27–29] In Ref. [27], Mei *et al.* investigated the identification and synchronization of finite-time structures in drive-response systems with uncertain parameters in finite time. In the case of delayed coupling networks, the Lyapunov stability theory has been used to obtain the outer synchronization criterion while the uncertain parameters were considered in Ref. [28].

Fractional calculus is an extension of calculus in the real number field. Compared to integer-order calculation, it takes into account the historical and non-local distributed influence. In other words, fractional calculus has the characteristics of memory and inheritance.[30,31] Because of this satisfactory property, fractional calculus can describe the physical model in practical applications more accurately, such as neural

†Corresponding author. E-mail: dwding@ahu.edu.cn

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network[32,33] and chaotic memristor circuit.[34,35] Therefore, it is more valuable to consider the dynamic behavior of CNs under fractional-order perspective.[36–38] In Ref. [36], Li *et al.* studied the synchronization phenomenon of fractional order CNs using the method of graph theory and proposed a new finite-time synchronization theorem. In Ref. [37], the author accomplished finite-time synchronization and parameter identification of uncertain fractional-order CNs using fractional calculus principles and certain analytical approaches.

To make the network reach synchronization, many effective control methods have been proposed, such as pinning control,[39] intermittent control,[40] and event-triggered control.[41] However, there are few studies on quantitative control at present. As is well known, the signal transmission between network nodes depends on the transmission channel and channel bandwidth. Sometimes, due to the limitation of the channel itself, the signal in the transmission process results in the decline of the performance of the network.[42,43] Therefore, the quantification of the signal before transmission is an effective method to improve the transmission performance and enhance the stability of network communication.[44–46] In Ref. [44], authors considered the finite-time quantized synchronization for CNs with quantized time-varying delayed coupling. In Ref. [45], according to the use quantized output control method, the synchronization of the neural network with output coupling was realized.

Motivated by the aforementioned considerations, the issue of finite-time synchronization of FMCNs under adaptive quantization control is mainly studied. At the same time, uncertain parameters, coupling delay, and external interference are also taken into account, which makes the results of the study more practical. The main contributions of this paper are listed as follows:

(i) Different from Refs. [20,22,23], this paper not only considers the synchronization of CNs under the properties of fractional calculus, but also considers the influence of multiple weights on the synchronization of CNs, which is more practical.

(ii) Based on the logarithmic quantizer, this paper designs a new adaptive quantized controller, which enables the network to achieve synchronization in finite time, and the upper bound of synchronization time can be calculated. This paper also considers the finite-time synchronization problem of adaptive quantized control under no-coupling delay and coupling delay conditions, making the theory obtained in this paper more universal.

(iii) Uncertain parameters and external disturbances are taken into account simultaneously in the FMCNs studied in this paper. Under the conditions of coupling delay and no-coupling delay, through the logarithmic quantizer, we obtain parameter update laws and control methods, which can accurately identify the uncertain parameters of the system and eliminate the influence of external disturbances.

The remainder content is structured as follows. In Section 2, some necessary definitions and lemmas are given, and the FMCNs model is established. In Section 3, finite-time synchronization theorems for no-coupling delay and coupling delay are given, respectively. The correctness of the theorem is shown in Section 4 via illustrations of numerical simulation. Finally, the conclusion is given in Section 5.

Notions The rational set of real numbers is denoted by $\mathbb{R}$, $\mathbb{R}_{m\times n}$ denotes the set of all $m\times n$ real matrices, and $\mathbb{R}^m$ represents the m-dimensional Euclidean space. $\|\cdot\| = (\sum_{p=1}^{m}|x_p^2|)^{1/2}$ denotes Euclidean norm. I_m denotes the m-order identity matrix. For any matrix A, $A>0$ represent the matrix A is positive while $A<0$ represents the matrix A is negative. Besides, the $\lambda_{\max}$ denotes the maximun eigenvalue of the matrix.

2. Preliminaries

In this section, with the purpose of setting the problem preferably, some essential definitions of logarithmic quantizer and Caputo fractional calculus are reviewed, some helpful lemmas are presented, and the mathematical model of FMCNs is developed.

2.1. Logarithmic quantizer and Caputo fractional calculus

Definition 1[22] The logarithmic quantizer $\chi(\tau):\mathbb{R}\to\Pi$ is built in the following way:

$$\chi(\tau)=\begin{cases}\varphi_p, & \text{if} \quad \frac{1}{1+\zeta}\varphi_p\le\tau\le\frac{1}{1-\zeta}\varphi_p,\\ 0, & \text{if} \quad \tau=0,\\ -\chi(-\tau), & \text{if} \quad \tau\le 0,\end{cases}$$

where $\Pi=\{\pm\varphi_p:\varphi_p=\rho^p\varphi_0,p=0,\pm1,\pm2,\ldots\}\cup\{0\}$, φ_p denote the set of quantization levels and $\varphi_0\ge0$, $\rho\in[0,1]$ is called quantized density and $\zeta=\frac{1-\rho}{1+\rho}$. For any $\tau(\cdot)$, we can obtain the following equation:

$$\chi(\tau)=(1+\Delta)\tau,$$

where $\Delta\in[-\zeta,\zeta]$ is the Filippov solution.[50]

Definition 2[11] There are Caputo fractional derivative definition of order γ for a function $f(t)$:

$${}_{t_0}^{c}D_t^{\gamma}f(t)=\frac{1}{\Gamma(m-\gamma)}\int_{t_0}^{t}\frac{f^{(m)}(s)}{(t-s)^{\gamma-m+1}}\,\mathrm{d}s,$$

where $m-1<\gamma<m$, and $\Gamma(\cdot)$ is the gamma function. Generally, if we let $m=1$,

$${}_{t_0}^{c}D_t^{\gamma}f(t)=\frac{1}{\Gamma(1-\gamma)}\int_{t_0}^{t}\frac{f(s)}{(t-s)^{\gamma}}\,\mathrm{d}s.$$

The usual operation rules of Caputo fractional derivative are as follows:

- If c is a constant, ${}_{t_0}^{c}D_t^{\gamma}c = 0$.
- For any function $f_1(t), f_2(t)$ and constants a and b, then ${}_{t_0}^{c}D_t^{\gamma}af_1(t) + {}_{t_0}^{c}D_t^{\gamma}bf_2(t) = a{}_{t_0}^{c}D_t^{\gamma}f_1(t) + b{}_{t_0}^{c}D_t^{\gamma}f_2(t)$.

Lemma 1[36] Given a continuous positive function $V(t)$, and consider it fulfills the inequality shown below:

$$ {}_{t_0}^{c}D_t^{\gamma}(V(t,x(t))) \le -cV(t)^{\sigma}, $$

where $\gamma \in (0,1)$, and $c \in (0,+\infty)$ and $\sigma \in (0,\gamma)$ are two positive definite constants. Then $V(t)$ fulfills the inequality in the following way:

$$ V^{\gamma-\sigma}(t) \le V^{\gamma-\sigma}(t_0) - \frac{c\Gamma(1+\gamma-\sigma)(t-t_0)^{\gamma}}{\Gamma(1+\gamma)\Gamma(1-\sigma)}, \quad t \in [t_0, t_1], $$

and $V(t) \equiv 0$ for all $t \in [t_1, +\infty)$, where t_1 is given by

$$ t_1 = t_0 + \left(\frac{\Gamma(1+\gamma)\Gamma(1-\sigma)V^{\gamma-\sigma}(t_0)}{c\Gamma(1+\gamma-\sigma)} \right)^{1/\gamma}. $$

2.2. Model description

Consider FMCNs with time-delay coulped by M nodes, which would be summarized by the following mathematical formula:

$$ \begin{aligned} {}_{t_0}^{c}D_t^{\gamma}x_p(t) &= \tilde{f}(x_p(t)) + \sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G x_q(t) \\ &= f(x_p(t)) + z(x_p(t))\Xi + \sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G x_q(t), \quad (1) \end{aligned} $$

where $\gamma \in (0,1), x_p(t) = (x_{p1}(t), x_{p2}(t), \ldots, x_{pm}(t))^{\mathrm{T}} \in \mathbb{R}^m$, $p = 1,2,\ldots,M$, represent the state vector of p-th node, $f(x_p(t)) : \mathbb{R}^m \to \mathbb{R}^m$ are continuous functions and $z(x_p(t)) : \mathbb{R}^m \to \mathbb{R}^m$ are continuous function matrix. $\Xi \in \mathbb{R}^m$ represents the vector of system parameters, and $\tilde{f}(x_p(t)) : \mathbb{R}^m \to \mathbb{R}^m$ are continuous vector functions which consist of two parts. h_k represents the coupling weight value of the k-th weight network, and $L^{(k)} = (l_{pq}^{(k)})_{M\times M}$ is the weighted matrix of k-th weight network. If a connection between q-th node to p-th node $(p \ne q)$ exists, then $l_{pq} \ge 0$, otherwise $l_{pq} = 0$, and the diagonal elements $l_{pp} = -\sum_{q=1,q\ne p}^{M} l_{pq}$, $(p = 1,2,\ldots,M)$. $G = \mathrm{diag}(g_1, g_2, \ldots, g_m) \in \mathbb{R}^m$ is the internal coupling strength matrix and $z_p \ge 0$.

