Reciprocal transformations of the space–time shifted nonlocal short pulse equations

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Reciprocal transformations of the space–time shifted nonlocal short pulse equations are elaborated. Covariance of dependent and independent variables involved in the reciprocal transformations is investigated. Exact solutions of the space–time shifted nonlocal short pulse equations are given in terms of double Wronskians. Realness of independent variables involved in the reciprocal transformations is verified. Dynamics of some obtained solutions are illustrated.

Keywords: reciprocal transformation, space–time shifted nonlocal short pulse equation, covariance of variable, solution

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1. Introduction

Reciprocal transformation is a technique used to transform nonlinear partial differential equations into other partial differential equations. It consists of interchanging the dependent and independent variables. In particular, when the achieved equation is linear, the transformation is known as a hodograph transformation. Reciprocal transformations play useful roles in the research of nonlinear partial differential equations. One apparent advantage of reciprocal transformations is that in many cases these transformations give rise to a procedure of relating allegedly new equations to the rest of their equivalent integrable sisters. Thus, the objective equations can be studied by studying the integrable equations linked via reciprocal transformations. For more details about the introduction of reciprocal transformations and their application examples, one can refer to Ref. [1] and the references therein. Some remarkable integrable equations that are investigated by means of reciprocal transformations are the Camassa–Holm equation,[2] the Vakhnenko equation,[3] the (ultra-)short pulse equation,[4] the constant astigmatism equation,[5] *etc*. The short pulse (SP) equation,

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \quad x,t \in \mathbb{R}, \tag{1}$$

provides a model to describe propagation of ultra-short pulses in nonlinear media with more accurate approximation.[4–6] Before the work,[4] this equation has been derived by Rabelo in 1987 as a pseudospherical-type equation.[7–9] It is integrable, characterized by theWadati–Konno–Ichikawa[10] (WKI) spectral problem,[8,11] and related to the sine-Gordon equation or its non-potential form (known also as finite-amplitude baroclinic wave equations[12] or coupled integrable dispersionless equations[13]) via reciprocal transformations.[11,14,15] The SP equation (1) allows both physical and mathematical two-component extension.[16–19] In this paper, the two-component form of our interest is[17,18]

$$\begin{aligned} &u_{zy} + 2u - (uvu_z)_z = 0, \\ &v_{zy} + 2v - (uvv_z)_z = 0, \quad z,y \in \mathbb{R}, \end{aligned} \tag{2}$$

of which Lax pair is given as (*cf*. Ref. [18])

$$\begin{aligned} &\boldsymbol{\Phi}_z = \lambda \begin{pmatrix} 1 & -u_z \\ v_z & -1 \end{pmatrix} \boldsymbol{\Phi}, \\ &\boldsymbol{\Phi}_y = \begin{pmatrix} \lambda uv - \frac{1}{2\lambda} & -\lambda uvu_z - u \\ \lambda uvv_z - v & -\lambda uv + \frac{1}{2\lambda} \end{pmatrix} \boldsymbol{\Phi}, \\ &\boldsymbol{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \end{aligned} \tag{3}$$

where $\lambda_z = \lambda_y = 0$. The system (2) allows various reductions. For example, by $u = \delta v$ it yields the SP equation (1); by $u = \delta v^*$ it yields a complex version of Eq. (1) (*cf*. Ref. [20]). It also allows nonlocal reductions $u(z,y) = \delta v(-z,-y)$ and $u(z,y) = \delta v^*(-z,-y)$, where $\delta = \pm 1$ and $*$ stands for complex conjugate.

