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Analytical formula describing the non-saturating linear magnetoresistance in inhomogeneous conductors

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The effective-medium theory (EMT) has proved successful in modeling the non-saturating linear magnetoresistance induced by inhomogeneity. However, calculating magnetoresistance using the EMT usually involves solving coupled integral equations which have no analytical solutions, and therefore, it is still difficult to directly compare the predictions of EMT with experimental data. Here we demonstrate that the linear magnetoresistance predicted by the EMT can be either exactly formulated or well approximated by a simple analytical equation $\Delta\rho/\rho_0 = \sqrt{k^2B^2+a^2} - a$ in a number of known situations. The relations between the EMT parameters and the phenomenological parameters k and a are evaluated. Our results provide a convenient and effective method for extracting the EMT parameters from experimental data.

Keywords: linear magnetoresistance, effective medium theory

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1. Introduction

The magnetoresistance (MR) of a conventional conductor usually increases quadratically with magnetic field B at low fields and saturates at high fields. However, in the last few decades, a non-saturating linear magnetoresistance (LMR) has been observed in a wide variety of materials, including narrow-gap semiconductors,[1–3] graphene,[4–8] topological insulators,[9,10] topological semimetals,[11–14] and materials with charge-density-wave or spin-density-wave order.[15–19] In contrast to the conventional MR, the LMR observed in these materials exhibits a non-saturating quadratic-to-linear field dependence: As magnetic field B increases, the function form of MR curves gradually changes from $\Delta\rho/\rho_0 \sim B^2$ to $\Delta\rho/\rho_0 \sim B$ and the linear B dependence keeps up to very high fields.

Several theories have been developed to explain the non-saturating LMR. These theories can be generally classified into two categories. The first category are quantum theories which take account of specific quantum mechanical processes of electrons, such as the condensation of electrons into the lowest Landau level[20,21] or the electron scattering at the hot spots of the Fermi surface due to charge-density-wave fluctuations.[15] The second category includes semiclassical theories which attribute the LMR to inhomogeneous distributions of carrier density or mobility in the sample.[22–27] Although quantum LMR has been experimentally reported in a few material systems,[3,5,15,16] the LMR observed in most materials were considered to be disorder-induced LMR which can be well understood within the framework of semiclassical theories.[3,4,6–10,19]

As one of the most successful semiclassical theories of disorder-induced LMR, the effective-medium theory (EMT) was first introduced by Stroud and Pan in 1975 to calculate the effective conductivity of an inhomogeneous system.[25] In 2005, Guttal and Stroud derived the EMT equations for two-dimensional (2D) systems consisting of two different components.[26] When the areal fractions of the two ingredients are equal, the two-componet EMT equations predict a non-saturating LMR very similar to the one observed in silver chalcogenides.[1,2,26] Soon after, the two-component EMT was employed to explain the MR of graphene near the Dirac point and satisfactory results were obtained.[28,29] In 2014, Ping *et al.* derived the EMT equations for disordered systems with a continuous distribution of carrier density n.[6] The predictions of the continuous EMT were found to be quantitatively consistent with the experimental data obtained in graphene.[6]

Although the EMT has proved successful in modeling the LMR of disordered systems, in practice, it is still difficult to directly compare the predictions of the EMT with experimental data, because calculating MR curves using the continuous EMT requires solving coupled integral equations which have no analytical solutions. In this paper, we propose that the LMR curves predicted by the EMT can be either exactly expressed or well approximated by an unified analytical formula

$$\frac{\Delta\rho}{\rho_0}(B) = \sqrt{k^2B^2+a^2} - a, \tag{1}$$

where k and a are two phenomenological parameters usually determined by curve fitting. The relations between the EMT parameters and the phenomenological parameters k and a are evaluated. Our results provide a convenient and effective tool

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for analyzing the LMR data with the EMT.

2. Theoretical methods

2.1. Effective medium theory

The Drude resistivity tensor $\hat{\rho}(\boldsymbol{r})$ of an inhomogeneous conductor varies with position $\boldsymbol{r}$ and thus cannot be measured experimentally by conventional methods. In this case, the results of resistivity measurements actually correspond to the elements of the effective resistivity tensor $\hat{\rho}_{\rm eff}$ which is defined by $\langle \boldsymbol{E} \rangle = \hat{\rho}_{\rm eff} \langle \boldsymbol{J} \rangle$, where $\boldsymbol{E}$ and $\boldsymbol{J}$ are electric field and current density, respectively. Here the angle brackets denote volume average, the same hereinafter. Similarly, the effective conductivity tensor $\hat{\sigma}_{\rm eff}$ is given by $\hat{\sigma}_{\rm eff} = \hat{\rho}_{\rm eff}^{-1}$, which follows $\langle \boldsymbol{J} \rangle = \hat{\sigma}_{\rm eff} \langle \boldsymbol{E} \rangle$.

