Rogue waves of a (3+1)-dimensional BKP equation

Yu-Qiang Yuan(袁玉强)[1], Xiao-Yu Wu(武晓昱)[2], and Zhong Du(杜仲)[3,†]

[1] *Department of Mathematics, China University of Mining and Technology (Beijing), Beijing* 100083, *China*
[2] *School of Science, Beijing Forestry University, Beijing* 100083, *China*
[3] *Department of Mathematics and Physics, North China Electric Power University, Baoding* 071003, *China*

(Received 10 March 2022; revised manuscript received 13 April 2022; accepted manuscript online 20 April 2022)

We investigate certain rogue waves of a $(3+1)$-dimensional BKP equation via the Kadomtsev–Petviashili hierarchy reduction method. We obtain semi-rational solutions in the determinant form, which contain two special interactions: (i) one lump develops from a kink soliton and then fuses into the other kink one; (ii) a line rogue wave arises from the segment between two kink solitons and then disappears quickly. We find that such a lump or line rogue wave only survives in a short time and localizes in both space and time, which performs like a rogue wave. Furthermore, the higher-order semi-rational solutions describing the interaction between two lumps (one line rogue wave) and three kink solitons are presented.

Keywords: $(3+1)$-dimensional BKP equation, Kadomtsev–Petviashvili hierarchy reduction, interaction, rogue wave, lump

PACS: 02.30.Ik, 04.20.Jb, 04.30.Nk **DOI:** 10.1088/1674-1056/ac6869

1. Introduction

Since nonlinear waves have extensive applications, such as the evolution of optical pulse and water wave packet, researches of the nonlinear waves have been attractive in the optical fibers, fluids, and plasmas.[1–3] The physical systems relating to those nonlinear waves have been described via those nonlinear evolution equations (NLEEs) such as the nonlinear Schrödinger equation, the three-wave resonant interaction system, the Kadomtsev–Petviashvili (KP) equation, the long-wave-short-wave resonant interaction system, the Ito equation, and the Fokas–Lenells equation.[4–7] The NLEEs play a critical part in explaining and handling problems in the fields of experiment and engineering.[8–10] Among these models, studies of the high dimensional NLEEs are concentrated to investigate the rich nonlinear waves interactions.[11–15]

In this paper, we will investigate a $(3+1)$-dimensional BKP equation,[16–20]

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz} = 0, \tag{1}$$

where u is an analytic function of the scaled spatial coordinates x, y, z and temporal coordinate t, with the subscripts denoting the partial differential derivatives. Equation (1) has been proposed as an opposite procedure on applying the linear superposition principle.[16] By using the multiple exp-function algorithm, the multiple wave solutions of Eq. (1) have been presented.[17] Rational lump solutions of Eq. (1) have been obtained via the symbolic computation.[18] The lump and line-rogue wave have been derived via the KP hierarchy reduction method.[19] Certain hybrid solutions and breather solutions have been constructed.[20]

Recently, some semi-rational solutions with certain parameter constraints which reveal special resonant interaction [The resonant interaction describing a special interaction between the N-th order rational solutions and $N+1$ solitons, which occurs that the phase of the lump (line rogue wave) varies to infinite suddenly when such lump (line rogue wave) interacts with a soliton (based on the semi-rational solutions obtained by the Hirota method)] between the lump (line rogue wave) and soliton have been investigated via the KP hierarchy reduction[21,22] and bilinear method for the KP I equation,[23,24] the Fokas system,[25] the Davey–Stewartson (DS) I equation,[26] and some other models.[27] The resonant interaction describes the pattern in which a lump arises from one soliton and then merges with the other soliton. What is more, such the lump (line rogue wave) is localized in both space and time, which performs like a rogue wave in the higher dimensional space.

Motivated by the above interesting interaction, in this paper, we will pay our attention to investigating such resonant interaction by constructing semi-rational solutions of Eq. (1) which have not been reported before, to our knowledge. In Section 2, via the KP hierarchy reduction, we will derive the semi-rational solutions in determinant form of Eq. (1). Resonant interactions between one lump (line rogue wave) and two solitons, and between the higher-order lump and three solitons will be discussed in Section 3. Our conclusions will be presented in Section 4.