Note that nonlocal reduction was first systematically introduced by Ablowitz and Mussilimani in 2013[21] and soon after nonlocal integrable systems received intensive attention from various aspects, *e.g*., Refs. [22–25]. More recently, Ablowitz and Mussilimani introduced space–time shifted nonlocal reductions.[56] For the SP system (2), via two space–time shifted nonlocal reductions

$$v(z,y) = \delta u(z_0 - z, y_0 - y), \quad \delta = \pm 1, \tag{4}$$

$$v(z,y) = \delta u^*(z_0 - z, y_0 - y), \quad \delta = \pm 1, \tag{5}$$

one can, respectively, obtain two nonlocal SP equations with space–time shifts:

$$u_{zy} + 2u - \delta(u\tilde{u}u_z)_z = 0, \quad \delta = \pm 1, \tag{6}$$

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$$u_{zy}+2u-\delta(u\tilde{u}^*u_z)_z=0,\quad \delta=\pm1, \tag{7}$$

where $\tilde{u}=u(z_0-z,y_0-y)$, (z,y) are real coordinates, and z_0,y_0 are real numbers.

Since reciprocal transformations mix dependent and independent variables and to recover the independent variables of the investigated equation one needs to make use of integration, the research will become complicated once a reverse space $-x$ is involved. In a recent paper,[41] covariance of dependent and independent variables in the reciprocal transformation of the nonlocal vector SP system was discussed. It can be imagined that the reciprocal transformation will be more complicated when space–time shifts are involved in nonlocal cases.

In this paper, with Ref. [41] as a pre-research, we will focus on the following: to elaborate the reciprocal transformations and the covariance of variables for the space–time shifted nonlocal SP equations (6) and (7); to present solutions in double Wronskian form for these two equations; and to present conditions that guarantee the realness of (z,y) in the reciprocal transformations.

This paper is organized as follows. In Section 2, we elaborate reciprocal transformation for the space–time shifted nonlocal SP equations and the covariance of variables involved in the transformation. Then in Section 3 we present double Wronskian solutions to the the space–time shifted nonlocal SP equations, and investigate the realness of independent variable z. As examples, dynamics of two obtained solutions are illustrated. Finally, conclusions are given in Section 4.

2. Nonlocal reciprocal transformation and covariance of variables

Let us consider the negative order Ablowitz–Kaup–Newell–Segur (AKNS) equation (AKNS(-1) for short)[57]

$$q_{x,t}-2q\partial_x^{-1}(qr)_t=q,\quad r_{x,t}-2r\partial_x^{-1}(qr)_t=r, \tag{8}$$

where $\partial_x=\frac{\partial}{\partial x}$, $\partial_x^{-1}=\frac{1}{2}(\int_{-\infty}^{x}-\int_{x}^{+\infty})\cdot\mathrm{d}x$. An alternative form of Eq. (8) is ($cf.$ Refs. [12,13])

$$q_{xt}=2q\rho,\quad r_{xt}=2r\rho,\quad \rho_x=(qr)_t. \tag{9}$$

Using the conserved form of the third equation (recalling qr is a mutual conserved quantity of the AKNS hierarchy), one can introduce a reciprocal transformation ($cf.$ Ref. [14])

$$\begin{aligned} &z(x,t)_t=\rho,\quad z(x,t)_x=qr,\\ &y(x,t)_x=-1,\quad y(x,t)_t=0, \end{aligned} \tag{10}$$

and redefine

$$\begin{aligned} &u(z,y)=u(z(x,t),y(x,t))=q(x,t),\\ &v(z,y)=v(z(x,t),y(x,t))=r(x,t). \end{aligned} \tag{11}$$

Then, it can be verified that the first two equations in Eq. (9) are converted to the coupled SP system (2).

The AKNS(-1) system (8) allows two space–time shifted nonlocal reductions,

$$r(x,t)=\delta q(x_0-x,t_0-t),\quad \delta=\pm1, \tag{12}$$

$$r(x,t)=\delta q^*(x_0-x,t_0-t),\quad \delta=\pm1, \tag{13}$$

which yield nonlocal equations with space–time shifts, respectively,

$$q_{xt}-2\delta q\partial_x^{-1}(q\tilde{q})_t=q,\quad \delta=\pm1, \tag{14}$$

$$q_{xt}-2\delta q\partial_x^{-1}(q\tilde{q}^*)_t=q,\quad \delta=\pm1, \tag{15}$$

where $\tilde{q}=q(x_0-x,t_0-t)$. Then, for reciprocal transformations that associate Eqs. (14) and (15) with Eqs. (16) and (17), we have the following.

