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Dynamics and intermittent stochastic stabilization of a rumor spreading model with guidance mechanism in heterogeneous network

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We propose a novel rumor propagation model with guidance mechanism in heterogeneous complex networks. Firstly, the sharp threshold of rumor propagation, global stability of the information-equilibrium and information-prevailing-equilibrium under $R_0 < 1$ and $R_0 > 1$ is carried out by Lyapunov method and LaSalle's invariant principle. Next, we design an aperiodically intermittent stochastic stabilization method to suppress the rumor propagation. By using the Itô formula and exponential martingale inequality, the expression of the minimum control intensity is calculated. This method can effectively stabilize the rumor propagation by choosing a suitable perturb intensity and a perturb time ratio, while minimizing the control cost. Finally, numerical examples are given to illustrate the analysis and method of the paper.

Keywords: guidance mechanism, heterogenous network, global dynamic analysis, aperiodically intermittent stochastic stabilization

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1. Introduction

Rumor can be defined as a social phenomenon, which shows an unsubstantiated elaboration or interpretation of a thing, event or issue by using various means to attract public interest.[1,2] With the rapid development of internet technology and the widespread use of mobile network social platforms, such as Facebook, Twitter, Weibo and other social APPs, online social networks have replaced traditional communication means and become one of the most important ways to publish and disseminate information, while also contributes to rumor spreading.[3,4] Unfortunately, the prevalence of rumors may damage individual reputation, affect financial market and lead to social panic and instability.[5–7] Therefore, understanding the mechanisms of rumor propagation in complex networks is of great importance to supervise and control the rumor propagation.

Rumor propagation models have evolved from disease spreading models, such as SI, SIS, and SIR. Although the mechanisms of rumor propagation are similar to epidemic spreading, they are also fundamentally different. When a rumor spreader encounters another spreader, it may stop diffusing the rumor because they feel that people know about the rumor and lose interest. Researchers have established many rumor propagation models and proposed many useful analysis methods, mainly in homogenous networks and heterogeneous networks (Table 1).[8–18] In homogeneous networks, the node degrees concentrate around the mean-degree and follow Poisson degree distribution. Zanette *et al.*[19] studied an epidemic like model for rumor propagation and put forward the critical threshold by mean-field approximation. In order to describe the internal relationship of the rumor spreading system, authors[9,10,16,17] established many double layer network models. Moreover, Dang *et al.*[10,30] introduced counter-attractors into category division. Tian *et al.*[13] considered debunking behavior in rumor spreading, and proved the stability of the rumor-demise equilibrium and the rumor-demise equilibrium. However, one disadvantage of homogeneous networks is its poor robustness. If a small number of nodes in the network are randomly removed, the whole network connectivity performance will be affected. For further addressing this problem, the heterogeneous networks has been proposed, where the node degrees obey power-law distribution and have scale-free property in most networks.[14,18,20–23] Considering the effect of people's psychological factors like hesitation mechanism,[22] the forgetting and remembering mechanisms,[18] education mechanism,[13] cross-transmitted mechanism[23] and self-growth mechanism,[14] authors analyzed the reproduction theory and the global stability of rumor propagation equilibrium, and they found that the rumor spreads faster and the final size of the rumor is smaller in heterogeneous network. It is worth noting that the above researchers studies the rumor propagation behavior with deterministic model, i.e., fixed parameters. However, due to the various interior and exterior noises on the propagation pro-

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cess, different stochastic models have been considered, such as Itô stochastic models,[24–28] stratonovich stochastic models,[8] and Levy process models.[29] Authors proved the dynamic property around equilibriums and the existence of stationary distribution.

As well known, stochastic noises can stabilize unstable systems.[34–36] Even for epidemic systems[37,40] and malware propagation systems,[38] it has been proved that stochastic perturbations on contact rate and infections death rate can suppress the spreading of virus and malware. However, the positive effect of stochastic noises has not been explored in rumor propagation systems, especially the quantitative effect of stochastic noise on suppressing rumor propagation. Motivated by this, we propose a new rumor propagation model and design a new control method of a random noise stabilization. The main contributions can be summarized as follows.

Compared with the previous rumor propagation models, our new model considers the guidance mechanism and the heterogeneity in complex network. Our theoretical results prove that correct guidance measures can suppress the spread of rumor spreading.

The control strategy is a stochastic stabilization method driven by Brownian motion and the control time is aperiodically intermittent, which makes the control mode and control time more flexibility. We can choose a start time and a work width arbitrarily to balance the control requirement and the control cost.

The rest of the paper is organized as follows. In Section 2, we establish a new rumor propagation model, then we prove the global existence and the uniqueness of the positive solutions. Section 3 analyzes the global dynamics of the new rumor propagation model. If $R_0 < 1$, the rumor will die down naturally; and if $R_0 > 1$, the rumor will spread steadily to a certain scale. In Section 4, the stochastic stabilization strategy is designed and the control effect has been proven in details. In Section 5, numerical simulations are provided to verify our theoretical results. Section 6 gives the conclusion.