2. Semi-rational solutions of Eq. (1)

In this section, we will construct the semi-rational solutions which generate the resonant interactions between the

†Corresponding author. E-mail: duzhong002@ncepu.edu.cn
© 2022 Chinese Physical Society and IOP Publishing Ltd

lump (line rogue wave) and the solitons (actually the kinks) via the KP hierarchy reduction method. The basic idea of such a reduction is to make the bilinear form of the studied equation as a reduction of the KP hierarchy under certain constraints. Then we begin with the bilinear equation of Eq. (1),[19,20]

$$\left(3D_xD_z - D_x^3D_y + D_yD_t\right) f\cdot f = 0, \tag{2}$$

with the transformation

$$u = 2(\ln f)_x, \tag{3}$$

where f is a real function of x, y, z, and t, D is the Hirota bilinear operator defined as Ref. [28].

On the other hand, the second KP hierarchy identity is[29]

$$\left(3D_{x_1}D_{x_4} - D_{x_1}^3D_{x_2} - 2D_{x_2}D_{x_3}\right)\tau\cdot\tau = 0, \tag{4}$$

where τ is a function of the variables x_j $(j = 1,2,3,4)$. The N-th order semi-rational solutions for KP hierarchy (4) are defined as the following $N+1$ determinant:

$$\tau = \det_{1\leqslant j,k\leqslant N+1}\left(m_{jk}\right), \tag{5}$$

with

$$\begin{aligned}
&m_{jk} = \delta_{jk} + \int \phi_j\psi_k\,\mathrm{d}x_1, \quad \phi_j = A_j\,\mathrm{e}^{\zeta_j}, \quad \psi_k = B_k\,\mathrm{e}^{\vartheta_k},\\
&A_j = \sum_{l=0}^{N+1-l} c_{jl}\partial_p^{N+1-j-l}, \quad B_k = \sum_{m=0}^{N+1-m} d_{km}\partial_p^{N+1-k-m},\\
&\zeta_j = px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \zeta_{j0},\\
&\vartheta_k = qx_1 - q^2x_2 + q^3x_3 - q^4x_4 + \vartheta_{k0},
\end{aligned}$$

δ_{jk} is the Kronecker delta notation, c_{jl}, d_{km}, p, q, ζ_{j0}, and ϑ_{k0} are complex constants.

It can be seen that by using the following assumption:

$$f = \tau, \quad x = x_1, \quad y = \mathrm{i}x_2, \quad t = -\frac{1}{2}x_3, \quad z = \mathrm{i}x_4, \tag{6}$$

KP hierarchy (4) can be reduced to bilinear form (2), namely, we can derive soutions of Eq. (1) from solutions (5). Then with setting $q = p^*$, $\vartheta_{k0}^* = \zeta_{k0}^*$, and $d_{km} = c_{km}^*$, we can rewrite m_{jk} as follows:

$$\begin{aligned}
m_{jk} \;=\; &\delta_{jk} + \mathrm{e}^{\xi+\xi^*+\xi_{j0}+\xi_{k0}^*}\sum_{l=0}^{N+1-l} c_{jl}(\partial_p + \xi')^{N+1-j-l}\\
&\times\sum_{m=0}^{N+1-m} c_{km}^*(\partial_p' + \xi'^*)^{N+1-k-m}\frac{1}{p+p^*},
\end{aligned} \tag{7}$$

with $\xi = px - \mathrm{i}p^2y - 2p^3t - \mathrm{i}p^4z$, $\xi' = x - 2\mathrm{i}py - 6p^2t - 4\mathrm{i}p^3z$. The above solutions combine the rational and the exponential functions, namely, they are the semi-rational solutions in nature. Moreover, it is worth noting the following:

(I) The above semi-rational solutions will be reduced to rational solutions if the constant δ_{jk} is removed, which are exactly those solutions reported in Ref. [19] of Eq. (1). We conclude that the order of operator ∂_p (or ∂_{p^*}) determines the order of rational solutions and the order of the determinant determines the number of solitons. More subtle discussions will be seen in Section 3. In addition, to obtain the interaction between the rational lump or rogue wave and solitons, we need at least two solitons. Thus, the fundamental solutions defined above should be second-order determinants.