Proposition 1 The reciprocal transformation (10), together with

$$\rho=\partial_x^{-1}(qr)_t+\rho_0,\quad \rho_0=\lim_{|x|\to+\infty}\rho=\frac{1}{2}, \tag{16}$$

converts Eqs. (14) and (15) to Eqs. (6) and (7), respectively.

Proof Let us consider the connection between Eqs. (14) and (6). First, it follows from Eq. (12) that

$$r(x,t)=\delta q(x_0-x,t_0-t),$$

and then from Eq. (16) we can verify that

$$\rho(x,t)=\rho(x_0-x,t_0-t).$$

Then, the reciprocal transformation (10) together with Eq. (16) defines

$$z(x,t)=\partial_x^{-1}(qr)+\frac{1}{2}t+z_1,\quad y(x,t)=-x+y_1, \tag{17}$$

where constants z_1 and y_1 are taken as

$$z_1=\partial_x^{-1}(qr)|_{(x,t)=(x_0,t_0)},\quad y_1=0. \tag{18}$$

Define

$$z_0=z(x_0,t_0)=2z_1+\frac{1}{2}t_0,\quad y_0=y(x_0,t_0)=-x_0.$$

Then we have

$$\begin{aligned} &z(x_0-x,t_0-t)\\ &=\frac{1}{2}\Big(\int_{-\infty}^{x_0-x}-\int_{x_0-x}^{+\infty}\Big)\Big[q(x,t_0-t)r(x,t_0-t)\Big]\mathrm{d}x\\ &\quad+\frac{1}{2}(t_0-t)+z_1\\ &=-\frac{1}{2}\Big(\int_{+\infty}^{x}-\int_{x}^{-\infty}\Big)\Big[q(x_0'-x',t_0-t)r(x_0'-x',t_0-t)\Big]\mathrm{d}x'\\ &\quad-\frac{1}{2}t+\frac{1}{2}t_0+z_1\\ &=-\frac{1}{2}\Big(\int_{-\infty}^{x}-\int_{x}^{+\infty}\Big)\Big[r(x',t)q(x',t)\Big]\mathrm{d}x'-\frac{1}{2}t+\frac{1}{2}t_0+z_1 \end{aligned}$$

$$= -\partial_x^{-1}(qr) - \frac{1}{2}t + \frac{1}{2}t_0 + z_1 = -z(x,t) + z_0, \tag{19a}$$

$$y(x_0 - x, t_0 - t)$$

$$= x - x_0 = -y - x_0 = -y(x,t) + y_0. \tag{19b}$$

With the above covariance, when q and r obey the relation (12), we have

$$\begin{aligned} v(z,y) = r(x,t) &= \delta q(x_0 - x, t_0 - t) \\ &= \delta u(z(x_0 - x, t_0 - t), y(x_0 - x, t_0 - t) \\ &= \delta u(z_0 - z, y_0 - y). \end{aligned}$$

In other words, the space–time shifted nonlocal reduction (4) holds, which means the reciprocal transformation (10) together with Eq. (16) does convert the space–time shifted nonlocal AKNS(-1) equation (14) into the space–time shifted nonlocal SP equation (6).

For the complex reduction (13), one has

$$r(x,t) = \delta q^*(x_0 - x, t_0 - t), \quad \rho(x,t) = \rho^*(x_0 - x, t_0 - t)$$

and the covariant correspondence (19) as well, from which the relation (5) holds and equation (15) is converted to Eq. (7).

Note that z and y defined by Eq. (17) must be real. This will be elaborated in the next section after we present explicit solutions of the nonlocal equations (14) and (15).