Table 1. Comparisons of some representative papers about rumor spreading.

Paper	States	Heterogeneous	Stochastic	Key contributions	Loopholes
Jiang *et al.* (2020)[11]	$SNP_1\ P_2R$	N	N	positive opinions in propagation process	theoretical analysis of the simulation results
Tian *et al.* (2019)[13]	$ILRDS$	N	N	debunking behavior	restrictive assumption of same node degrees
Li *et al.* (2019)[15]	$IS_1S_2R_1R_2$	N	N	different languages mechanism	restrictive assumption of same node degrees
Wang *et al.* (2019)[31]	$XYISR$	N	N	optimal control on deleting rumor posts, popular science education mechanism	restrictive assumption of same node degrees
Huang *et al.* (2021)[16]	UA_1A_2	N	N	two-layer network	theoretical analysis of dynamic behaviour
Yu *et al.* (2021)[32]	$S_1S_2I_1I_2R_1R_2$	N	N	double-rumors spreading models	theoretical analysis of dynamic behaviour
Li *et al.* (2019)[15]	$I_kS_k^1S_k^2R_k^1R_k^2$	Y	N	education mechanism	influence analysis of education mechanism
Wang *et al.* (2019)[23]	$I_kS_kR_k$	Y	N	cross-transmitted mechanism	theoretical proof of the influential force of cross transmitted rates
Huo *et al.* (2020)[28]	$I_kW_kS_kR_k$	Y	N	self-growth mechanism indiscernible degree	difference analysis of homogeneous model and heterogeneous model
Li *et al.* (2020)[33]	XYZ	N	Y	forgetting mechanism stochastic contact rate	stability analysis of the rumor-free equilibrium
Dauhoo *et al.* (2016)[27]	$XWYZ$	N	Y	incubation and constant immigration mechanism	stochastic reproduction number for rumor propagation
This paper	$I_kC_kS_kT_k$	Y	Y	guidance mechanism and stochastic control strategy	debunking term

2. Model formulation

In order to investigate the influence of rumor attraction on rumor spreading, we propose a heterogeneous ignoramus-commentators-spreaders-stiflers (ICST) model. $N(t)$ denotes the size of whole individuals in heterogeneous networks at time t, which is divided into four categories: Ignoramus $I(t)$, denoting individuals who do not know the rumor but vulnerable to affect; commentators $C(t)$, representing individuals that comment the rumor if they contact with a rumor; spreaders $S(t)$, denoting the rumor-propagating individuals whose behavior are believing and propagating after contacting with a

rumor; and stiflers $T(t)$, representing individuals that have no response to contact with a rumor. Considering the heterogeneity induced by different abilities of absorbing and disseminating information, the population is further divided into different categories by the connectivity degree k at time t, where $k = 1, 2, \ldots, n$. The topology structure of individuals in heterogeneous networks can be described in Fig. 1. For simplicity, the rumor propagation model on heterogeneous network is named as the ICST model. The rumor propagation mechanism of the ICST model can be described as follows:

(R1) In a heterogeneous network, we assume that rumor propagate in a reasonable time, during which we can consider the number of immigrate individuals per unit of time as a constant related to degree k, named as b_k. The emigration of each categories is μ_k.

(R2) When an ignorant individual contacts with a spreader, the ignorant may comment the rumor with probability α_k, namely, the commenting rate; the ignoramus may be interested in the rumor and spreading the rumor with probability β_k, namely, spreading rate; and the ignorant may have no response to the rumor with probability γ_k, namely, the stifling rate.

(R3) A comment individual has interest in the rumor, and he or she may become a spreader with probability η_k. Before forwarding information, if he or she is guided to distinguish the truth with probability h_k, then he or she may become a stifler with probability $h_k\eta_k$, or a spreader with probability $(1-h_k)\eta_k$.

(R4) When a spreader contacts with another spreader, he or she could think the rumor has been propagated generally and becomes a stifler with probability δ_k.

Based on the above dynamics rules, we construct a transition diagram of the rumor spreading in Fig. 1. The rumor spreading model in the heterogeneous network can be expressed in the following equations:

$$\begin{cases} I_k(t)' = b_k - \mu_k I_k(t) - \alpha_k k\Theta(t) I_k(t) - \beta_k i\Theta(t) I_k(t) \\ \qquad\quad - \gamma_k k\Theta(t) I_k(t), \\ C_k(t)' = \alpha_k k\Theta(t) I_k(t) - \eta_k C_k(t) - \mu_k C_k(t), \\ S_k(t)' = \beta_k k\Theta(t) I_k(t) + (1-h_k)\eta_k C_k(t) \\ \qquad\quad - \delta_k S_k(t) - \mu_k S_k(t), \\ T_k(t)' = \delta_k S_k(t) + \gamma_k k\Theta(t) I_k(t) + h_k\eta_k C_k(t) - \mu_k T_k(t), \end{cases} \tag{1}$$