(II) Dynamics of the rational solutions contain entirely different two cases (a stable propagating lump and a line rogue wave), consequently, the above semi-rational will lead to two interaction patterns, *i.e.*, the interaction between the lump and the solitons, and between the line rogue wave and the solitons.

3. Interaction between the lump (line rogue wave) and solitons

3.1. The first order lump and rogue wave

By setting $N = 1$ in solutions (6), we derive

$$\begin{aligned}
m_{11} \;&=\; 1 + \frac{\mathrm{e}^{\xi+\xi^*+\zeta_{10}+\zeta_{10}^*}}{p+p^*}\left[\eta\eta^* + \frac{1}{(p+p^*)^2}\right],\\
m_{22} \;&=\; 1 + \frac{\mathrm{e}^{\xi+\xi^*+\zeta_{20}+\zeta_{20}^*}}{p+p^*},\\
m_{12} \;&=\; \frac{\mathrm{e}^{\xi+\xi^*+\zeta_{10}+\zeta_{20}^*}}{p+p^*}\eta, \quad m_{21} = \frac{\mathrm{e}^{\xi+\xi^*+\zeta_{20}+\zeta_{10}^*}}{p+p^*}\eta^*,\\
f \;&=\; 1 + \frac{\mathrm{e}^{\xi+\xi^*+\zeta_{10}+\zeta_{10}^*}}{p+p^*}(R + \mathrm{e}^{\zeta_{20}+\zeta_{20}^*-\zeta_{10}-\zeta_{10}^*})\\
&\quad + \frac{\mathrm{e}^{2(\xi+\xi^*)+\zeta_{20}+\zeta_{20}^*+\zeta_{10}+\zeta_{10}^*}}{(p+p^*)^4},
\end{aligned} \tag{8}$$

with $R = \eta\eta^* + 1/(p+p^*)^2$, $\eta = \xi' - 1/p + p^*$, and $c_{10} = c_{20} = 1$, $c_{11} = 0$. It can be seen that ζ_{j0} is always accompanied by ζ_{j0}^*, which means that the imaginary part of ζ_{j0} can be removed, thus we will take ζ_{j0} as real constants hereafter.

Separating the rational and exponential functions, we rewrite the fundamental semi-rational solutions of Eq. (1) as

$$u = 2\frac{\mathrm{e}^{2(\xi_\mathrm{R}+\zeta_{10})}\left[\eta_\mathrm{R}^2 + \eta_\mathrm{I}^2 + \mathrm{e}^{2(\zeta_{20}-\zeta_{10})} - 2\eta_\mathrm{R}/(p+p^*)\right] + \dfrac{2\mathrm{e}^{2(2\xi_\mathrm{R}+\zeta_{10}+\zeta_{20})}}{(p+p^*)^3}}{1 + \dfrac{\mathrm{e}^{2(\xi_\mathrm{R}+\zeta_{10})}}{p+p^*}\left[\eta_\mathrm{R}^2 + \eta_\mathrm{I}^2 + \mathrm{e}^{2(\zeta_{20}-\zeta_{10})}\right] + \dfrac{\mathrm{e}^{2(2\xi_\mathrm{R}+\zeta_{10}+\zeta_{20})}}{(p+p^*)^4}}, \tag{9}$$

$$\eta_\mathrm{R} = -x - 2by + 6(a^2-b^2)t - 4b(3a^2-b^2)z - \frac{1}{2a}, \tag{10}$$

$$\eta_\mathrm{I} = 2ay + 12abt + 4a(a^2-3b^2)z, \quad p = a + \mathrm{i}b, \tag{11}$$

with the subscripts R and I denoting the real and imaginary parts of constants. We derive that properties of the two solitons are determined by $e^{2(\xi_R+\zeta_{10})}$ and $e^{2(\xi_R+\zeta_{20})}$ respectively, while the properties of the rational solutions are determined by $\eta_R^2+\eta_I^2$.