3. Solutions

In this section, we first begin with by presenting double Wronskian solutions for the unreduced AKNS(-1) system (8) as well as for the coupled SP system (2). Next, by means of a reduction technique, solutions of the reduced equations (14) and (15) will be derived. After that, we will check the realness of the independent variable $z(x,t)$ defined in Eq. (17). Then, solutions to the reduced SP equations (6) and (7) will be achieved via Eqs. (11) and (17). Finally, as examples, we will also give explicit one-soliton solution and breather solution with illustrations for the space–time shifted nonlocal SP equation (6).

3.1. Solutions to the unreduced AKNS(-1) system (8) and SP system (2)

The AKNS(-1) system (8) admits bilinear forms[57]

$$D_x^2 f \cdot f = 2gh, \quad D_x D_t g \cdot f = gf, \quad D_x D_t h \cdot f = hf, \tag{20}$$

via transformation

$$q = -\frac{g}{f}, \quad r = \frac{h}{f}, \tag{21}$$

and the double Wronskian solutions to the bilinear equation (20) are[58]

$$f = |\widehat{\varphi^{[N]}}; \widehat{\psi^{[M]}}|, \quad g = 2|\widehat{\varphi^{[N+1]}}; \widehat{\psi^{[M-1]}}|,$$

$$h = 2|\widehat{\varphi^{[N-1]}}; \widehat{\psi^{[M+1]}}|, \tag{22}$$

where φ and ψ are the following $(N+M+2)$-th order column vectors

$$\begin{aligned} \varphi &= \exp\left(-Ax - \frac{1}{4}A^{-1}t\right)C^+, \\ \psi &= \exp\left(Ax + \frac{1}{4}A^{-1}t\right)C^-, \end{aligned} \tag{23}$$

$A \in \mathbb{C}_{(N+M+2)\times(N+M+2)}$, $C^{\pm}$ are $(N+M+2)$-th order constant complex column vectors, and $\widehat{\varphi^{[N]}}$ denotes the consecutive columns

$$\widehat{\varphi^{[N]}} = (\varphi, \partial_x\varphi, \partial_x^2\varphi, \ldots, \partial_x^N\varphi). \tag{24}$$

Here D is the Hirota bilinear operator defined as[59]

$$\begin{aligned} &D_x^m D_y^n f(x,y) \cdot g(x,y) \\ &= (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n f(x,y) g(x',y')|_{x'=x, y=y'}. \end{aligned}$$

It is notable that matrix A and any matrix similar to it lead to the same q and r. Then, with the transformation (21) and Wronskians (22), and in light of the reciprocal transformations between the AKNS(-1) system (8) and the coupled SP system (2) that we have described in the previous section, one can find that the unreduced SP system (2) admits solutions

$$\begin{aligned} u = q(x,t) &= -2\frac{|\widehat{\varphi^{[N+1]}}; \widehat{\psi^{[M-1]}}|}{|\widehat{\varphi^{[N]}}; \widehat{\psi^{[M]}}|}, \\ v = r(x,t) &= 2\frac{|\widehat{\varphi^{[N-1]}}; \widehat{\psi^{[M+1]}}|}{|\widehat{\varphi^{[N]}}; \widehat{\psi^{[M]}}|}, \end{aligned} \tag{25a}$$

$$\begin{aligned} z(x,t) &= -\frac{f_x}{f} + \frac{1}{2}t + z_1 = -\frac{\partial_x|\widehat{\varphi^{[N]}}; \widehat{\psi^{[M]}}|}{|\widehat{\varphi^{[N]}}; \widehat{\psi^{[M]}}|} + \frac{1}{2}t + z_1, \\ y(x,t) &= -x. \end{aligned} \tag{25b}$$

3.2. Solutions to the reduced SP equations (6) and (7)

Note that the reduction technique based on bilinear form and double Wronskians was first developed in Refs. [32,33] and has proved effective in deriving solutions for nonlocal integrable equations,[34,41,42,51,60–63] including the discrete[34] and space–time shifted nonlocal cases[63] as well. The main idea of this technique is to impose certain constraints on matrix A and column vector ψ in double Wronskian solution of the unreduced system, such that the defined functions q and r (or u and v) satisfy desired reductions. In the following, let us directly present solutions for the reduced equations.