with the initial values $I_k(0) \geq 0$, $C_k(0) \geq 0$, $S_k(t) \geq 0$, and $T_k(t) \geq 0$, $k = 0, \ldots, n$. $\Theta(t)$ denotes the probability of an ignoramus contacting with a sharer at time t, which satisfies the relation

$$\Theta(t) = \sum_{k=1}^{n} \frac{kp(k)}{\langle k \rangle} \frac{S_k(t)}{N_k(t)}, \tag{2}$$

where the factor $1/k$ stands for the probability that one of the spread neighbors of a node, with degree k, will contact this node at the present time step. The function $p(k)$ is the forwarder node with degree k, which represents the average number of edges from which a node with degree i can transmit information. $\sum_{k=1}^{n} p(k) = 1$, and $\langle k \rangle = \sum_{k=1}^{n} kp(k)$ denote the average degree. The notations used in the ICST model are listed in Table 2.

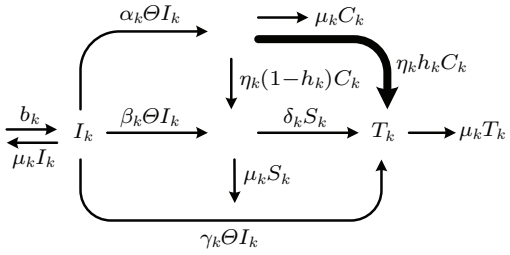


Fig. 1. Structure of rumor propagation under the influence of positive guidance from institutions or government department.

Table 2. Notations.

Symbol	Description	Value range
$N(t)$	the total population in heterogeneous networks	finite positive integer N
$I_k(t)$	numbers of ignoramuses with degree k at time t	$[0, N]$
$C_k(t)$	numbers of commentators with degree k at time t	$[0, N]$
$S_k(t)$	numbers of spreaders with degree k at time t	$[0, N]$
$T_k(t)$	numbers of stiflers with degree k at time t	$[0, N]$
$p(k)$	probability of an individual having degree k	[0, 1]
$\langle k \rangle$	average degree of the network	finite positive integer
$\Theta(t)$	probability of an ignoramus contacting with a sharer at time t	[0, 1]
b_k	number of immigrate individuals per unit of time t	finite positive integer
μ_k	probability of an individual emigrating to heterogeneous network	[0,1]
α_k	probability of an ignoramus commenting the rumor	[0, 1)
β_k	probability of an ignoramus spreading the rumor	[0, 1)
γ_k	probability of an ignoramus ignoring the rumor	[0, 1)
η_k	probability of a commentator emigrating to other categories	[0, 1]
h_k	guidance rate of the rumor	[0, 1]
δ_k	probability of a spreader becoming to a stifler	[0, 1)

Remark 1 Compared with the work of [9,15], the guidance mechanism is introduced in the link from commentator to spreader, which invents guidance or education to the commentator by an organization or individual. A commentator will loss interest in a rumor by the man-made correctly guide, such as science refuting rumor, set forwarding permissions, etc. The guidance mechanism provides a new link from commentators to stiflers, and the number of spreaders will reduce subsequently. It should be stressed in particular that the guidance mechanism is different from the education mechanism in Ref. [9], which focused on the restraining of spreader. However, the guidance mechanism is targeted at commentator and the guidance rates are allowed to be different for different individuals.

Let $N_k(t)$ be the size of whole individuals with degree k at time t, then $N_k(t) = I_k(t) + C_k(t) + S_k(t) + T_k(t)$ and $N(t) = \sum_{k=1}^{n} N_k(t)$. All parameter values are assumed to be nonnegative and μ_k, $b_k > 0$ for all k. Adding the four equations in system (1) gives

$$\begin{aligned} N'(t) &= \big(I_k(t)+C_k(t)+S_k(t)+T_k(t)\big)' \\ &= b_k - \mu_k(I_k(t)+C_k(t)+S_k(t)+T_k(t)) \\ &= b_k - \mu_k N(t). \end{aligned}$$

Hence $\limsup_{t\to\infty} N_k(t) \le b_k/\mu_k$. Therefore, all solution are bounded and the feasible region for model (1) is

$$\begin{aligned} \Gamma = \Big\{ &\big(I_k(t),C_k(t),S_k(t),T_k(t)\big) \in R_+^{4n} | I_k(t) \\ &+ C_k(t)+S_k(t)+T_k(t) \le \frac{b_k}{\mu_k} \Big\} \end{aligned}$$

with $k = 1,\ldots,n$.

To guarantee the negativeness of the solutions of the new model (1), we make the following conclusion.

Theorem 1 For any initial values $I_k(0) > 0$, $C_k(0) > 0$, $S_k(0) > 0$ and $T_k(0) > 0$, $k = 1,\ldots,n$, the solutions of the model (1) are positive for all $t \ge 0$.