System (1) can be used as a drive system, and system (2) can be used as a response system with controller and external disturbance, which is formulated in the following way:

$$ \begin{aligned} {}_{t_0}^{c}D_t^{\gamma}y_p(t) &= \tilde{f}(y_p(t)) + \sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G y_q(t) + \rho(t) + u_p(t) \\ &= f(y_p(t)) + z(y_p(t))\hat{\Xi}(t) + \sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G y_q(t) + \rho(t) + u_p(t), \quad (2) \end{aligned} $$

where $\gamma \in (0,1)$, $y_p(t) = (y_{p1}(t), y_{p2}(t), \ldots, y_{pm}(t))^{\mathrm{T}} \in \mathbb{R}^m$ represent the state vector of p-th node, $u_p(t)$ is the controller to be design, $\rho : \mathbb{R} \to \mathbb{R}^m$ represents the external disturbance function, and $\hat{\Xi}(t) \in \mathbb{R}^m$ represents the estimation of uncertain parameter vector Ξ.

Definition 3[25] We define the synchronization error as $e(t) = (e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \ldots, e_M^{\mathrm{T}}(t))^{\mathrm{T}}$, and the systems (1) and (2) are said to be synchronized in finite time if a positive constant $T \in (0,+\infty)$ exists, which could be described by the following formula:

$$ \lim_{t\to T} \|y_p(t) - x_p(t)\| = \lim_{t\to T} \|e_p(t)\| = 0, \quad (p = 1,2,\ldots,m), $$

and $\|y_p(t) - x_p(t)\| \equiv 0$ if $t \ge T$, where T is called the setting time.

After that, error system between the drive network (1) and the response network (2) can be provided by the formula

$$ \begin{aligned} {}_{t_0}^{c}D_t^{\gamma}e_p(t) &= f(y_p(t)) - f(x_p(t)) + z(y_p(t))\hat{\Xi}(t) - z(x_p(t))\Xi \\ &\quad + \sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G e_q(t-\kappa) + \rho(t) + u_p(t). \quad (3) \end{aligned} $$

Definition 4[36] For any vector

$$ x_p(t) = (x_{p1}(t), x_{p2}(t), \ldots, x_{pm}(t))^{\mathrm{T}} \in \mathbb{R}^m, \quad p = 1,2,\ldots,M, $$

and constant $\beta \in \mathbb{R}$, we define function $\mathrm{SIGN}(x(t)), \overline{\mathrm{SIGN}}(x(t))$, and $|x_p(t)|^{\beta}$, which are listed as follows:

$$ \begin{aligned} &\overline{\mathrm{SIGN}}(x_p(t)) = (\mathrm{sign}(x_{p1}(t)), \mathrm{sign}(x_{p2}(t)), \ldots, \mathrm{sign}(x_{pm}(t)))^{\mathrm{T}}, \\ &\mathrm{SIGN}(x_p(t)) \\ &\quad = \mathrm{diag}(\mathrm{sign}(x_{p1}(t)), \mathrm{sign}(x_{p2}(t)), \ldots, \mathrm{sign}(x_{pm}(t))), \\ &|x_p(t)|^{\beta} = \left[|x_{p1}(t)|^{\beta}, |x_{p2}(t)|^{\beta}, \ldots, |x_{pm}(t)|^{\beta} \right]^{\mathrm{T}}. \end{aligned} $$

Definition 5[44] For given matrices $\Psi = (\psi_{pq})_{m\times n}, N \in \mathbb{R}_{s\times r}$, the Kronecker product of Ψ and N is defined as

$$ \Psi \otimes N = \begin{pmatrix} \psi_{11}N & \cdots & \psi_{1n}N \\ \vdots & \ddots & \vdots \\ \psi_{m1}N & \cdots & \psi_{mn}N \end{pmatrix} \in \mathbb{R}_{ms\times nr}. $$

The Kronecker product has the following property:

$$ (\Psi \otimes N)^{\mathrm{T}} = \Psi^{\mathrm{T}} \otimes N^{\mathrm{T}}. $$

Assumption 1 There exists a non-negative constant $\upsilon \in (0,+\infty)$ such that for any $x(t), y(t) \in \mathbb{R}^m$,

$$ \begin{aligned} &(y(t) - x(t))^{\mathrm{T}}(\tilde{f}(y(t)) - \tilde{f}(x(t))) \\ &\le \upsilon(y(t) - x(t))^{\mathrm{T}}(y(t) - x(t)). \end{aligned} $$

Assumption 2 Given a positive definite constant $\overline{\rho}$ and an external disturbance function $\rho_p(t), (p = 1,2,\ldots,m)$, the relationship between them satisfies the following inequality:

$$ \|\rho_p(t)\| \le \overline{\rho}. $$

Lemma 2[47] If $x(t)$ is a continuous vector value function that can be derivalbe, the following inequality holds when the

condition of $t \geq t_0$ is satisfied:

$$\frac{1}{2}\left({}_{t_0}^{c}D_t^{\gamma}x^{\mathrm{T}}(t)x(t)\right) \leq x^{\mathrm{T}}(t){}_{t_0}^{c}D_t^{\gamma}x(t).$$

Lemma 3[48] Give any vectors x and $y \in \mathbb{R}^m$, a constant $\varepsilon \in (0,+\infty)$, and any $m \times m$ real matrix $Q \in (0,+\infty)$, then we have the following inequality:

$$2x^{\mathrm{T}}y \leq \varepsilon^{-1}x^{\mathrm{T}}Q^{-1}x + \varepsilon y^{\mathrm{T}}Qy.$$

Lemma 4[49] For any $\forall a_p \in \mathbb{R}, (p = 1,2,\ldots,m)$, if $u, v \in \mathbb{R}$ and $0<u \leq 1, 0<v<2$, we can get the inequality as follows:

$$\sum_{p=1}^{m}|a_p|^v \geq \left(\sum_{p=1}^{m}|a_p|^2\right)^{v/2}, \quad \left(\sum_{p=1}^{m}|a_p|\right)^u \leq \sum_{p=1}^{m}|a_p|^u.$$

3. Main result

In this section, fractional calculus and the Lyapunov stability theory are used to establish the CNs model, and the sufficient synchronization criteria of FMCNs with uncertain parameters in finite time are obtained.

3.1. Finite-time synchronization of FMCNs with uncertain parameter

Theorem 1 Suppose that Assumptions 1 and 2 are satisfied. If there exist constant $\varepsilon_1 > 0$ and gain $m^* > 0$ such that

$$\frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_1^{-1}\lambda_{\max}(\Omega\Omega^{\mathrm{T}}) + v \leq (1-\zeta)m^*, \tag{4}$$

where $\Omega = \sum_{k=1}^{s}h_k(L^{(k)} \otimes G)$, the response network (2) and drive network (1) can be synchronized in finite time with the controller $u_p(t) = u_p^{(1)}(t) + u_p^{(2)}(t) + u_p^{(3)}(t)$ given by

$$\begin{aligned}
u_p^{(1)}(t) &= -m_p(t)\chi(e_p(t)) - m^*\chi(e_p(t)),\\
u_p^{(2)}(t) &= -\overline{\rho}\,\overline{\mathrm{SIGN}}(\chi(e_p(t))),\\
u_p^{(3)}(t) &= -\mathrm{SIGN}(\chi(e_p(t)))|\chi(e_p(t))|^{\sigma},
\end{aligned} \tag{5}$$

where $\chi(\cdot)$ is logarithmic quantizer, and updating laws $m_p(t)$ and parameter estimate laws $\hat{\Xi}(t)$ are given, respectively,