Theorem 1 Assume

$$\varphi = \exp\left(-Ax - \frac{1}{4}A^{-1}t\right)C^+, \tag{26}$$

where $A \in \mathbb{C}_{2(N+1)\times 2(N+1)}$, $C^+ \in \mathbb{C}_{2(N+1)}$, and define ψ by, respectively,

$$\psi(x,t) = T\varphi(x_0 - x, t_0 - t), \tag{27a}$$

$$AT - TA = 0, \quad T^2 = -\delta I, \tag{27b}$$

and

$$\psi(x,t) = T\varphi^*(x_0 - x, t_0 - t), \tag{28a}$$
$$AT - TA^* = 0, \quad TT^* = -\delta I, \tag{28b}$$

where $T \in \mathbb{C}_{2(N+1)\times 2(N+1)}$ and I is the identity matrix of order $2N+2$. Then,

$$q(x,t) = -2\frac{|\widehat{\varphi^{[N+1]}}; \widehat{\psi^{[N-1]}}|}{|\widehat{\varphi^{[N]}}; \widehat{\psi^{[N]}}|} \tag{29}$$

solves the space–time shifted nonlocal AKNS(-1) equations (14) and (15), respectively; and

$$u(z,y) = q(x,t) = -2\frac{|\widehat{\varphi^{[N+1]}}; \widehat{\psi^{[N-1]}}|}{|\widehat{\varphi^{[N]}}; \widehat{\psi^{[N]}}|} \tag{30a}$$

with

$$z(x,t) = -\frac{\partial_x|\widehat{\varphi^{[N]}}; \widehat{\psi^{[N]}}|}{|\widehat{\varphi^{[N]}}; \widehat{\psi^{[N]}}|} + \frac{1}{2}t + z_1, \quad y(x,t) = -x, \tag{30b}$$

if $z(x,t)$ is real, solves the space–time shifted nonlocal SP equations (6) and (7), respectively.

We skip details of the proof. In fact, when A and T obey Eq. (27b) and taking $C^- = T\exp(-Ax_0 - \frac{1}{4}A^{-1}t_0)C^+$, the vector ψ defined by Eq. (27) takes the form as given in Eq. (23) with $M = N$. In addition, under the constraint Eq. (27) one can find

$$f(x_0 - x, t_0 - t) = \delta^{N+1}|T|f(x,t),$$
$$g(x_0 - x, t_0 - t) = -\delta^N|T|h(x,t),$$

which indicates the reduction (12) holds. For the constraint (28), one can find (with $C^- = T\exp(-A^*x_0 - \frac{1}{4}A^{*^{-1}}t_0)C^{+^*}$)

$$f^*(x_0 - x, t_0 - t) = \delta^{N+1}|T^*|f(x,t),$$
$$g^*(x_0 - x, t_0 - t) = -\delta^N|T^*|h(x,t),$$

which indicates the reduction (13) holds. One can refer to Ref. [63] which contains similar details.

Note that nonlocal equations with shifted space and time have also been studied based on Hirota's bilinear forms and low order soliton (*e.g.*, 1-soliton and 2-soliton) solutions can be explicitly presented in terms of exponential polynomials.[64]

3.3. Explicit form of solutions and realness of $z(x,t)$

For the sake of consistency of reduction, the independent variable z defined in Eq. (25b) must be real. In the following, we present explicit form of φ and ψ with which we can clarify the realness of z.

Following the treatment in Ref. [32], we consider T and A of the following block matrix form

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \quad A = \begin{pmatrix} K_1 & \mathbf{0} \\ \mathbf{0} & K_4 \end{pmatrix}, \tag{31}$$

and list out solutions of Eqs. (27b) and (28b) in Table 1, where T_i and K_i are $(N+1)\times(N+1)$ matrices, $\boldsymbol{I}_{N+1}$ is the identity matrix of order $N+1$ and $\mathrm{i}^2 = -1$.