Proof For any given initial values $I_k(0) > 0$, $C_k(0) > 0$, $S_k(0) > 0$ and $T_k(0) > 0$, $k = 1,\ldots,n$, we denote $\mathcal{X}(t) = \min_t \{I_k(t), C_k(t), S_k(t), T_k(t)\}$. Using reduction to absurdity, we assume there exists a constant $\tau > 0$, such that

$$\mathcal{X}(t) > 0, \quad t \in (0,\tau), \quad \text{and} \quad \mathcal{X}(\tau) = 0, \quad \dot{\mathcal{X}}(\tau) \le 0. \tag{3}$$

From the definition of $\mathcal{X}(t)$, $\mathcal{X}(\tau) = 0$ implies $I_k(t) = 0, C_k(t) = 0, S_k(t) = 0$ or $T_k(t) = 0$.

If $\mathcal{X}(\tau) = I_k(\tau)$, model (1) implies that

$$\frac{\mathrm{d}I_k(\tau)}{\mathrm{d}t} = b_k > 0,$$

a contradiction to $\dot{\mathcal{X}}(\tau) \le 0$. Hence, $I_k(t) > 0$, $k = 1,\ldots,n$.

If $\mathcal{X}(\tau) = C_k(\tau)$, model (1) implies that

$$\frac{\mathrm{d}C_k(\tau)}{\mathrm{d}t} = \alpha_k k\Theta(\tau) I_k(\tau) > 0,$$

a contradiction to $\dot{\mathcal{X}}(\tau) \le 0$. Hence, $C_k(t) > 0, k = 1,\ldots,n$.

If $\mathcal{X}(\tau) = S_k(\tau)$, the model (1) implies that

$$\frac{\mathrm{d}S_k(\tau)}{\mathrm{d}t} = \beta_k k\Theta(\tau) I_k(\tau) + \eta_k h C_k(\tau) > 0,$$

a contradiction to $\dot{\mathcal{X}}(\tau) \le 0$. Hence, $S_k(t) > 0$, $k = 1,\ldots,n$.

If $\mathcal{X}(\tau) = T_k(\tau)$, then one has $I_k(\tau) > 0$, $C_k(\tau) > 0$, $S_k(\tau) > 0$ and

$$\frac{\mathrm{d}T_k(\tau)}{\mathrm{d}t} = \delta_k S_k(\tau) + \gamma_k k\Theta(\tau) I_k(\tau) + \eta_k(1-h_k) C_k(\tau) > 0.$$

This still contradicts with the inequality (3), that is, $T_k(t) > 0$, $k = 1,\ldots,n$. The proof is completed.

3. Dynamic analysis of the deterministic ICST model

3.1. Equilibria and the basic reproduction number

In this section, we calculate a threshold R_0 which determines the rumor dies out or out in nature, and prove the existence of a positive equilibrium under $R_0 > 1$. Firstly, we calculate the equilibrium points of the deterministic ICST model. Letting the right side of model (1) equal to zero, we get the equilibrium equations as follows:

$$\begin{cases} b_k - \mu_k I_k(t) - \alpha_k k\Theta(t) I_k(t) - \beta_k i\Theta(t) I_k(t) \\ \quad - \gamma_k k\Theta(t) I_k(t) = 0, \\ \alpha_k k\Theta(t) I_k(t) - \eta_k C_k(t) - \mu_k C_k(t) = 0, \\ \beta_k k\Theta(t) I_k(t) + (1-h_k)\eta_k C_k(t) - \delta_k S_k(t) - \mu_k S_k(t) = 0, \\ \delta_k S_k(t) + \gamma_k k\Theta(t) I_k(t) + h_k \eta_k C_k(t) - \mu_k T_k(t) = 0. \end{cases}$$

Since $S_k(t) = 0$, it can derive $\Theta(t) = 0$, and model (1) has a unique rumor-free equilibrium $E_0(b_k/\mu_k, 0, 0, 0)$. Then we prove the existence of the positive equilibrium, which is called a rumor-prevailing equilibrium, and denoted by $E^*\left(I_k^*, C_k^*, S_k^*, T_k^*\right)$, where $I_k^* > 0, C_k^* > 0, S_k^* > 0, T_k^* > 0$ satisfies

$$\begin{cases} b_k - \mu_k I_k^*(t) - \alpha_k k\Theta^*(t) I_k^*(t) - \beta_k k\Theta^*(t) I_k^*(t) \\ \quad - \gamma_k k\Theta^*(t) I_k^*(t) = 0, \\ \alpha_k k\Theta^*(t) I_k^*(t) - \eta_k C_k^*(t) - \mu_k C_k^*(t) = 0, \\ \beta_k k\Theta^*(t) I_k^*(t) + (1-h_k)\eta_k C_k^*(t) - \delta_k S_k^*(t) - \mu_k S_k^*(t) = 0, \\ \delta_k S_k^*(t) + \gamma_k k\Theta^*(t) I_k^*(t) + h_k \eta_k C_k^*(t) - \mu_k T_k^*(t) = 0. \end{cases} \tag{4}$$