Based on the above solutions, we can obtain the asymptotic of u as

$$\lim_{\xi_R+\xi_R^*\to-\infty} u=0, \qquad \lim_{\xi_R+\xi_R^*\to+\infty} u=4(p+p^*). \tag{12}$$

The result shows the soliton with two different asymptotic planes, which implies the above soliton is a kink one. Besides, for studying the interaction, we will take $\zeta_{10}>\zeta_{20}$ (also in the following discussions), which makes the rational function $\eta_R^2+\eta_I^2$ hold the prominent effect in the intermediate time. Since η_R and η_I bear different variables, then with the different choices of parameters and planes, we will derive two different kinds of nonlinear waves: (i) when $a^2-3b^2\neq 0$, the rational solutions generate a lump that develops along the trajectory of $\eta_R=0$ and $\eta_I=0$; (ii) when $a^2-3b^2=0$ and on the x–z plane, η_I is just a constant, thus the rational solutions generate a line wave (along the line trajectory $\eta_R=0$). Then combining with the kink solitons, we will discuss the interactions between the lump and kink solitons, and between the line rogue wave and kink solitons on the x–z plane.

In Fig. 1, we present the interaction between one lump and two kink solitons on the x–y plane. The semi-rational solutions generate a novel nonlinear wave structure which shows: (i) as the two kink solitons own the same wavenumber, they are nearly parallel with each other; (ii) the lump only exists in a short time and emerges from one kink soliton and disappears into the other one later. It should pay attention that during the interaction, such lump localizes not only in space (the nature of the lump) but also in time, which behaves like a rogue wave. Similar structure has also been revealed in Refs. [23,26] for the KP and DSI equations (that lump is called a lump-type rogue wave). In addition, as the lump will disappear finally instead of passes through the kink soliton, such interaction is quite different from that interaction pattern reported in Ref. [20]. Interactions of the lump and kink solitons on the y–z plane are similar to those on the x–y plane, so we will omit those analyses.

When we choose $a^2-3b^2=0$ on the x–z plane, η_I can be seen as a constant, then solutions (9) will generate the interaction between line rogue wave and kink solitons. As shown in Figs. 2, we find that (i) the two kink solitons propagate stably; (ii) the line rogue wave rises up at a certain time, reaches higher amplitude at $t=0$, and then disappears soon; (iii) especially, the line rogue wave only exists in a segment between the two kink solitons, which implies that the line rogue wave is also localized in space. Such line rogue wave is different from that proposed in Ref. [21] (which extends to infinity in space) for DSI equation. To discuss such interaction more clearly, in Fig. 3 we present some details such as the density plot of interaction at $t=0$ and the trajectories at different positions in Fig. 2(c). We compare the evolution of the interaction at different times at a fixed position at $x=-5$. A localized wave with a sharp shape rises between the two kink solitons (as shown in the real line in Fig. 3(b)) and vanishes after a short time.

In general, in this paper, we obtain two novel kinds of interactions: between the lump and two kink solitons and between the rogue wave and two kink solitons. Evolutions of the lump and the line rogue wave are both confined in time and to space (the segment between two kink solitons), which perform like a rogue wave. Thus, we show two kinds of potential rogue wave for Eq. (1) via the semi-rational solutions, which are different from that in Refs. [19,20].