Table 1. T and A for constraints (27b) and (28b).

	δ	T	A
(27b)	-1	$T_1 = -T_4 = \boldsymbol{I}_{N+1}, T_3 = T_2 = \boldsymbol{0}_{N+1}$	$K_1 = \boldsymbol{K}_{N+1}, K_4 = \boldsymbol{H}_{N+1}$
	1	$T_1 = -T_4 = \mathrm{i}\boldsymbol{I}_{N+1}, T_3 = T_2 = \boldsymbol{0}_{N+1}$	
(28b)	-1	$T_1 = T_4 = \boldsymbol{0}_{N+1}, T_3 = T_2 = \boldsymbol{I}_{N+1}$	$K_1 = K_4^* = \boldsymbol{K}_{N+1}$
		$T_1 = -T_4 = \boldsymbol{I}_{N+1}, T_3 = T_2 = \boldsymbol{0}_{N+1}$	$K_1 = K_1^* = \boldsymbol{K}_{N+1}, K_4 = K_4^* = \boldsymbol{H}_{N+1}$
	1	$T_1 = T_4 = \boldsymbol{0}_{N+1}, T_3 = -T_2 = \boldsymbol{I}_{N+1}$	$K_1 = K_4^* = \boldsymbol{K}_{N+1}$

For convenience, we introduce notations

$$\mathrm{Diag}[k_i]_{i=1}^{N+1} \doteq \mathrm{Diag}(k_1, k_2, \ldots, k_{N+1}),$$
$$\boldsymbol{J}_{N+1}[k] \doteq \begin{pmatrix} k & 0 & \cdots & 0 \\ 1 & k & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & k \end{pmatrix}_{(N+1)\times(N+1)}.$$

The realness of z defined by Eq. (25b) can be guaranteed by $f = \varepsilon f^*$, where $\varepsilon = \pm 1$.

Proposition 2 For both cases (27) and (28), z is real when

(i) $\boldsymbol{K}_{N+1}$ and $\boldsymbol{H}_{N+1}$ are real and therefore φ can be a real vector.

(ii) In the case $\boldsymbol{K}_{N+1}$ and $\boldsymbol{H}_{N+1}$ are complex, we take $\boldsymbol{H}_{N+1} = \boldsymbol{K}_{N+1}^*$, and thus we have the following subcases: when $\boldsymbol{K}_{N+1} = \mathrm{Diag}[k_i]_{i=1}^{N+1}$, φ is given as

$$\varphi = \left(c_1\mathrm{e}^{\eta(k_1)}, c_2\mathrm{e}^{\eta(k_2)}, \cdots, c_{N+1}\mathrm{e}^{\eta(k_{N+1})}, c_1^*\mathrm{e}^{\eta(k_1^*)}, c_2^*\mathrm{e}^{\eta(k_2^*)}, \cdots, c_{N+1}^*\mathrm{e}^{\eta(k_{N+1}^*)}\right)^{\mathrm{T}}, \tag{32}$$

where $c_i, k_i \in \mathbb{C}$, $\eta(k) = -kx - t/4k$; when $\boldsymbol{K}_{N+1} = \boldsymbol{J}_{N+1}(k)$, φ takes the form

$$\varphi = \left(c\,\mathrm{e}^{\eta(k)}, \frac{\partial_k}{1!}(c\,\mathrm{e}^{\eta(k)}), \cdots, \frac{\partial_k^N}{N!}(c\,\mathrm{e}^{\eta(k)}), c^*\,\mathrm{e}^{\eta(k^*)}, \frac{\partial_{k^*}}{1!}(c^*\,\mathrm{e}^{\eta(k^*)}), \cdots, \frac{\partial_{k^*}^N}{N!}(c^*\,\mathrm{e}^{\eta(k^*)})\right)^{\mathrm{T}}, \tag{33}$$

where $c, k \in \mathbb{C}$.