The aim of the EMT is to calculate $\hat{\sigma}_{\rm eff}$ and $\hat{\rho}_{\rm eff}$ by the mothed of mean-field approximation. In the approach of the EMT, an inhomogeneous system is considered to be locally homogenous, and each local region is treated as a circular area embedded in an effective medium with conductivity $\hat{\sigma}_{\rm eff}$, which can be calculated self-consistently using the EMT equation.[25,26] The EMT equations for three different types of inhomogeneous 2D systems are listed in the following.

For a system consisting of w types of discrete components, the EMT equation reads[25,26]

$$\sum_{i=1}^{w} p_i (\hat{\sigma}_i - \hat{\sigma}_{\rm eff}) \left[\hat{I} + \frac{(\hat{\sigma}_i - \hat{\sigma}_{\rm eff})}{2\sigma_{{\rm eff},xx}} \right]^{-1} = 0, \tag{2}$$

where $\hat{I}$ is the 2×2 identity matrix, and $\hat{\sigma}_i$ and p_i are the Drude conductivity tensor and areal fraction of the i-th component, respectively.

For a system with a continuous distribution of carrier density n, the EMT equation is given as[6,27]

$$\int {\rm d}n P_n(n) \left[\hat{\sigma}(n) - \hat{\sigma}_{\rm eff} \right] \left[\hat{I} + \frac{(\hat{\sigma}(n) - \hat{\sigma}_{\rm eff})}{2\sigma_{{\rm eff},xx}} \right]^{-1} = 0, \tag{3}$$

where P_n is the distribution function of n.

Similarly, for a 2D system with a continuous distribution of mobility μ, the EMT equation is written as

$$\int {\rm d}\mu P_\mu(\mu) \left[\hat{\sigma}(\mu) - \hat{\sigma}_{\rm eff} \right] \left[\hat{I} + \frac{(\hat{\sigma}(\mu) - \hat{\sigma}_{\rm eff})}{2\sigma_{{\rm eff},xx}} \right]^{-1} = 0, \tag{4}$$

where P_μ is the distribution functions of μ.

2.2. Calculation of magnetoresistance

In the presence of a perpendicular field B, the elements of Drude conductivity tenser are given as

$$\sigma_{xx} = \sigma_{yy} = \frac{e|n|\mu}{1+(\mu B)^2} \quad \text{and} \quad \sigma_{xy} = -\sigma_{yx} = \frac{en\mu^2 B}{1+(\mu B)^2}. \tag{5}$$

Here we assume that μ takes only positive values and the type of carriers is determined by the sign of n. It is noteworthy that equation (5) needs to be modified accordingly in the case where n is chosen to be positive and the carrier type is specified by the sign of μ. With the $\hat{\sigma}(B)$ given in Eq. (5), $\hat{\sigma}_{\rm eff}(B)$ can be calculated by solving Eq. (2), Eq. (3) or Eq. (4), which subsequently gives $\hat{\rho}_{\rm eff}(B)$ through $\hat{\rho}_{\rm eff} = \hat{\sigma}_{\rm eff}^{-1}$.

Finally, the MR is calculated using

$$\frac{\Delta\rho}{\rho_0}(B) = \frac{\rho_{{\rm eff},xx}(B) - \rho_{{\rm eff},xx}(0)}{\rho_{{\rm eff},xx}(0)}. \tag{6}$$

2.3. Empirical equation for linear magnetoresistance

Equation (1) is an empirical equation for the LMR. It describes LMR curves which have a quadratic B dependence near zero field and a linear B dependence in high fields.

At the low-field limit ($B \to 0$), equation (1) reduces to $\Delta\rho/\rho \approx (k^2/2a)B^2$, while it approaches $\Delta\rho/\rho \approx kB - a$ at the high-field limit ($B \to \infty$). Therefore, the low-field quadratic coefficient $(k^2/2a)$ and the high-field slope k of LMR curves can be easily extracted by curve fitting using Eq. (1). In the following section, we will show that the LMR curves calculated from the EMT equations can be either exactly formulated or well fitted using the empirical equation given by Eq. (1). This finding will greatly facilitate the data analysis process of disorder-induced LMR.

3. Results and discussion

3.1. Systems consisting of two equally distributed components

As discussed in Ref. [26], a 2D system consisting of two discrete components exhibits a non-saturating LMR only when the areal fractions of the two components are equal. In the following, we will prove that the LMR of such a system has exactly the function form of Eq. (1).