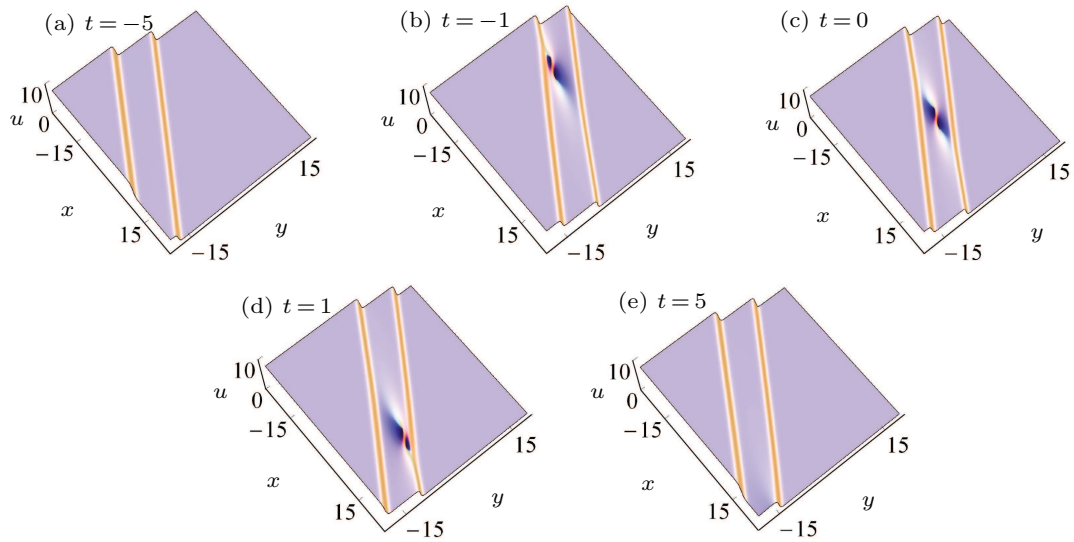


Fig. 1. Interaction between a lump and two kink solitons via solutions (9) with $z=0$, $a=1$, $b=1$, $\zeta_{10}=2\pi$, $\zeta_{20}=-2\pi$.

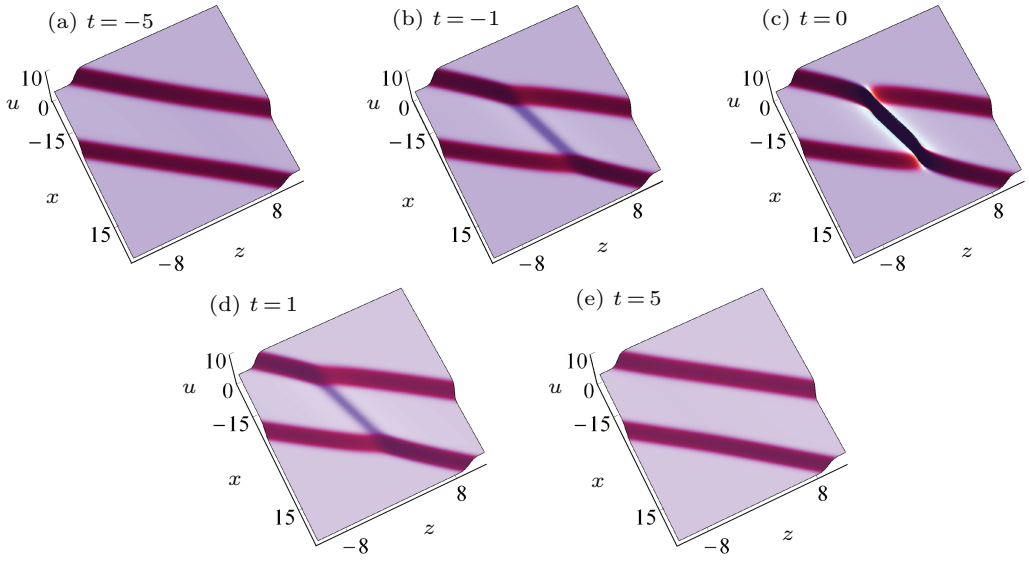


Fig. 2. Interaction between a line rogue wave and two kink solitons via solutions (9) with $y=0$, $a=1$, $b=-1/\sqrt{3}$, $\zeta_{10}=2\pi$, $\zeta_{20}=-2\pi$.

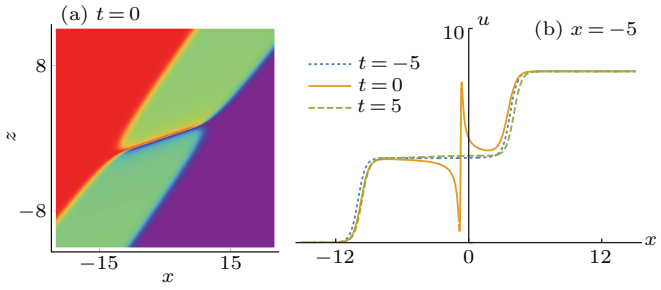


Fig. 3. (a) The density plot of interaction at $t=0$, (b) trajectories at different positions in Fig. 2(a). The parameters are the same as those in Fig. 2.