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\gamma}m_p(t) &= (1-\zeta)d_p e_p^{\mathrm{T}}(t)e_p(t)\\
&-(1-\zeta)^{\sigma}\left(\frac{1}{d_p}\right)^{\frac{\sigma-1}{2}}\mathrm{SIGN}(m_p(t))|m_p(t)|^{\sigma},
\end{aligned} \tag{6}$$

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\gamma}\hat{\Xi}(t) &= -\sum_{p=1}^{M}z^{\mathrm{T}}(y_p(t))e_p(t)\\
&-(1-\zeta)^{\sigma}\mathrm{SIGN}(\hat{\Xi}(t)-\Xi)|\hat{\Xi}(t)-\Xi|^{\sigma},
\end{aligned} \tag{7}$$

where $\hat{\Xi}(t)$ denote the estimations of uncertain parameter Ξ, d_p is a positive constant, and σ satisfies $\sigma \in (0,\gamma)$. Furthermore, the finite time t_1 can be estimated by

$$t_1 = t_0 + \left(\frac{\Gamma(1+\gamma)\Gamma(1-(1+\sigma)/2)V^{\gamma-(1+\sigma)/2}(t_0)}{2^{(1+\sigma)/2}(1-\zeta)^{\sigma}\Gamma(1+\gamma-(1+\sigma)/2)}\right)^{1/\gamma}. \tag{8}$$

Proof Construct the following Lyapunov function:

$$\begin{aligned}
V(t) &= \frac{1}{2}\sum_{p=1}^{M}e_p^{\mathrm{T}}(t)e_p(t) + \frac{1}{2}\sum_{p=1}^{M}\frac{1}{d_p}m_p^2(t)\\
&+\frac{1}{2}\left(\hat{\Xi}(t)-\Xi\right)^{\mathrm{T}}\left(\hat{\Xi}(t)-\Xi\right).
\end{aligned} \tag{9}$$

By Assumptions 1 and 2 and Lemma 2, the fractional derivative of $V(t)$ could be calculated as follows:

$$\begin{aligned}
&{}_{t_0}^{c}D_t^{\gamma}V(t)\\
&\leq \sum_{p=1}^{M}e_p^{\mathrm{T}}(t){}_{t_0}^{c}D_t^{\gamma}e_p(t) + \sum_{p=1}^{M}\frac{1}{d_p}m_p(t){}_{t_0}^{c}D_t^{\gamma}m_p(t)\\
&+(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}({}_{t_0}^{c}D_t^{\gamma}\hat{\Xi}(t))\\
&\leq \sum_{p=1}^{M}e_p^{\mathrm{T}}(t)\Big[f(y_p(t)) - f(x_p(t)) + z(y_p)(t)\hat{\Xi}(t)\\
&-z(x_p)(t)\Xi - \sum_{k=1}^{s}\sum_{q=1}^{M}h_k l_{pq}^{(k)}Ge_q(t) + \rho_p(t)\Big]\\
&+(1-\zeta)\sum_{p=1}^{M}m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\
&-(1-\zeta)^{\sigma}\sum_{p=1}^{M}\left(\frac{1}{d_p}\right)^{\frac{1+\sigma}{2}}m_p(t)\mathrm{SIGN}(m_p(t))|m_p(t)|^{\sigma}\\
&-(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}\sum_{p=1}^{M}z^{\mathrm{T}}(y_p(t))e_p(t)\\
&-(1-\zeta)^{\sigma}(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}(\mathrm{SIGN}(\hat{\Xi}(t)-\Xi)|\hat{\Xi}(t)-\Xi|^{\sigma})\\
&+\sum_{p=1}^{M}e_p^{\mathrm{T}}(t)u_p(t)\\
&= V_1 + V_2 + V_3.
\end{aligned} \tag{10}$$

In equality (10), we have

$$\begin{aligned}
V_1 &= \sum_{p=1}^{M}e_p^{\mathrm{T}}(t)\left[\tilde{f}(y_p(t)) - \tilde{f}(x_p(t)) - \sum_{k=1}^{s}\sum_{q=1}^{M}h_k l_{pq}^{(k)}Ge_q(t)\right]\\
&+(1-\zeta)\sum_{p=1}^{M}m_p(t)e_p^{\mathrm{T}}(t)e_p(t) + \sum_{p=1}^{M}e_p^{\mathrm{T}}(t)u_p^{(1)}(t),
\end{aligned} \tag{11}$$

$$V_2 = \sum_{p=1}^{M}e_p^{\mathrm{T}}(t)\rho_p(t) + \sum_{p=1}^{M}e_p^{\mathrm{T}}(t)u_p^{(2)}(t), \tag{12}$$

$$\begin{aligned}
V_3 &= -(1-\zeta)^{\sigma}\sum_{p=1}^{M}\left(\frac{1}{d_p}\right)^{\frac{1+\sigma}{2}}m_p(t)\mathrm{SIGN}(m_p(t))|m_p(t)|^{\sigma}\\
&+\sum_{p=1}^{M}e_p^{\mathrm{T}}(t)u_p^{(3)}(t).
\end{aligned} \tag{13}$$

For $V_1(t)$, by Assumption 1, Definitions 4 and 5, we can rewrite it as follows:

$$V_1 = \sum_{p=1}^{M}e_p^{\mathrm{T}}(t)\left[\tilde{f}(y_p(t)) - \tilde{f}(x_p(t)) + \sum_{k=1}^{s}\sum_{q=1}^{M}h_k l_{pq}^{(k)}Ge_q(t)\right]$$

$$
\begin{aligned}
&+(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)-\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)u_p^{(1)}(t)\\
=&\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\left[\tilde{f}(y_p(t))-\tilde{f}(x_p(t))+\sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G e_q(t)\right]\\
&+(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\
&-\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)\chi(e_p(t))-m^*\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\chi(e_p(t))\\
\le& \nu\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)e_p(t)+\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G e_q(t)\\
&+(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\
&-(1-\zeta)m^*\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)e_p(t)\\
&-(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\
\le& (\nu-(1-\zeta)m^*)e^{\mathrm{T}}(t)e(t)+e^{\mathrm{T}}(t)\Omega e(t)\\
\le& (\nu-(1-\zeta)m^*)e^{\mathrm{T}}(t)e(t)\\
&+\frac{1}{2}\varepsilon_1^{-1}e(t)\Omega\Omega^{\mathrm{T}}e(t)+\frac{1}{2}\varepsilon_1 e(t)e(t)\\
\le& \left[\nu-(1-\zeta)m^*+\frac{1}{2}\varepsilon_1+\frac{1}{2}\varepsilon_{-1}\lambda_{\max}(\Omega\Omega^{\mathrm{T}})\right]e^{\mathrm{T}}(t)e(t)\\
\le& 0.
\end{aligned}
$$

For the $V_2(t)$, according to Lemma 4, we have

$$
\begin{aligned}
V_2(t) =& \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\rho(t)+\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)u_p^{(2)}(t)\\
=& \sum_{p=1}^{M}\sum_{q=1}^{m}(e_{pq}(t)\rho_q(t))-\overline{\rho}\sum_{p=1}^{M}\sum_{q=1}^{m}|e_{pq}(t)|\\
\le& \sum_{p=1}^{M}\left[\left(\sum_{q=1}^{m}|e_{pq}(t)|^2\right)^{1/2}\left(\sum_{q=1}^{m}|\rho_q(t)|^2\right)^{1/2}\right]\\
&-\overline{\rho}\sum_{p=1}^{M}\left(\sum_{q=1}^{m}|e_{pq}(t)|^2\right)^{1/2}\\
=& (\|\rho(t)\|-\overline{\rho})\left(\sum_{p=1}^{M}(\sum_{q=1}^{m}|e_{pq}(t)|^2)^{1/2}\right).
\end{aligned}
$$