Through calculating, one has

$$\begin{aligned} S_k^* = \frac{1}{\delta_k + \mu_k} \Bigg[&\frac{\beta_k b_k k\Theta^*}{\mu_k + (\alpha_k + \beta_k + \gamma_k) k\Theta^*} \\ &+ \frac{(1-h_k)\eta_k b_k \alpha_k k\Theta^*}{(\eta_k + \mu_k)(\mu_k + (\alpha_k + \beta_k + \gamma_k) k\Theta^*)} \Bigg]. \end{aligned} \tag{5}$$

Then, from the definition of $\Theta^*(t) = \frac{1}{\langle k \rangle} \sum_{k=1}^{n} k p(k \mid i) \frac{S_k^*(t)}{N_k(t)}$, we obtain the following self-consistent equation with respect to $\Theta^*(t)$:

$$\Theta^* = \frac{1}{\langle k \rangle} \sum_{k=1}^{n} \frac{k p(k \mid i) b_k}{(\delta_k + \mu_k)\mu_k} \Bigg[\frac{\beta_k b_k k\Theta^*}{\mu_k + (\alpha_k + \beta_k + \gamma_k) k\Theta^*}$$

$$+\frac{(1-h_k)\eta_k b_k \alpha_k k\Theta^*}{(\eta_k+\mu_k)\left(\mu_k+(\alpha_k+\beta_k+\gamma_k)k\Theta^*\right)}\Bigg]. \tag{6}$$

Let $\Theta^* = f(\Theta^*)$, and it is easy to check that $f(0)=0, f(1)<1$. So, a nonzero $\Theta^* \in (0,1)$ exists for the Eq. (6) if $f'(0)>1$, which leads to

$$\sum_{k=1}^{n} \frac{k^2 p(k)\left[\beta_k(\eta_k+\mu_k)+(1-h_k)\eta_k\alpha_k\right]}{\langle k\rangle(\mu_k+\delta_k)(\eta_k+\mu_k)} > 1.$$

Therefore, the rumor-prevailing equilibrium exists under the above condition. Define

$$\sum_{k=1}^{n} \frac{k^2 p(k)\left[\beta_k(\eta_k+\mu_k)+(1-h_k)\eta_k\alpha_k\right]}{\langle k\rangle(\mu_k+\delta_k)(\eta_k+\mu_k)} = R_*,$$

then we conclude that the rumor-prevailing equilibrium exists if $R_* > 1$.

Next, we calculate the threshold R_0. Based on the next generation matrix method, the propagation system can be divided into $\mathcal{F}$ part and $\mathcal{V}$ part, which mean the rate of new infections to the compartment and the rate of transfer from the compartment, respectively. So we write system (2.1) as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{F}(x) - \mathcal{V}(x),$$

where

$$x = \begin{pmatrix} C_k & S_k \end{pmatrix}^{\mathrm{T}},$$
$$\mathcal{F}(x) = \begin{pmatrix} \alpha_k k\Theta(t) I_k(t) \\ \beta_k k\Theta(t) I_k(t) \end{pmatrix},$$
$$\mathcal{V}(x) = \begin{pmatrix} (\eta_k+\mu_k)C_k(t) \\ -\eta_k(1-h_k)C_k(t) + (\delta_k+\mu_k)S_k(t) \end{pmatrix}.$$

The Jacobian matrices of $F(x)$ and $V(x)$ at the rumor-free equilibrium E_0 are as following:

$$F = D\mathcal{F}(E_0) = \begin{pmatrix} \mathbf{0} & F_{12} \\ \mathbf{0} & F_{22} \end{pmatrix},$$
$$V = D\mathcal{V}(E_0) = \begin{pmatrix} V_{11} & \mathbf{0} \\ V_{21} & V_{22} \end{pmatrix},$$

where

$$F_{12} = \frac{1}{\langle k\rangle}\begin{pmatrix} \alpha_1 P(1) & \frac{2P(2)\alpha_1 b_1 \mu_2}{b_2\mu_1} & \cdots & \frac{nP(n)\alpha_1 b_1 \mu_n}{b_n\mu_1} \\ \frac{2P(1)\alpha_2 b_2 \mu_1}{b_1\mu_2} & 2^2 P(2)\alpha_2 & \cdots & \frac{2nP(n)\alpha_2 b_2 \mu_n}{b_n\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{nP(1)\alpha_n b_n \mu_1}{b_1\mu_n} & \frac{2nP(2)\alpha_n b_n \mu_2}{b_2\mu_n} & \cdots & n^2 P(n)\alpha_n \end{pmatrix},$$

$$F_{22} = \frac{1}{\langle k\rangle}\begin{pmatrix} \beta_1 P(1) & \frac{2P(2)\beta_1 b_1 \mu_2}{b_2\mu_1} & \cdots & \frac{nP(n)\beta_1 b_1 \mu_n}{b_n\mu_1} \\ \frac{2P(1)\beta_2 b_2 \mu_1}{b_1\mu_2} & 2^2 P(2)\beta_2 & \cdots & \frac{2nP(n)\beta_2 b_2 \mu_n}{b_n\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{nP(1)\beta_n b_n \mu_1}{b_1\mu_n} & \frac{2nP(2)\beta_n b_n \mu_2}{b_2\mu_n} & \cdots & n^2 P(n)\beta_n \end{pmatrix},$$