3.2. Higher-order lump and rogue wave

With $N=2$ in solutions (7), we can derive the second-order semi-rational solutions of Eq. (1):

$$u=2(\ln f)_x, \quad f=\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}, \tag{13}$$

where m_{jk} are given by solutions (7). The same as the above analyses, those solutions will generate interactions between three solitons and higher-order rational solutions, including (I) between three kink solitons and the line rogue wave on the x–z plane with constraint $a^2-3b^2=0$, as shown in Fig. 5; (II) between three kink solitons and two lumps in other cases, as shown in Fig. 4. The three kink solitons are of the same wavenumber $(a+\mathrm{i}b)$, thus they are all nearly parallel. In addition to the different kinds of rational solutions, the distance between the two kink solitons will also remarkably affect such wave interactions, which can be adjusted via the phase shifts constants ζ_{j0}. Thus, in what follows, we will discuss the interactions between the lumps (line rogue waves) and three kink solitons graphicly.

We present the interactions between two lumps and three kink solitons on the x–y plane as an example. With different choices of ζ_{j0}, there are two kinds of interaction. The one is, by setting $\zeta_{10}>\zeta_{20}>\zeta_{30}$, we see that two lumps arise from the leftmost (in the x-axis direction) kink soliton, go through the middle one, and fuse into the rightmost kink soliton, as shown in Figs. 4(a)–4(d); The other is, while setting $\zeta_{10}>\zeta_{30}>\zeta_{20}$, as shown in Figs. 4(e)–4(h), we see only one lump arises from the leftmost kink soliton, with its shape varying when it goes through the middle kink soliton. And later, it shows a two-lumps outline (which means that there is another lump arising from the middle kink soliton). Subsequently, the two lumps fuse into the rightmost kink soliton.

In the same way, with different choices of ζ_{j0}, we will present two interactions between line rogue wave and three kink solitons by setting $a^2-3b^2=0$ on the x–z plane in Fig. 5. With $\zeta_{10}>\zeta_{20}>\zeta_{30}$, it can be seen that one line rogue wave rises up in the segment between the leftmost and rightmost kink solitons, crosses the middle one, and arranges on a line nearly; and soon it reaches higher amplitude at $t=0$, in the meantime, these two segments of the rogue wave connect together; and then the two segments retreat back with amplitudes decreased, and finally disappear. While by setting $\zeta_{10}>\zeta_{30}>\zeta_{20}$, the rogue wave behaves almost the same as the former case except for that one segment of the rogue wave increases its amplitude after retreating back, as shown in Fig. 5(g). Thus, the dynamics of these two interactions between the rogue wave and kinks are nearly the same, different from those in Fig. 4. Besides, it is noteworthy that the second-order rational solutions do not generate two-line rogue waves (arranging on two lines) or a second-order rogue wave (with three wave arms). Hence, in this paper, we call such localized line rogue waves as one rogue wave, distinctly different from those reported in Refs. [23,26].