For the Assumption 2, we can get that $V_2(t)<0$. Then considering $V_3(t)$, we obtain

$$
\begin{aligned}
V_3(t) =& -(1-\zeta)^{\sigma}\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\mathrm{SIGN}(e_p(t))(|e_p(t)|^{\sigma})\\
&-(1-\zeta)^{\sigma}\sum_{p=1}^{M}(\frac{1}{d_p})^{\frac{\sigma-1}{2}}m_p(t)\mathrm{SIGN}(m_p(t))(|m_p(t)|^{\sigma})\\
&-(1-\zeta)^{\sigma}\mathrm{SIGN}(\hat{\Xi}(t)-\Xi)|\hat{\Xi}(t)-\Xi|^{\sigma}\\
\le& -(1-\zeta)^{\sigma}\Bigg[\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\mathrm{SIGN}(e_p(t))(|e_p(t)|^{\sigma})\\
&+\sum_{p=1}^{M}(\frac{1}{d_p})^{\frac{\sigma-1}{2}}m_p(t)\mathrm{SIGN}(m_p(t))(|m_p(t)|^{\sigma})\\
&+\mathrm{SIGN}(\hat{\Xi}(t)-\Xi)|\hat{\Xi}(t)-\Xi|^{\sigma}\Bigg]\\
\le& -(1-\zeta)^{\sigma}\Bigg(\sum_{p=1}^{M}\sum_{q=1}^{M}|e_{pq}(t)|^{1+\sigma}\\
&+\sum_{p=1}^{M}(\frac{1}{d_p})^{\frac{\sigma-1}{2}}|m_p(t)|^{1+\sigma}+\sum_{p=1}^{M}|\hat{\Xi}(t)-\Xi|^{1+\sigma}\Bigg)\\
\le& -(1-\zeta)^{\sigma}\Bigg[\left(\sum_{p=1}^{M}\sum_{q=1}^{M}|e_p(t)|\right)^{\frac{1+\sigma}{2}}\\
&+\left(\sum_{p=1}^{M}\frac{1}{d_p}|m_p(t)|^2\right)^{\frac{1+\sigma}{2}}+\left(\sum_{p=1}^{M}|\hat{\Xi}(t)-\Xi|^2\right)^{\frac{1+\sigma}{2}}\Bigg]\\
\le& -2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Bigg[\left(\frac{1}{2}\sum_{p=1}^{M}\sum_{q=1}^{M}|e_p(t)|^2\right)^{\frac{1+\sigma}{2}}\\
&+\left(\frac{1}{2}\sum_{p=1}^{M}\frac{1}{d_p}|m_p(t)|^2\right)^{\frac{1+\sigma}{2}}+\left(\sum_{p=1}^{M}|\hat{\Xi}_p(t)-\Xi|^2\right)^{\frac{1+\sigma}{2}}\Bigg]\\
\le& -2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Bigg[\left(\frac{1}{2}\sum_{p=1}^{M}\sum_{q=1}^{M}|e_p(t)|^2\right)\\
&+\left(\frac{1}{2}\sum_{p=1}^{M}\frac{1}{d_p}|m_p(t)|^2\right)+\left(\frac{1}{2}\sum_{p=1}^{M}|\hat{\Xi}_p(t)-\Xi|^2\right)\Bigg]^{\frac{1+\sigma}{2}}\\
\le& -2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Bigg[\frac{1}{2}\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)e_p(t)\\
&+\frac{1}{2}\sum_{p=1}^{M}\frac{1}{d_p}m_p^2(t)+\frac{1}{2}(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}(\hat{\Xi}(t)-\Xi)\Bigg]^{\frac{1+\sigma}{2}}\\
\le& -2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}V(t)^{\frac{1+\sigma}{2}}.
\end{aligned}
$$

By Lemma 1, the following results are achieved easily:

$$
\begin{aligned}
&V^{\gamma-\frac{1+\sigma}{2}}(t)\le V^{\gamma-\frac{1+\sigma}{2}}(t_0)\\
&-\frac{2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Gamma(1+\gamma-\frac{1+\sigma}{2})(t-t_0)^{\gamma}}{\Gamma(1+\gamma)\Gamma(1-\frac{1+\sigma}{2})},\ t\in[t_0,t_1],
\end{aligned}
\tag{14}
$$

then $V(t)\equiv 0$ for $t>t_1$, and the drive and response network can be synchronized in finite time t_1, which can be estimated as follows:

$$
t_1=t_0+\left(\frac{\Gamma(1+\gamma)\Gamma(1-\frac{1+\sigma}{2})V^{\gamma-\frac{1+\sigma}{2}}(t_0)}{2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Gamma(1+\gamma-\frac{1+\sigma}{2})}\right)^{1/\gamma}. \tag{15}
$$

This completes the proof.

3.2. Finite-time synchronization of FMCNs with coupling delay and uncertain parameter

Consider the following FMCNs composing of M identical nodes with uncertain parameter and coupling delay described by

$$\begin{aligned} {}_{t_0}^{c}D_t^{\gamma}x_p(t) &= \tilde{f}(x_p(t))+\sum_{k=1}^{s}\sum_{q=1}^{M}h_k l_{pq}^{(k)}Gx_q(t-\kappa)\\ &= f(x_p(t))+z(x_p(t))\Xi+\sum_{k=1}^{s}\sum_{q=1}^{M}h_k l_{pq}^{(k)}Gx_q(t-\kappa), \end{aligned} \tag{16}$$

where $\gamma\in(0,1)$, $x_p(t)=(x_{p1}(t),x_{p2}(t),\ldots,x_{pm}(t))^{\mathrm{T}}\in\mathbb{R}^m$, $(p=1,2,\ldots,M)$ represents the state vector of p-th node, $f(x_p(t)):\mathbb{R}^m\to\mathbb{R}^m$are continuous functions and $z(x_p(t)):\mathbb{R}^m\to\mathbb{R}^m$ are continuous function matrix. $\Xi\in\mathbb{R}^m$ represents the vector of system uncertain parameters, and $\tilde{f}(x_p(t)):\mathbb{R}^m\to\mathbb{R}^m$ are continuous vector functions which consists of two parts. h_k represents the coupling weight value of the k-th weight network, and $L^{(k)}=(l_{pq}^{(k)})$ is the weight matrix of k-th weight network. If a connection between q-th node to p-th node $(p\neq q)$ exist, then $l_{pq}\geq 0$, otherwise $l_{pq}=0$, and the diagonal elements $l_{pp}=-\sum_{q=1,q\neq p}^{M}l_{pq}$, $(p=1,2,\ldots,M)$, $G=\mathrm{diag}(g_1,g_2,\ldots,g_m)\in\mathbb{R}^m$ is the internal coupling strength matrix and $g_p\geq 0$, and κ represents the coupling delays.

The system (16) can be used as a drive system, and the system (17) can be used as a response system with controller and external disturbance, which is formulated in the following way:

$$\begin{aligned} {}_{t_0}^{c}D_t^{\gamma}y_p(t) &= f(y_p(t))+z(y_p(t))\hat{\Xi}(t)\\ &\quad+\sum_{k=1}^{s}\sum_{q=1}^{M}h_k l_{pq}^{(k)}Gy_q(t-\kappa)+\rho(t)+u_p(t), \end{aligned} \tag{17}$$

where fractional-order $\gamma\in(0,1)$,

$$y_p(t)=(y_{p1}(t),y_{p2}(t),\ldots,y_{pm}(t))^{\mathrm{T}}\in\mathbb{R}^m$$

represents the state vector of p-th node, $u_p(t)$ is the controller to be design, $\rho:\mathbb{R}\to\mathbb{R}^m$ represents the external disturbance function, and $\hat{\Xi}(t)\in\mathbb{R}^m$ represents the estimation of uncertain parameter vector Ξ. Then the error system with coupling delay is also provided by the following formula:

$$\begin{aligned} {}_{t_0}^{c}D_t^{\gamma}e_p(t) &= f(y_p(t))-f(x_p(t))+z(y_p(t))\hat{\Xi}(t)-z(x_p(t))\Xi\\ &\quad+\sum_{k=1}^{s}\sum_{q=1}^{M}h_k l_{pq}^{(k)}Ge_q(t-\kappa)+\rho(t)+u_p(t). \end{aligned} \tag{18}$$