With $w = 2$ and $p_1 = p_2$, equation (2) is simplified to

$$\sum_{i=1,2} (\hat{\sigma}_i - \hat{\sigma}_{\rm eff}) \left[\hat{I} + \frac{(\hat{\sigma}_i - \hat{\sigma}_{\rm eff})}{2\sigma_{{\rm eff},xx}} \right]^{-1} = 0. \tag{7}$$

Equation (7) contains two coupled algebraic equations and can be solved analytically. The authors in Ref. [26] gave a solution of Eq. (7) under the condition of $n_1 = -n_2$ and $\mu_1 = \mu_2$, but the full solution was not shown. We have solved Eq. (7) in full and here we present the full solution in the following:

$$\begin{aligned} \sigma_{{\rm eff},xx} &= \frac{\sqrt{\sigma_{1,xx}\sigma_{2,xx}\left[(\sigma_{1,xx}+\sigma_{2,xx})^2 + (\sigma_{1,xy}-\sigma_{2,xy})^2\right]}}{\sigma_{1,xx}+\sigma_{2,xx}}, \\ \sigma_{{\rm eff},xy} &= \frac{\sigma_{2,xx}\sigma_{1,xy} + \sigma_{1,xx}\sigma_{2,xy}}{\sigma_{1,xx}+\sigma_{2,xx}}. \end{aligned} \tag{8}$$

Together with Eq. (5) and $\hat{\rho}_{\rm eff} = \hat{\sigma}_{\rm eff}^{-1}$, we get

$$\begin{aligned} \rho_{{\rm eff},xx} &= \frac{1}{\sqrt{|n_1||n_2|\mu_1\mu_2 e^2}} \sqrt{\left(\frac{(n_1-n_2)\mu_1\mu_2}{|n_1|\mu_1 + |n_2|\mu_2} \right)^2 B^2 + 1}, \\ \rho_{{\rm eff},xy} &= \frac{n_1\mu_1/|n_1| + n_2\mu_2/|n_2|}{|n_1|\mu_1 e + |n_2|\mu_2 e} B. \end{aligned} \tag{9}$$

Here n_i and μ_i denote the carrier density and mobility of the i-th component, respectively.

Finally the MR is calculated to be

$$\frac{\Delta\rho}{\rho_0}(B) = \frac{\rho_{\mathrm{eff},xx}(B)-\rho_{\mathrm{eff},xx}(0)}{\rho_{\mathrm{eff},xx}(0)} = \sqrt{\left(\frac{(n_1-n_2)\mu_1\mu_2}{|n_1|\mu_1+|n_2|\mu_2}\right)^2 B^2+1}-1. \quad (10)$$

Obviously, the LMR given in Eq. (10) has the function form of $\Delta\rho/\rho_0=\sqrt{k^2B^2+a^2}-a$, with phenomenological parameters $k=\dfrac{(n_1-n_2)\mu_1\mu_2}{|n_1|\mu_1+|n_2|\mu_2}$ and $a=1$.

3.2. Systems with Gaussian distribution of n

In 2014, Ping *et al.* proposed a continuous model to explain the LMR in graphene.[6] In this model, the distribution of n is taken to be a Gaussian function $P_n=\mathrm{e}^{-(n-\langle n\rangle)^2/\Delta n^2}$, where $\langle n\rangle$ and Δn denote the average value and standard deviation of n, respectively. In this case, the EMT equation of the system is given as

$$\int \mathrm{d}n\mathrm{e}^{-(n-\langle n\rangle)^2/\Delta n^2}[\hat{\sigma}(n)-\hat{\sigma}_{\mathrm{eff}}] \times\left[\hat{I}+\frac{(\hat{\sigma}(n)-\hat{\sigma}_{\mathrm{eff}})}{2\sigma_{\mathrm{eff},xx}}\right]^{-1}=0. \quad (11)$$

As discussed in Ref. [27], when plotted as a function of the dimensionless magnetic field μB, the LMR curves calculated from Eq. (11) depend only on the relative deviation of carrier density, *i.e.*, $\Delta n/\langle n\rangle$.

As shown in Fig. 1(a), we have numerically calculated the LMR curves using Eq. (11) with various values of $\Delta n/\langle n\rangle$ and fitted them using Eq. (1). The fitting results are reasonably good in the entire range of μB, which indicates that equation (1) works very well as an empirical equation. The obtained parameters k/μ and a are plotted as a function of μB as shown in Fig. 1(b). As $\Delta n/\langle n\rangle$ increases, the high-field slope k/μ first increases rapidly and then gradually saturates at around $\Delta n/\langle n\rangle=6$.