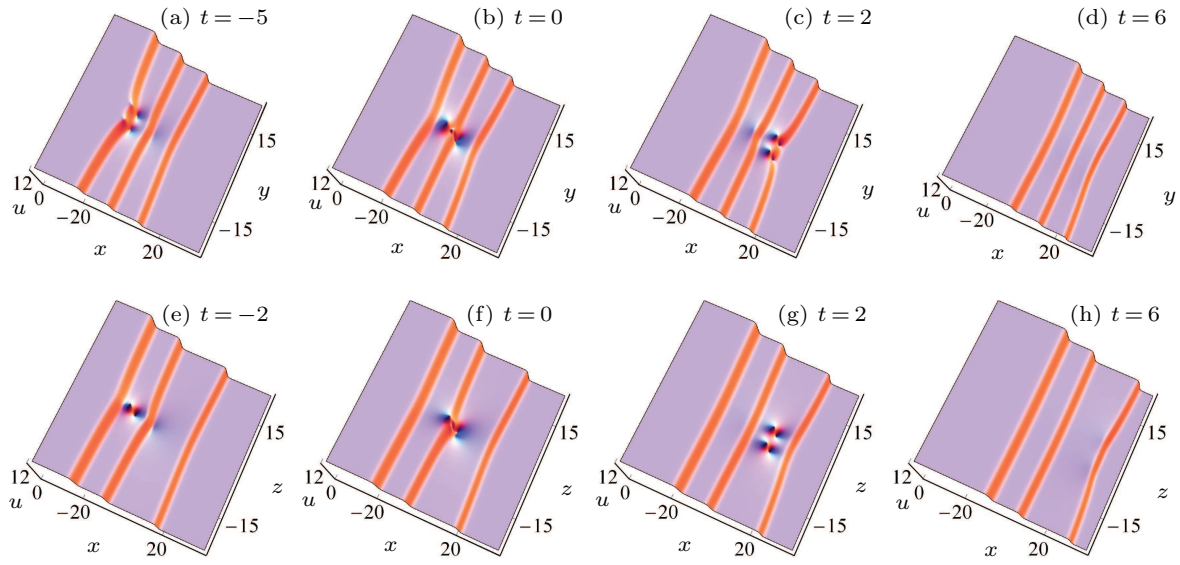


Fig. 4. Interactions between two lumps and three kinks via solutions (13) with $a=1$, $b=0$. (a)–(d) $\zeta_{10}=6$, $\zeta_{20}=0$. and $\zeta_{30}=-6$; (e)–(h): $\zeta_{10}=6\pi$, $\zeta_{20}=-6\pi$, and $\zeta_{30}=0$.

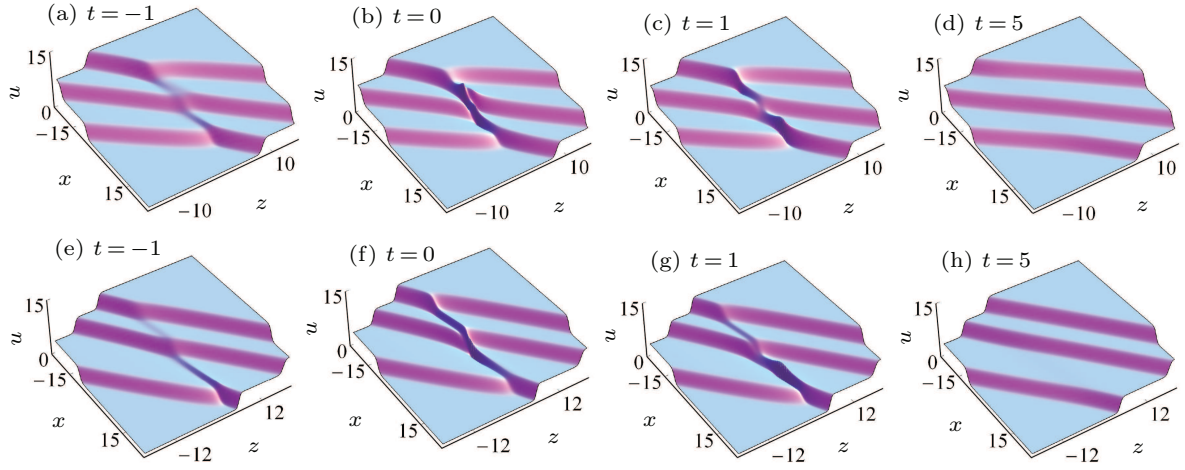


Fig. 5. Interactions between one rogue wave and three kinks via solutions (13) with $y=1$, $a=1$, $b=-1/\sqrt{3}$, and (a)–(d): $\zeta_{10}=5$, $\zeta_{20}=0$ and $\zeta_{30}=-5$; (e)–(h): $\zeta_{10}=4\pi$, $\zeta_{20}=-4\pi$, and $\zeta_{30}=0$.