Theorem 2 Suppose that Assumptions 1 and 2 are satisfied, if there exist gain m^* such that

$$v\leq(1-\zeta)m^*, \tag{19}$$

the response network (17) and drive network (16) can be synchronized in finite time t_2 with the controller $u_p(t)=u_p^{(1)}(t)+u_p^{(2)}(t)+u_p^{(3)}(t)+u_p^{(4)}(t)$ given by

$$\begin{aligned} u_p^{(1)}(t) &= -m_p(t)\chi(e_p(t))-m^*\chi(e_p(t)),\\ u_p^{(2)}(t) &= -\varepsilon_2^{-1}\frac{(\sum_{q=1}^{M}\|\chi(e_q(t-\kappa))\|^2)\overline{\mathrm{SIGN}}(\chi(e_p(t)))}{2(1-\zeta)^2}\\ &\quad-\frac{1}{2}\varepsilon_2\lambda_{\max}(\Omega\Omega^{\mathrm{T}})\overline{\mathrm{SIGN}}(\chi(e_p(t))),\\ u_p^{(3)}(t) &= -\overline{\rho}\overline{\mathrm{SIGN}}(\chi(e_p(t))),\\ u_p^{(4)}(t) &= -\mathrm{SIGN}(\chi(e_p(t)))|\chi(e_p(t))|^{\sigma}, \end{aligned} \tag{20}$$

where $\Omega=\sum_{k=1}^{s}h_k(L^{(k)}\otimes G)$, $\chi(\cdot)$ is logarithmic quantizer, and updating laws $m_p(t)$ and parameter estimate laws $\hat{\Xi}(t)$ given respectively by

$$\begin{aligned} {}_{t_0}^{c}D_t^{\gamma}m_p(t) &= (1-\zeta)d_p e_p^{\mathrm{T}}(t)e_p(t)\\ &\quad-(1-\zeta)^{\sigma}\left(\frac{1}{d_p}\right)^{\frac{\sigma-1}{2}}\mathrm{SIGN}(m_p(t))|m_p(t)|^{\sigma}, \end{aligned} \tag{21}$$

$$\begin{aligned} {}_{t_0}^{c}D_t^{\gamma}\hat{\Xi}(t) &= -\sum_{p=1}^{M}z^{\mathrm{T}}(y_p(t))e_p(t)\\ &\quad-(1-\zeta)^{\sigma}\mathrm{SIGN}(\hat{\Xi}(t)-\Xi)|\hat{\Xi}(t)-\Xi|^{\sigma}, \end{aligned} \tag{22}$$

where $\hat{\Xi}(t)$ denote the estimations of uncertain parameter Ξ, ε_2 and d_p are positive constants, and $\sigma\in(0,\gamma)$, then the finite time t_2 is estimated by

$$t_2=t_0+\left(\frac{\Gamma(1+\gamma)\Gamma(1-\frac{1+\sigma}{2})V^{\gamma-\frac{1+\sigma}{2}}(t_0)}{2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Gamma(1+\gamma-\frac{1+\sigma}{2})}\right)^{1/\gamma}. \tag{23}$$

Proof Construct the following Laypunov function:

$$\begin{aligned} V(t) &= \frac{1}{2}\sum_{p=1}^{M}e_p^{\mathrm{T}}(t)e_p(t)+\frac{1}{2}\sum_{p=1}^{M}\frac{1}{d_p}m_p^2(t)\\ &\quad+\frac{1}{2}(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}(\hat{\Xi}(t)-\Xi). \end{aligned} \tag{24}$$

By Assumptions 1 and 2 and Lemma 2, we can calculate the fractional derivative of $V(t)$ as follows:

$$\begin{aligned} &{}_{t_0}^{c}D_t^{\gamma}V(t)\\ &\leq\sum_{p=1}^{M}e_p^{\mathrm{T}}(t){}_{t_0}^{c}D_t^{\gamma}e_p(t)+\sum_{p=1}^{M}\frac{1}{d_p}m_p(t){}_{t_0}^{c}D_t^{\gamma}m_p(t)\\ &\quad+(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}({}_{t_0}^{c}D_t^{\gamma}\hat{\Xi}(t))\\ &\leq\sum_{p=1}^{M}e_p^{\mathrm{T}}(t)\Big[f(y_p(t))-f(x_p(t))+z(y_p)(t)\hat{\Xi}(t)\\ &\quad-z(x_p)(t)\Xi-\sum_{k=1}^{s}\sum_{q=1}^{M}h_k l_{pq}^{(k)}Ge_q(t-\kappa)+\rho_p(t)+u_p(t)\Big]\\ &\quad+(1-\zeta)\sum_{p=1}^{M}\frac{1}{d_p}m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\ &\quad-(1-\zeta)^{\sigma}\sum_{p=1}^{M}\left(\frac{1}{d_p}\right)^{\frac{1+\sigma}{2}}m_p(t)\mathrm{SIGN}(m_p(t))|m_p(t)|^{\sigma}\\ &\quad-(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}\sum_{p=1}^{M}z^{\mathrm{T}}(y_p(t))e_p(t)\\ &\quad-(1-\zeta)^{\sigma}(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}(\mathrm{SIGN}(\hat{\Xi}(t)-\Xi)|\hat{\Xi}(t)-\Xi|^{\sigma}) \end{aligned}$$

$$= V_1+V_2+V_3+V_4. \tag{25}$$

In equality (25), we have

$$\begin{aligned}
V_1(t) &= \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\big[f(y_p(t))-f(x_p(t))+z(y_p)(t)\hat{\Xi}(t)\\
&\quad -z(x_p)(t)\big]-(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}\sum_{p=1}^{M} z^{\mathrm{T}}(y_p(t))e_p(t)\\
&\quad +(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)+\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)u_p^{(1)}(t),\\
V_2(t) &= \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G e_q(t-\kappa)+\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)u_p^{(2)}(t),\\
V_3(t) &= \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\rho_p(t)+\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)u_p^{(3)}(t),\\
V_4(t) &= -(1-\zeta)^{\sigma}\sum_{p=1}^{M}\Big(\frac{1}{d_p}\Big)^{\frac{1+\sigma}{2}}\mathrm{SIGN}(m_p(t))|m_p(t)|^{\sigma}\\
&\quad +\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)u_p^{(3)}(t)-(1-\zeta)^{\sigma}(\hat{\Xi}(t)-\Xi)^{\mathrm{T}}\\
&\quad \times(\mathrm{SIGN}(\hat{\Xi}(t)-\Xi)|\hat{\Xi}(t)-\Xi|^{\sigma})+\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)u_p^{(4)}(t).
\end{aligned}$$

For $V_1(t)$, we can rewrite it as follows:

$$\begin{aligned}
V_1(t) &= \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\big[f(y_p(t))-f(x_p(t))+z(y_p(t))\hat{\Xi}(t)\\
&\quad -z(x_p(t))\Xi\big]+(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\
&\quad -\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)z(y_p(t))(\hat{\Xi}(t)-\Xi)+\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)u_p^{(1)}(t)\\
&= \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\big[f(y_p(t))-f(x_p(t))+z(y_p(t))\Xi\\
&\quad -z(x_p(t))\Xi\big]+(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\
&\quad -m_p(t)\chi(e_p(t))-m^*\chi(e_p(t))\\
&= \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\big[\tilde{f}(x)-\tilde{f}(y)\big]+(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\
&\quad -m_p(t)\chi(e_p(t))-m^*\chi(e_p(t))\\
&\le \nu\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)e_p(t)+(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\
&\quad -(1-\zeta)\sum_{p=1}^{M} m_p(t)e_p^{\mathrm{T}}(t)e_p(t)\\
&\quad -(1-\zeta)m^*\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)e_p(t)\\
&= (\nu-(1-\zeta)m^*)\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)e_p(t).
\end{aligned}$$

Then, according to condition (19), we can get that $V_1\le 0$.