Note that there can be more choices for case (27) to have a real z. For example, one may take $A = \mathrm{Diag}(k_1, k_2, h_1, h_2)$ where $k_1, k_2 \in \mathbb{R}$ and $h_1 = h_2^* \in \mathbb{C}$.

3.4. Examples

As an example we investigate dynamics of the space–time shifted nonlocal SP equation (6) with $\delta = -1$, *i.e.*,

$$u_{zy} + 2u + (u\tilde{u}u_z)_z = 0, \quad \tilde{u} = u(z_0 - z, y_0 - y). \tag{34}$$

From Eq. (30), taking $N = 0$, one obtains 1-soliton solution in parametric form of equation (34),

$$\begin{aligned}
u &= \frac{2(k_1 - h_1)\exp\left[-(k_1 + h_1)x - \dfrac{k_1 + h_1}{4k_1h_1}t\right]}{\exp\left[-(k_1 - h_1)x + \dfrac{k_1 - h_1}{4k_1h_1}t - h_1x_0 - \dfrac{t_0}{4h_1}\right] + \exp\left[(k_1 - h_1)x - \dfrac{k_1 - h_1}{4k_1h_1}t - k_1x_0 - \dfrac{t_0}{4k_1}\right]}, \\
z &= \frac{(k_1 - h_1)\left(\exp\left[-(k_1 - h_1)x + \dfrac{k_1 - h_1}{4k_1h_1}t - h_1x_0 - \dfrac{t_0}{4h_1}\right] - \exp\left[(k_1 - h_1)x - \dfrac{k_1 - h_1}{4k_1h_1}t - k_1x_0 - \dfrac{t_0}{4k_1}\right]\right)}{\exp\left[-(k_1 - h_1)x + \dfrac{k_1 - h_1}{4k_1h_1}t - h_1x_0 - \dfrac{t_0}{4h_1}\right] + \exp\left[(k_1 - h_1)x - \dfrac{k_1 - h_1}{4k_1h_1}t - k_1x_0 - \dfrac{t_0}{4k_1}\right]} + \frac{t}{2} + z_1, \\
y &= -x,
\end{aligned} \tag{35}$$

where $k_1,\ h_1 \in \mathbb{R}$ or $k_1 = h_1^* \in \mathbb{C}$. Consider a special case where taking $k_1 = -h_1 \in \mathbb{R}$, and then we have

$$\begin{aligned}
&u = 2k\ \mathrm{sech}\ (\xi_1 + \delta_1), \\
&z = 2k\tanh(\xi_1 + \delta_1) - 2k_1^2x - k_1\xi_1 + z_1, \quad y = -x,
\end{aligned} \tag{36}$$

where $\xi_1 = -2k_1x - t/2k_1$, $\delta_1 = k_1x_0 + t_0/4k_1$, $z_1 = 2k_1\tanh(-k_1x_0 - t_0/4k_1)$, and both u and z are presented by the parameter ξ_1. As illustrated in Fig. 1, equation (36) describes a 1-loop soliton or 1-antiloop soliton (by taking k to be positive or negative) with amplitude $2|k|$ which appears at $\xi_1 = -\delta_1$. In $\{z, y\}$ coordinates, the soliton propagates along $z = 2k_1^2y + \frac{1}{2}z_0 - k_1^2y_0$ with speed $2k_1^2$. One can also see that the space–time shifted parameters z_0 and y_0 only affect initial phase of the soliton.