4. Conclusions

In this paper, we investigated certain rogue waves of the $(3+1)$-dimensional BKP equation. Via the KP hierarchy reduction, semi-rational solutions in the determinant form of Eq. (1) have been constructed, which generate two novel nonlinear wave behaviors: (i) the interaction between one lump and two kink solitons with the lump developing from one kink soliton and then fusing into the other one; (ii) the interaction between one line rogue wave and two kink solitons on the x–z plane with the line rogue wave growing from and then decaying into the constant background between the segment of the two solitons. We have found that evolutions of the lump and line rogue wave are localized both in time and space, which means that such lump and line rogue wave play as a rogue wave. In addition, we have also discussed the second-order semi-rational solutions graphicly, which describe two types of interactions between two lumps (one line rogue wave) and three kink solitons.

Acknowledgements

Project supported by the Fundamental Research Funds for the Central Universities (Grant Nos. 2021XJLX01 and BLX201927), China Post-doctoral Science Foundation (Grant No. 2019M660491), and the Natural Science Foundation of Hebei Province, China (Grant No. A2021502003).

References

[1] Ablowitz M J, Kaup D J, Newell A C and Segur H 1973 *Phys. Rev. Lett.* **31** 125
[2] Zhang H S, Wang L, Sun W R, Wang X and Xu T 2021 *Physica D* **419** 132849
[3] Fang Y, Wu G Z, Wang Y Y and Dai C Q 2021 *Nonlinear Dyn.* **105** 603
[4] Xu T, Li L, Li M, Li C and Zhang X 2021 *Proc. R. Soc. A* **477** 2254
[5] Yang B and Yang J 2021 *IMA J. Appl. Math.* **86** 378
[6] Lü X and Chen S J 2021 *Nonlinear Dyn.* **103** 947

[7] Wang B H, Wang Y Y, Dai C Q and Chen Y X 2020 *Alex. Eng. J.* **59** 4699
[8] Yan Y Y and Liu W J 2021 *Chin. Phys. Lett.* **38** 094201
[9] Dai C Q and Zhang J F 2020 *Nonlinear Dyn.* **100** 1621
[10] Yang Y X, Yao S K, Zhang C H and Zhao W Z 2015 *Chin. Phys. Lett.* **32** 40202
[11] He X J and Lü X 2022 *Math. Comput. Simulat.* **197** 327
[12] Lü X and Chen S J 2021 *Commun. Nonlinear Sci.* **103** 105939
[13] Zhang H and Liu D Y 2020 *Appl. Math. Lett.* **102** 106102
[14] Chen S J, Lü X, Li M G and Wang F 2021 *Phys. Scripta* **96** 095201
[15] Liu Y K and Li B 2017 *Chin. Phys. Lett.* **34** 10202
[16] Ma W X and Fan E G 2011 *Comput. Math. Appl.* **61** 950
[17] Ma W X and Zhu Z 2012 *Appl. Math. Comput.* **218** 11871
[18] Ma W X, Qin Z and Lü X 2016 *Nonlinear Dyn.* **84** 923
[19] Sun Y, Tian B, Liu L, Chai H P and Yuan Y Q 2017 *Commun. Theor. Phys.* **68** 693
[20] Ding C C, Gao Y T and Deng G F 2019 *Nonlinear Dyn.* **97** 2023
[21] Ohta Y and Yang J 2012 *Phys. Rev. E* **86** 036604
[22] Chen J, Feng B F, Maruno K I and Ohta Y 2018 *Stud. Appl. Math.* **141** 145
[23] Rao J, Chow K W, Mihalache D and He J 2021 *Stud. Appl. Math.* **147** 1007
[24] Lü X and Chen S J 2021 *Commun. Nonlinear Sci.* **103** 105939
[25] Rao J, He J and Mihalache D 2021 *Appl. Math. Lett.* **121** 107435
[26] Rao J, Fokas A S and He J 2021 *J. Nonlinear Sci.* **31** 1
[27] Lü X and Chen S J 2022 *Commun. Nonlinear Sci.* **109** 106103
[28] Hirota R 2004 *The direct method in soliton theory* (Cambridge University Press) p. 23
[29] Date E, Jimbo M, Kashiwara M and Miwa T 1982 *Publ. Res. I. Math. Sci.* **18** 1077