For $V_2(t)$, we can get

$$\begin{aligned}
V_2(t) &= \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G e_q(t-\kappa)+\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)u_p^{(2)}(t)\\
&= \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G e_q(t-\kappa)\\
&\quad -\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\left[\varepsilon_2^{-1}\frac{(\sum_{q=1}^{M}\|\chi(e_q(t-\kappa))\|^2)\overline{\mathrm{SIGN}}(\chi(e_p(t)))}{2(1-\zeta)^2}\right.\\
&\quad \left.+\frac{1}{2}\varepsilon_2\lambda_{\max}(\Omega\Omega^{\mathrm{T}})\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\overline{\mathrm{SIGN}}(e_p(t))\right].
\end{aligned}$$

According the Definitions 4 and 5, $V_2(t)$ can be rewritten as

$$\begin{aligned}
V_2(t) &= \sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\sum_{k=1}^{s}\sum_{q=1}^{M} h_k l_{pq}^{(k)} G e_q(t-\kappa)\\
&\quad -\frac{1}{2}\varepsilon_2^{-1}\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\left(\sum_{q=1}^{M}\|e_q(t-\kappa)\|^2\right)\overline{\mathrm{SIGN}}(e_p(t))\\
&\quad -\frac{1}{2}\varepsilon_2\lambda_{\max}(\Omega\Omega^{\mathrm{T}})\sum_{p=1}^{M} e_p^{\mathrm{T}}(t)\overline{\mathrm{SIGN}}(e_p(t))\\
&= e^{\mathrm{T}}(t)\Omega e(t-\kappa)\\
&\quad -\frac{1}{2}\varepsilon_2^{-1}e^{\mathrm{T}}(t)(e^{\mathrm{T}}(t-\kappa)e(t-\kappa))\overline{\mathrm{SIGN}}(e(t))\\
&\quad -\frac{1}{2}\varepsilon_2\lambda_{\max}(\Omega\Omega^{\mathrm{T}})e^{\mathrm{T}}(t)\overline{\mathrm{SIGN}}(e(t)).
\end{aligned}$$

According to Lemma 3, it follows that

$$\begin{aligned}
&e^{\mathrm{T}}(t)\Omega e(t-\kappa)\\
&\le \frac{1}{2}\varepsilon_2 e^{\mathrm{T}}(t)\Omega(e^{\mathrm{T}}(t)\overline{\mathrm{SIGN}}(e(t))\otimes I_{Nn})^{-1}\Omega^{\mathrm{T}}e(t)\\
&\quad +\frac{1}{2}\varepsilon_2^{-1}e^{\mathrm{T}}(t-\kappa)(e^{\mathrm{T}}(t)\overline{\mathrm{SIGN}}(e(t))\otimes I_{Nn})^{-1}e(t-\kappa)\\
&\le \frac{1}{2}\varepsilon_2\lambda_{\max}(\Omega\Omega^{\mathrm{T}})e^{\mathrm{T}}(t)e(t)\left[e^{\mathrm{T}}(t)\overline{\mathrm{SIGN}}(e(t))\right]^{-1}\\
&\quad +\frac{1}{2}\varepsilon_2^{-1}e^{\mathrm{T}}(t-\kappa)e(t-\kappa)e^{\mathrm{T}}(t)\overline{\mathrm{SIGN}}(e(t))\\
&\le \frac{1}{2}\varepsilon_2\lambda_{\max}(\Omega\Omega^{\mathrm{T}})e^{\mathrm{T}}(t)\overline{\mathrm{SIGN}}(e(t))\\
&\quad +\frac{1}{2}\varepsilon_2^{-1}e^{\mathrm{T}}(t-\kappa)e(t-\kappa)e^{\mathrm{T}}(t)\overline{\mathrm{SIGN}}(e(t)).
\end{aligned}$$

After that, $V_2(t)\le 0$ can be easily gotten. Similar to the proof of Theorem 1, $V_3(t)<0$ and $V_4(t)<0$ can be obtained easily. Therefore, the proof process is omitted here.Based on the above discussion, we can get

$$V(t)\le -2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}V(t)^{\frac{1+\sigma}{2}}.$$

By Lemma 1, the following results are attained easily:

$$\begin{aligned}
&V^{\gamma-\frac{1+\sigma}{2}}(t)\le V^{\gamma-\frac{1+\sigma}{2}}(t_0)\\
&-\frac{2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Gamma(1+\gamma-\frac{1+\sigma}{2})(t-t_0)^{\gamma}}{\Gamma(1+\gamma)\Gamma(1-\frac{1+\sigma}{2})},\quad t\in[t_0,t_2],
\end{aligned}\tag{26}$$

then $V(t)\equiv 0$ for $t>t_2$, and drive and response network can be synchronized in finite time t_2, which can be estimated as

follows

$$t_2 = t_0 + \left(\frac{\Gamma(1+\gamma)\Gamma(1-\frac{1+\sigma}{2})V^{\gamma-\frac{1+\sigma}{2}}(t_0)}{2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Gamma(1+\gamma-\frac{1+\sigma}{2})} \right)^{1/\gamma}. \quad (27)$$

This completes the proof.

4. Numerical simulations

This section contains two simulation examples that demonstrate the correctness of Theorem 1 and Theorem 2 respectively.

Example 1 Consider the driver network composed of 4 nodes, each of which is a fractional-order Lorenze system

$$\begin{aligned} &{}_{t0}^{c}D_t^{\gamma}x_p(t) \\ &= \underbrace{\begin{pmatrix} 0 \\ -x_{i1}x_{i3}-x_{i2} \\ x_{i1}x_{i2} \end{pmatrix}}_{f(x_p(t))} + \underbrace{\begin{pmatrix} x_{i2}-x_{i1} & 0 & 0 \\ 0 & 0 & x_{i1} \\ 0 & -x_{i3} & 0 \end{pmatrix}}_{z(x_p(t))} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\Xi} \\ &+h_1\sum_{q=1}^{M} l_{pq}^{(1)}Gx_q(t) + h_2\sum_{q=1}^{M} l_{pq}^{(2)}Gx_q(t) + h_3\sum_{q=1}^{M} l_{pq}^{(3)}Gx_q(t), \quad (28) \end{aligned}$$

where $\gamma = 0.995$, $G = \mathrm{diag}(1,1,1)$, $h_1 = 0.1$, $h_2 = 0.2$, and $h_3 = 0.3$, then the chaotic attractor of Lorenze system is shown in Fig. 1. System parameter $\Xi = (a,b,c)^{\mathrm{T}} = (10, 8/3, 28)$, $x_p(0) = (6.2+0.2p, 6.4+0.2p, 6.6+0.2p)^{\mathrm{T}}$, $(p = 1,2,3,4)$, where three weighted matrices are listed as follows:

$$L^{(1)} = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix},$$

$$L^{(2)} = \begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$L^{(3)} = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

Figure 2 depicts the topological structures of network, then the corresponding response network is constructed by the following formula:

$$\begin{aligned} &{}_{t_0}^{c}D_t^{\gamma}y_p(t) = f(y_p(t)) + z(y_p(t))\hat{\Xi}(t) + h_1\sum_{q=1}^{M} l_{pq}^{(1)}Gx_q(t) \\ &+h_2\sum_{q=1}^{M} l_{pq}^{(2)}Gx_q(t) + h_3\sum_{q=1}^{M} l_{pq}^{(3)}Gx_q(t) + \rho(t) + u_p(t), \quad (29) \end{aligned}$$

where $\hat{\Xi}(t)$ is the estimation for vector Ξ, and external disturbance $\rho(t) = (\sin(t), \cos(t), \sin(t)\cos(t))$.

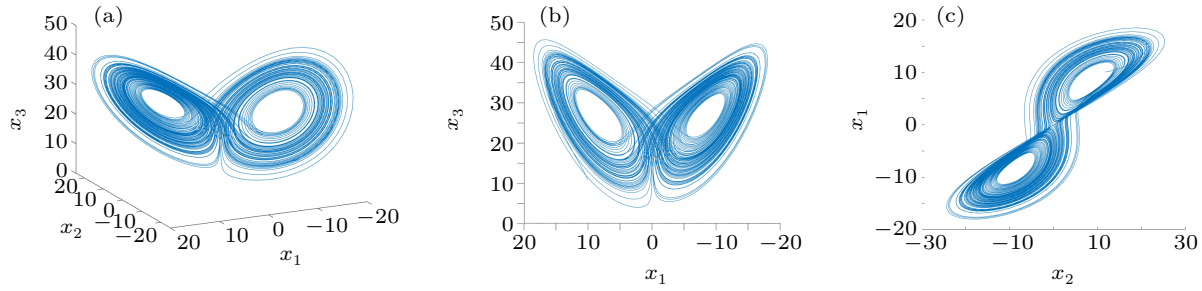


Fig. 1. Phase plot of the Lorenze system ($\gamma = 0.995$).

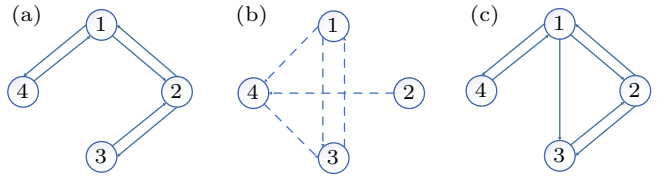


Fig. 2. The topological structures of weighted matrices $L^{(k)}$, $k = 1,2,3$. (a) is matrices $L^{(1)}$, (b) is matrices $L^{(1)}$, and (c) is matrices $L^{(3)}$.