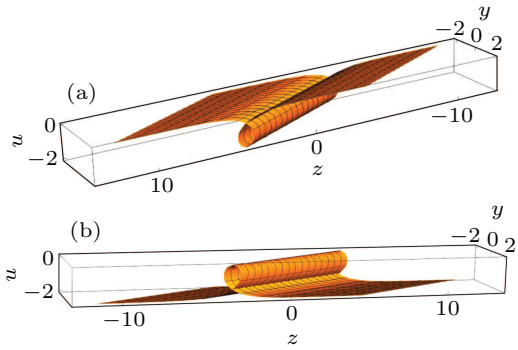


Fig. 1. (a) Profile of 1-antiloop soliton solution u given by Eq. (36) for $k_1 = -1, x_0 = -1, t_0 = -1$. (b) Profile of 1-loop soliton solution u given by Eq. (36) for $k_1 = 1, x_0 = -1, t_0 = -1$.

Next, let us consider breather solution obtained by taking $N = 1$ in Eq. (30) with $h_1 = -k_1$, $h_2 = -k_2$, $k_1 = a + \mathrm{i}b$, $k_2 = a - \mathrm{i}b$, $a, b \in \mathbb{R}$, where the explicit forms of g and f are

$$\begin{aligned}
g &= 64c_1c_2d_1d_2ab\left(-a\sin\gamma_2\sinh\gamma_1 + b\cos\gamma_2\cosh\gamma_1\right), \\
f &= -16c_1c_2d_1d_2\left(b^2\cosh^2\gamma_1 + a^2\sin^2\gamma_2\right),
\end{aligned}$$

with

$$\begin{aligned}
&\theta_1 = a(2x + \frac{t}{2(a^2+b^2)}), \quad \theta_2 = b(2x - \frac{t}{2(a^2+b^2)}), \\
&\sigma_1 = ax_0 + \frac{at_0}{4(a^2+b^2)}, \quad \sigma_2 = bx_0 - \frac{bt_0}{4(a^2+b^2)}, \\
&\gamma_i = \theta_i - \sigma_i, \quad c_i, d_i \in \mathbb{C}, \quad i = 1, 2.
\end{aligned}$$

Then u, z, y are given as

$$\begin{aligned}
u &= \frac{4ab\left(-a\sin\gamma_2\sinh\gamma_1 + b\cos\gamma_2\cosh\gamma_1\right)}{b^2\cosh^2\gamma_1 + a^2\sin^2\gamma_2}, \\
z &= -\frac{4ab^2\cosh\gamma_1\sinh\gamma_1 + 4a^2b\sin\gamma_2\cos\gamma_2}{b^2\cosh^2\gamma_1 + a^2\sin^2\gamma_2} \\
&\quad + (a^2+b^2)\left(\frac{\theta_1}{a} - \frac{b\theta_1 + a\theta_2}{2ab}\right) + z_1, \\
y &= -\frac{b\theta_1 + a\theta_2}{4ab}.
\end{aligned} \tag{37}$$

One can see that u contains two different phases θ_1 and θ_2, of which the former governs the envelope of the breather while the latter characterizes the internal oscillation, as depicted in Fig. 2.

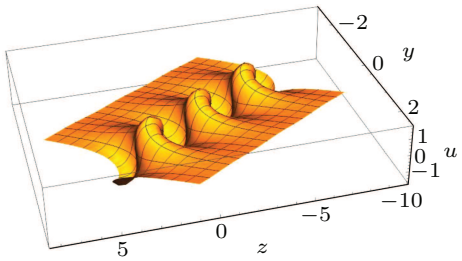


Fig. 2. Profile of 1-breather solution u given in Eq. (37) for $a = 0.4$, $b = -0.8$, $x_0 = 1$, $t_0 = 1$.

4. Conclusions

In this paper we have elaborated the reciprocal transformations that convert the space–time shifted AKNS(-1) equations (14) and (15) to the space–time shifted nonlocal SP equations (6) and (7), explained covariance of variables, presented double Wronskian solutions, clarified realness of the independent variable z, and illustrated some obtained solutions. Because the nonlocal integrable systems introduced by Ablowitz and Musslimani[22] have drawn intensive attention, the SP-type equations are a special type of integrable systems, and in nonlocal case reciprocal transformations become more complicated, thus the investigation of this paper is significant.

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