$$V_{11} = \begin{pmatrix} \frac{(\eta_1+\mu_1)b_1}{\mu_1} & & & \\ & \frac{(\eta_2+\mu_2)b_2}{\mu_2} & & \\ & & \ddots & \\ & & & \frac{(\eta_n+\mu_n)b_n}{\mu_n} \end{pmatrix},$$

$$V_{21} = \begin{pmatrix} -\eta_1(1-h_1) & & & \\ & -\eta_2(1-h_2) & & \\ & & \ddots & \\ & & & -\eta_n(1-h_n) \end{pmatrix},$$

$$V_{22} = \begin{pmatrix} \delta_1+\mu_1 & & & \\ & \delta_2+\mu_2 & & \\ & & \ddots & \\ & & & \delta_n+\mu_n \end{pmatrix},$$

where $\mathbf{0}$ is zero matrix. According to the concept of next generation matrix, the reproduction number of model (1) equals to $R_0 = \rho\left(FV^{-1}\right)$, where $\rho\left(FV^{-1}\right)$ denotes the spectral radius of the matrix FV^{-1}. Then

$$R_0 = \sum_{k=1}^{n} \frac{\left((1-h_k)\eta_k\alpha_k + \beta_k(\eta_k+\mu_k)\right)k^2 p(k)}{(\delta_k+\mu_k)(\eta_k+\mu_k)\langle k\rangle},$$

where $\langle k^2\rangle = \sum_i i^2 P(i)$, $i = 1,2,3,\ldots,n$.

Comparing the expressions of R_0 and R_*, we get $R_0 = R_*$. We conclude the rumor-prevailing equilibrium exists if $R_0 > 1$, and the rumor-free equilibrium always exists.

3.2. Global dynamics when $R_0 \le 1$

In this section, we show the rumor naturally disappears when $R_0 \le 1$, and the rumor persists otherwise.

Theorem 2 If $R_0 \le 1$, E_0 is globally asymptotically stable in Γ. If $R_0 > 1$, E_0 is unstable, and the rumor will be of uniform persistence.

Proof Define a Lyapunov function

$$V_1(t) = \sum_{k=1}^{n} \frac{kp(k)}{\delta_k+\mu_k}\left[\frac{(1-h_k)\eta_k}{\eta_k+\mu_k}C_k(t) + S_k(t)\right],$$

differentiation of which gives

$$\begin{aligned}\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} &= \sum_{k=1}^{n} \frac{kp(k)}{\delta_k+\mu_k}\left[\frac{(1-h_k)\eta_k}{\eta_k+\mu_k}\frac{\mathrm{d}C_k(t)}{\mathrm{d}t} + \frac{\mathrm{d}S_k(t)}{\mathrm{d}t}\right] \\ &= \sum_{k=1}^{n} \frac{kp(k)}{\delta_k+\mu_k}\left[\frac{((1-h_k)\eta_k\alpha_k}{\eta_k+\mu_k} + \beta_k\right]k\Theta(t)I_k(t) \\ &\quad - \sum_{k=1}^{n} kp(k)S_k(t).\end{aligned}$$

For $1 \le k \le n$, $0 \le I_k \le b_k/\mu_k$, we have

$$\begin{aligned}\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} &\le \sum_{k=1}^{n} \frac{kp(k)}{\delta_k+\mu_k}\left[\frac{(1-h_k)\eta_k\alpha_k}{\eta_k+\mu_k} + \beta_k\right]k \\ &\quad \times \sum_{i=1}^{n} \frac{ip(i)S_i(t)}{\langle k\rangle} - \sum_{k=1}^{n} kp(k)S_k(t) \\ &= \sum_{k=1}^{n} \frac{(1-h_k)\eta_k\alpha_k + \beta_k(\eta_k+\mu_k)}{(\delta_k+\mu_k)(\eta_k+\mu_k)\langle k\rangle}k^2 p(k) \\ &\quad \times \sum_{i=1}^{n} ip(i)S_i(t) - \sum_{k=1}^{n} kp(k)S_k(t)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \frac{(1-h_k)\eta_k\alpha_k + \beta_k(\eta_k+\mu_k)}{(\delta_k+\mu_k)(\eta_k+\mu_k)\langle k\rangle} k^2 p(k) \\
&\quad \times \sum_{i=1}^{n} ip(i)S_i(t) - \sum_{i=1}^{n} ip(i)S_i(t) \\
&= (R_0-1)\sum_{i=1}^{n} ip(i)S_i(t) \le 0, \ \text{ if } \ R_0 \le 1.
\end{aligned}
$$

Therefore, all limit points are contained in the largest invariant subset of

$$
\Omega = \left\{ (I_k, C_k, S_k, T_k) \in \Gamma | \dot{V}_1 = 0 \right\}.
$$

By the expression of $\dot{V}_1$, we obtain $S_k(t) = 0$ or $R_0 = 1$. If $S_k(t) = 0$, using the equations in system (1), we can conclude that along any solution in the largest invariant subset, we have

$$
I_k = \frac{b_k}{\mu_k}, C_k = S_k = T_k = 0.
$$

This implies that the largest invariant subset is E_0. If $R_0 = 1$ and $I_k = b_k/\mu_k$, by using the same argument as in the previous case, the largest invariant subset is E_0 as well. Therefore, all the limit points are contained in the largest invariant subset E_0. By the LaSalle's invariance principle, E_0 is globally asymptotically stable if $R_0 < 1$.