For fractional-order Lorenze system, we can choose $v_1 = 25.12$ such that Assumption 1 holds,[36] and we choose $\varepsilon_1 = 1$, $m^* = 38$, and $\overline{\rho} = 1$. It is simple to get that $\overline{\rho} \geq \|\rho(t)\|$, then through simple calculation, we can get that $\lambda_{\max}(\Omega\Omega^{\mathrm{T}}) = 3.5266$. Therefore the Assumption 2 and inequality (4) of Theorem 1 hold.

In this example, the parameters of the logarithmic quantizer are selected as $\rho = 0.7$, $\zeta = \frac{1-\rho}{1+\rho} = 0.18$, control parameters are given as $\sigma = 0.7$, $d_p = 1$, and the default settings of all parameters are selected as follows, $x_p(0) = (-2.2+p, -3.4+p, -4.6+p)$, $y_p(0) = (3.2+p, 3.4+p, 3.6+p)$, $m_p(0) = (1+0.5p, 1+0.5p, 1+0.5p)$, and $\Xi_p(0) = (1,1,1)$.

By Theorem 1, when $t > t_1$, the response network (2) can synchronize with drive network (1), and the time t_1 can be estimated as follows:

$$\begin{aligned} t_1 &= t_0 + \left(\frac{\Gamma(1+\gamma)\Gamma(1-\frac{1+\sigma}{2})V^{\gamma-\frac{1+\sigma}{2}}(t_0)}{2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Gamma(1+\gamma-\frac{1+\sigma}{2})} \right)^{1/\gamma} \\ &= 0 + \left(\frac{\Gamma(1+0.995)\Gamma(1-\frac{1+0.7}{2})V^{0.995-\frac{1+0.7}{2}}(0)}{2^{\frac{1+0.7}{2}}(1-0.18)^{0.7}\Gamma(1+0.995-\frac{1+0.7}{2})} \right)^{1/0.995} \\ &= 10.8376. \quad (30) \end{aligned}$$

Figure 3 depicts the synchronization errors $e_p(t)$ converge to zero, which indicates that synchronization has been realized. Figure 4 shows the synchronization quantized errors

$\chi(e_p(t))$ converge to zero. Figure 5 displays the identification curve of the uncertain system parameter vector $\Xi(t)$. We can see that the uncertain parameters of the system are successfully identified in finite time. Figure 6 shows the trajectory of adaptive feedback gain $m_p(t)$, which converges to zero eventually. Figure 11(a) describes the surface graph of maximum synchronization time t_1 of Theorem 1 with the change of parameter σ and parameter γ.

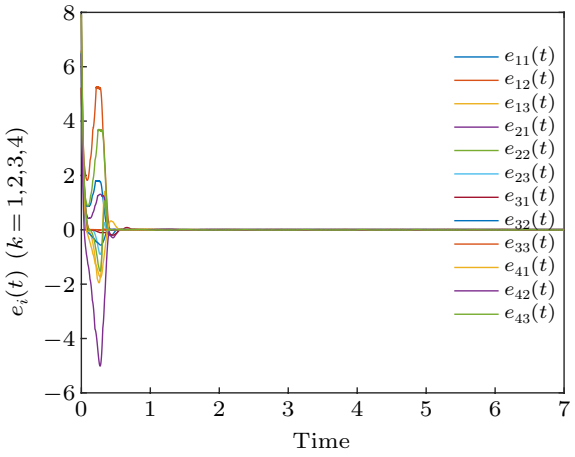


Fig. 3. The $e_p(t)$ of the error system.

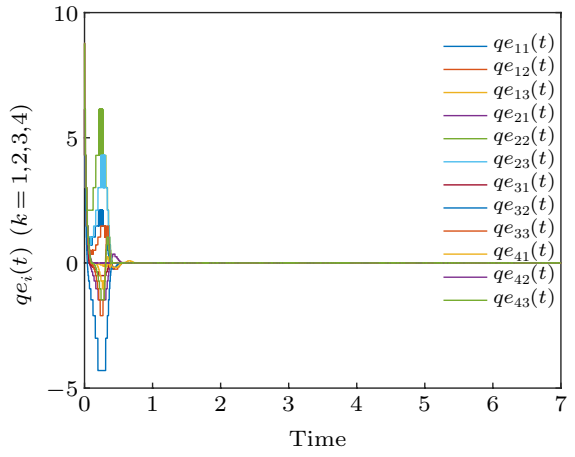


Fig. 4. The $\chi(e_p(t))$ of the error system.

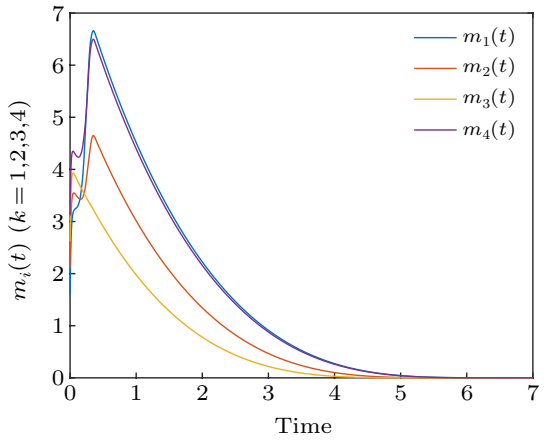


Fig. 5. The feedback gain $m_p(t)$, $p=1,2,3,4$ of the adaptive controller.

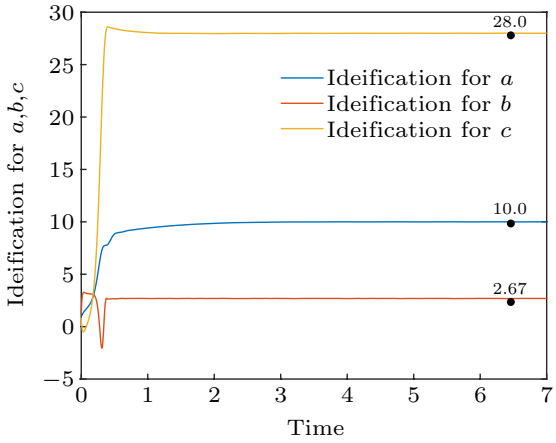


Fig. 6. Identification of the system parameter Ξ.

Example 2 The uncertain FMCNs with coupling delay $\kappa=0.1$ are considered, and the drive network can be shown as

$$
\begin{aligned}
&{}_{t0}^{c}D_t^{\gamma}x_p(t)\\
&=\underbrace{\begin{pmatrix}0\\ -x_{i1}x_{i3}-x_{i2}\\ x_{i1}x_{i2}\end{pmatrix}}_{f(x_p(t))}+\underbrace{\begin{pmatrix}x_{i2}-x_{i1} & 0 & 0\\ 0 & 0 & x_{i1}\\ 0 & -x_{i3} & 0\end{pmatrix}}_{z(x_p(t))}\underbrace{\begin{pmatrix}a\\ b\\ c\end{pmatrix}}_{\Xi}\\
&+h_1\sum_{q=1}^{M}l_{pq}^{(1)}Gx_q(t-\kappa)+h_2\sum_{q=1}^{M}l_{pq}^{(2)}Gx_q(t-\kappa)\\
&+h_3\sum_{q=1}^{M}l_{pq}^{(3)}Gx_q(t-\kappa). \qquad (31)
\end{aligned}
$$

The corresponding response network is designed as

$$
\begin{aligned}
{}_{t0}^{c}D_t^{\gamma}y_p(t) \;=\; & f(y_p(t))+z(y_p(t))\hat{\Xi}(t)\\
&+h_1\sum_{q=1}^{M}l_{pq}^{(1)}Gx_q(t-\kappa)+h_2\sum_{q=1}^{M}l_{pq}^{(2)}Gx_q(t-\kappa)\\
&+h_3\sum_{q=1}^{M}l_{pq}^{(3)}Gx_q(t-\kappa)+\rho(t)+u_p(t), \qquad (32)
\end{aligned}
$$

where $\hat{\Xi}(t)$ is the estimation for vector Ξ, and external disturbance $\rho(t)=(\sin(t),\cos(t),\sin(t)\cos(t))$.

For fractional-order Lorenze system, we can choose $\nu_1=25.12$ such that Assumption 1 holds,[36] and we choose $\varepsilon_2=1$, $m^*=38$, and $\overline{\rho}=1$. It is simple to get that $\overline{\rho}\geq\|\rho(t)\|$, then through simple calculation, we can get that $\lambda_{\max}(\Omega\Omega^{\mathrm{T}})=3.5266$. Therefore the Assumption 2 and inequality (19) of Theorem 2 hold.