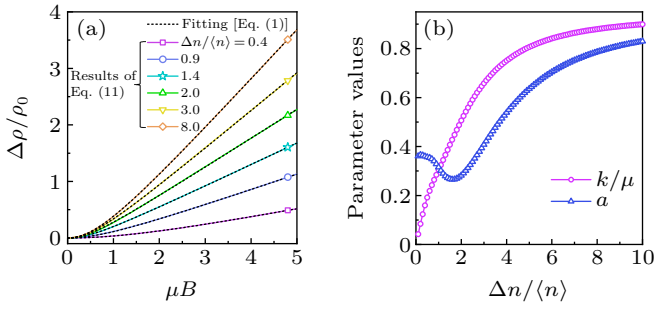


Fig. 1. (a) Normalized magnetoresistance $\Delta\rho/\rho_0$ calculated using Eq. (11) with various $\Delta n/\langle n\rangle$, plotted against the dimensionless magnetic field μB. Dashed lines show the fits to the empirical equation $\Delta\rho/\rho_0=\sqrt{(k/\mu)^2(\mu B)^2+a^2}-a$. (b) Fitting parameters k and a as a function of $\Delta n/\langle n\rangle$, obtained from the empirical fitting shown in panel (a).

3.3. Systems with Gaussian distribution of μ

Non-saturating LMR can also be realized in inhomogeneous systems with a Gaussian distribution of μ when $\langle\mu\rangle=0$, as discussed in Refs. [22,23] using the random-resistor model. Here we reproduce the LMR with the EMT and then demonstrate that the obtained LMR curves can be well fitted using Eq. (1).

With $P_\mu=\mathrm{e}^{-\mu^2/\Delta\mu^2}$, equation (4) becomes

$$\int \mathrm{d}\mu\mathrm{e}^{-\mu^2/\Delta\mu^2}[\hat{\sigma}(\mu)-\hat{\sigma}_{\mathrm{eff}}] \times\left[\hat{I}+\frac{(\hat{\sigma}(\mu)-\hat{\sigma}_{\mathrm{eff}})}{2\sigma_{\mathrm{eff},xx}}\right]^{-1}=0. \quad (12)$$

LMR curves are obtained by numerically solving Eq. (12) with various values of $\Delta\mu$, as plotted in Fig. 2(a). In this case, the obtained LMR curves are independent of n because n is canceled out in the step of calculating $\Delta\rho/\rho_0$ using Eq. (10). Fitting Eq. (1) to the LMR curves gives satisfactory results in the full range of B, proving the validity of Eq. (1) as an empirical formula.

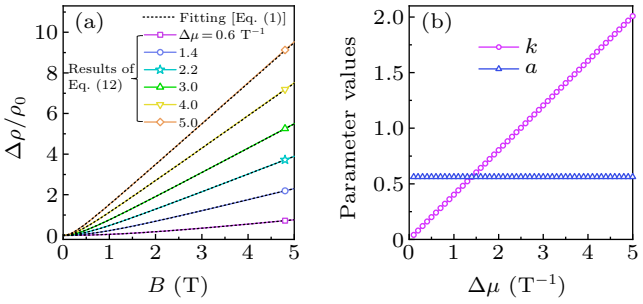


Fig. 2. (a) Normalized magnetoresistance $\Delta\rho/\rho_0$ calculated using Eq. (12) with various $\Delta\mu$. Dashed lines show the fits to the empirical equation $\Delta\rho/\rho_0=\sqrt{k^2B^2+a^2}-a$. (b) Fitting parameters k and a as a function of $\Delta\mu$, obtained from the empirical fitting shown in panel (a).

The fitting parameters k and a were found to follow $k\approx 0.4\Delta\mu$ and $a\approx 0.5626$ in the entire investigated range of $\Delta\mu$, as shown in Fig. 2(b). Such a linear relationship between k and $\Delta\mu$ is consistent with previous results obtained using the random-resistor model.[22,23] In this situation, equation (1) is rewritten as

$$\frac{\Delta\rho}{\rho_0}(B)=\sqrt{0.16\Delta\mu^2B^2+0.3165}-0.5626, \quad (13)$$

equation (13) can be directly used as an empirical equation for analyzing the LMR data of systems with a continuous distribution of μ.

3.4. Further discussions

As shown in Figs. 1(b) and 2(b), the phenomenological parameters k and a take different values for the LMR curves originated from fluctuations of n or μ. Therefore, these two situations can be easily distinguished by fitting Eq. (1) to the LMR data.

4. Conclusion

In conclusion, we have demonstrated that the disorder-induced LMR described by the EMT can be either exactly formulated or well approximated by an analytical formula $\Delta\rho/\rho_0 = \sqrt{k^2B^2 + a^2} - a$ in a couple of known situations. Our results provide a simple and effective approach to extracting the EMT parameters from experimental data.

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