If $R_0 > 1$, $\dot{V}_1 > 0$, and E_0 is unstable. By the definition of the uniform persistence in Ref. [8], it is easy to check that the instability of E_0 derives the uniform persistence of model (1).

Remark 2 Theorem 2 implies that the rumor naturally dies out from all stages and any size of the initial outbreak if $R_0 \le 1$. Otherwise the rumor persists at a constant state. Therefore, R_0 is a sharp threshold for the rumor persistence.

3.3. Global dynamics when $R_0 > 1$

As it is proved in Section 3.1, there is an information-prevailing equilibrium E^* if $R_0 > 1$. In this section, we show the global stability behavior of the equilibrium E^* by constructing a Lyapunov functions.[39] Generally, this implies the rumor persists at a unique scale even the initial size of information source is small.

Theorem 3 If $R_0 > 1$, then the information-prevailing equilibrium E^* is globally asymptotically stable in Γ.

Proof For convenient, we denote $S_k(t) = S_k, C_k(t) = C_k$ and $I_k(t) = I_k$. The Lyapunov function for the related single-group model of system(1) can be constructed as follows:

$$
\begin{aligned}
&V_k(S_k(t), C_k(t), I_k(t)) \\
&= \frac{(\alpha_k+\beta_k)I_k^*}{\alpha_k+\beta_k+\gamma_k} \Phi\left(\frac{I_k(t)}{I_k^*}\right) + C_k^* \Phi\left(\frac{C_k(t)}{C_k^*}\right) + S_k^* \Phi\left(\frac{S_k(t)}{S_k^*}\right),
\end{aligned}
$$

where $\Phi(x) = x - 1 - \ln x$, and $\Phi(x) > 0$ for $x > 0$. The derivative of $\Phi(I_k(t)/I_k^*)$ is

$$
\begin{aligned}
\frac{\mathrm{d}\Phi\left(\frac{I_k(t)}{I_k^*}\right)}{\mathrm{d}t} I_k^* &= \left(1 - \frac{I_k^*}{I_k}\right)\left[b_k - \mu_k I_k - \alpha_k k\Theta I_k \right. \\
&\quad \left. - \beta_k k\Theta I_k - \gamma_k k\Theta I_k\right].
\end{aligned}
$$

By the equilibrium equations (4), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}\Phi\left(\frac{I_k(t)}{I_k^*}\right)}{\mathrm{d}t} I_k^* &= \left(1 - \frac{I_k^*}{I_k}\right)\left[\mu_k(I_k^* - I_k) \right. \\
&\quad \left. - (\alpha_k+\beta_k+\gamma_k)k(\Theta^* I_k^* - \Theta I_k)\right] \\
&= -\frac{\mu_k}{I_k}(I_k - I_k^*)^2 - (\alpha_k+\beta_k+\gamma_k) \\
&\quad \times k\left(1 - \frac{I_k^*}{I_k}\right)(\Theta^* I_k^* - \Theta I_k).
\end{aligned}
$$

It can be verified similarly that

$$
\begin{aligned}
\frac{\mathrm{d}\Phi\left(\frac{C_k(t)}{C_k^*}\right)}{\mathrm{d}t} C_k^* &= \left(1 - \frac{C_k^*}{C_k}\right)\left[\alpha_k k\Theta I_k - \eta_k C_k - \mu_k C_k\right] \\
&= \left(1 - \frac{C_k^*}{C_k}\right)\left[\alpha_k k\Theta I_k - \alpha_k k\Theta^* I_k^* \frac{C_k}{C_k^*}\right], \\
\frac{\mathrm{d}\Phi\left(\frac{S_k(t)}{S_k^*}\right)}{\mathrm{d}t} S_k^* &= \left(1 - \frac{S_k^*}{S_k}\right)\left[\beta_k k\Theta I_k(t) + (1-h_k) \right. \\
&\quad \left. \times \eta_k C_k - \delta_k S_k - \mu_k S_k\right] \\
&= \left(1 - \frac{S_k^*}{S_k}\right)\left[\beta_k k\Theta I_k - \beta_k k\Theta^* I_k^* \frac{S_k}{S_k^*} \right. \\
&\quad \left. + (1-h_k)\eta_k C_k - \eta_k h_k C_k^* \frac{S_k}{S_k^*}\right] \\
&= \left(1 - \frac{S_k^*}{S_k}\right)\left[\beta_k k\Theta I_k - \beta_k k\Theta^* I_k^* \frac{S_k}{S_k^*} \right. \\
&\quad \left. + \alpha_k k\Theta^* I_k^*\left(\frac{C_k}{C_k^*} - \frac{S_k}{S_k^*}\right)\right].
\end{aligned}
$$