In this example, the parameters of the logarithmic quantizer are selected as $\rho=0.7$, $\zeta=\frac{1-\rho}{1+\rho}=0.18$, control parameters are given as $\sigma=0.7$, $d_p=1$, and the default settings of all parameters are selected as follows, $x_p(0)=(3.2+p,3.4+p,3.6+p)$, $y_p(0)=(2.2+p,2.4+p,2.6+p)$, $m_p(0)=(1+0.1p,1+0.1p,1+0.1p)$, and $\Xi_p(0)=(1,1,1)$.

By Theorem 2, when $t > t_2$, the response network (17) can synchronize with drive network (16), and the time t_2 can be estimated as follows:

$$\begin{aligned} t_2 &= t_0 + \left(\frac{\Gamma(1+\gamma)\Gamma(1-\frac{1+\sigma}{2})V^{\gamma-\frac{1+\sigma}{2}}(t_0)}{2^{\frac{1+\sigma}{2}}(1-\zeta)^{\sigma}\Gamma(1+\gamma-\frac{1+\sigma}{2})} \right)^{1/\gamma} \\ &= 0 + \left(\frac{\Gamma(1+0.995)\Gamma(1-\frac{1+0.7}{2})V^{0.995-\frac{1+0.7}{2}}(0)}{2^{\frac{1+0.7}{2}}(1-0.18)^{0.7}\Gamma(1+0.995-\frac{1+0.7}{2})} \right)^{1/0.995} \\ &= 10.3351. \end{aligned} \tag{33}$$

When synchronization is implemented, both synchronization error $e_p(t)$ and synchronization quantization error $\chi(e_p(t))$ converge to zero, as shown in Figs. 7 and 8, respectively. Figure 9 shows the identification curve of the uncertain system parameter vector $\Xi(t)$, and it can be seen that in the case of coupling delay, the uncertain parameters of the system are still identified correctly. Figure 10 shows the trajectory of adaptive feedback gain $m_p(t)$, which also converges to zero eventually. Figure 11(b) describes the surface graph of maximum synchronization time t_2 of Theorem 2 with the change of parameter σ and parameter γ.

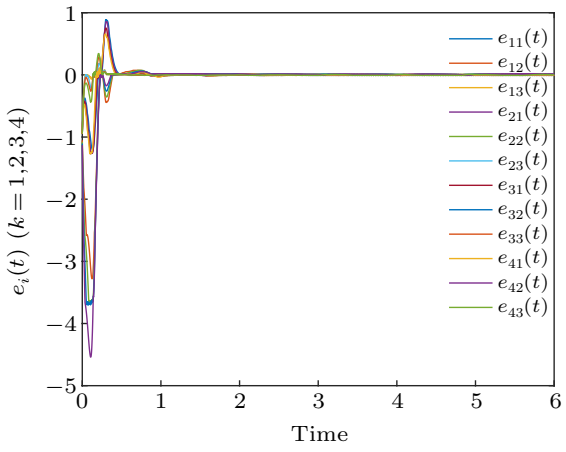


Fig. 7. The $e_p(t)$ of the error system with coupling delay $\kappa = 0.1$.

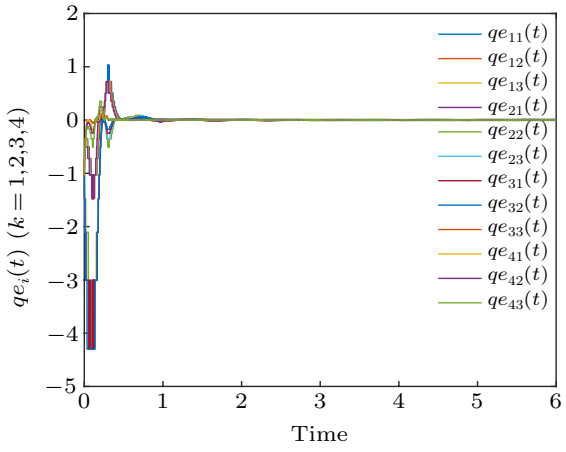


Fig. 8. The $\chi(e_p(t))$ of the error system with coupling delay $\kappa = 0.1$.

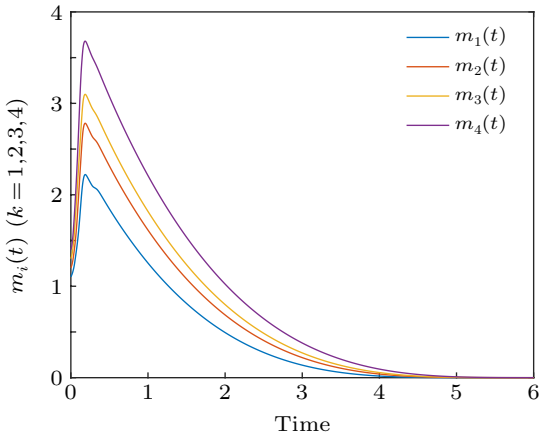


Fig. 9. The feedback gain $m_p(t)$, $p = 1,2,3,4$ of the adaptive controller with coupling delay $\kappa = 0.1$.

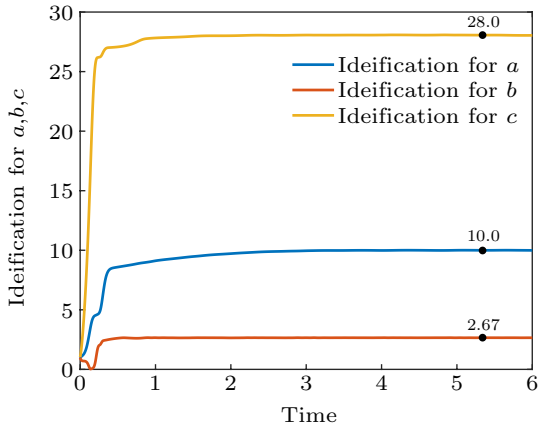


Fig. 10. Identification of the system parameter Ξ with coupling delay $\kappa = 0.1$.

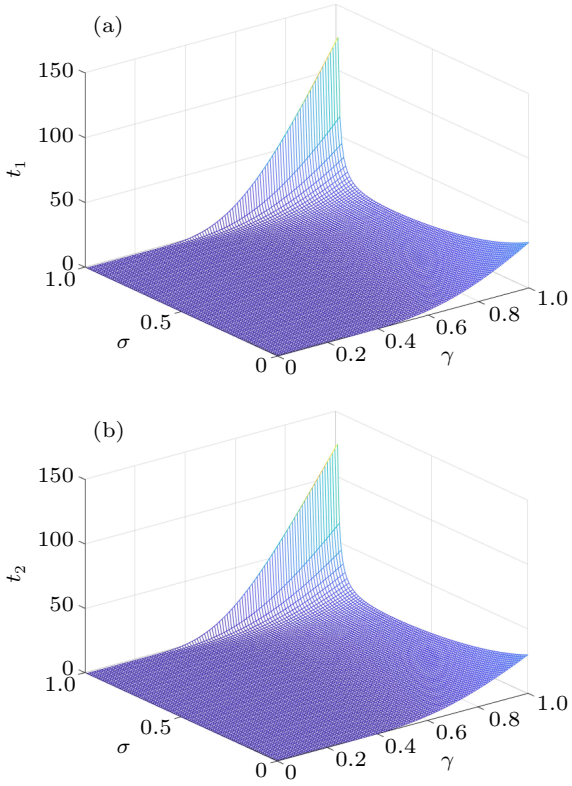


Fig. 11. (a) Setting time t_1 of Theorem 1, (b) setting time t_2 of Theorem 2.

5. Conclusion

This paper is engaged in the finite-time synchronization problem of FMCNs with uncertain parameters and external disturbances. Firstly, considering the two different cases of no-coupling delay and coupling delay, the corresponding adaptive quantized controller is designed to solve this problem. Then fractional calculus and Lyapunov stability theory are used to obtain theoretical results related to finite-time synchronization and parameter identification. It is worth noting that the time required for parameter identification and network synchronization can be estimated. Finally, numerical simulations of several chaotic systems are given, demonstrating the correctness and effectiveness of the theorem we obtained. Our future work will focus on designing appropriate quantization controllers to solve exponential synchronization problems of multi-layer networks.

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