Then the derivation of $V_k(S_k, C_k, I_k)$ is

$$
\begin{aligned}
\frac{\mathrm{d}V_k(S_k, C_k, I_k)}{\mathrm{d}t} &= -\frac{\mu_k(\alpha_k+\beta_k)}{(\alpha_k+\beta_k+\gamma_k)I_k}(I_k - I_k^*)^2 \\
&\quad - (\alpha_k+\beta_k)k\left(1 - \frac{I_k^*}{I_k}\right)(\Theta^* I_k^* - \Theta I_k) \\
&\quad + \left(1 - \frac{C_k^*}{C_k}\right)\left[\alpha_k k\Theta I_k - \alpha_k k\Theta^* I_k^* \frac{C_k}{C_k^*}\right] \\
&\quad + \left(1 - \frac{S_k^*}{S_k}\right)\left[\beta_k k\Theta I_k - \beta_k k\Theta^* I_k^* \frac{S_k}{S_k^*} \right. \\
&\quad \left. + \alpha_k k\Theta^* I_k^*\left(\frac{C_k}{C_k^*} - \frac{S_k}{S_k^*}\right)\right].
\end{aligned}
$$

Based on the definition of $\Theta(t)$, we can further derive

$$
\begin{aligned}
&\frac{\mathrm{d}V_k(S_k, C_k, I_k)}{\mathrm{d}t} \\
&= -\frac{\mu_k(\alpha_k+\beta_k)}{(\alpha_k+\beta_k+\gamma_k)I_k}(I_k - I_k^*)^2 + \sum_{i=1}^{n} \frac{kip(i)S_i^* I_k^*}{N_i} \\
&\quad \times \left[\alpha_k + \beta_k - (\alpha_k+\beta_k)\frac{I_k^*}{I_k} + (\alpha_k+\beta_k)\frac{S_k}{S_k^*} \right. \\
&\quad - \alpha_k \frac{S_i I_k C_k^*}{S_i^* I_k^* C_k} + \alpha_k - \beta_k \frac{S_k}{S_k^*} - \beta_k \frac{S_i I_k S_k^*}{S_i^* I_k^* S_k} + \beta_k \\
&\quad \left. - \alpha_k \frac{S_k}{S_k^*} - \alpha_k \frac{S_k^* C_k}{S_k C_k^*} + \alpha_k\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\mu_k(\alpha_k+\beta_k)}{(\alpha_k+\beta_k+\gamma_k)I_k}(I_k-I_k^*)^2+\sum_{i=1}^{n}\frac{kip(i)S_i^*I_k^*}{N_i}\\
&\times\left[-\alpha_k\Phi\left(\frac{I_k^*}{I_k}\right)-\alpha_k\Phi\left(\frac{S_iI_kC_k^*}{S_i^*I_k^*C_k}\right)\right.\\
&-\alpha_k\Phi\left(\frac{S_k^*C_k}{S_kC_k^*}\right)-\beta_k\Phi\left(\frac{I_k^*}{I_k}\right)-\beta_k\Phi\left(\frac{S_iI_kS_k^*}{S_i^*I_k^*S_k}\right)\\
&\left.+(\alpha_k+\beta_k)\left(\frac{S_i}{S_i^*}-\frac{S_k}{S_k^*}-\ln\frac{S_iS_k^*}{S_i^*S_k}\right)\right].
\end{aligned}
$$

Since $-\frac{\mu_k}{I_k}(I_k-I_k^*)^2<0$,and $\Phi(x)>0$, the derivative of $V_k(S_k,C_k,I_k)$ satisfies

$$
\begin{aligned}
&\frac{\mathrm{d}V_k(S_k,C_k,I_k)}{\mathrm{d}t}\\
&\le\sum_{i=1}^{n}\frac{kip(i)S_i^*I_k^*}{N_i}(\alpha_k+\beta_k)\left(\frac{S_i}{S_i^*}-\frac{S_k}{S_k^*}-\ln\frac{S_iS_k^*}{S_i^*S_k}\right).
\end{aligned}
$$

Let $a_{ki}=(\alpha_k+\beta_k)kip(i)S_i^*I_k^*/N_i$, and $G_k(x_k)=x_k-\ln x_k$, then we have

$$
\frac{\mathrm{d}V_k(S_k,C_k,I_k)}{\mathrm{d}t}\le\sum_{i=1}^{n}a_{ki}\left[G_i\left(\frac{S_i}{S_i^*}\right)-G_k\left(\frac{S_k}{S_k^*}\right)\right].
$$

So V_k, G_k and a_{ki} satisfy the assumption of Theorem 3.1 and Corollary 3.3 by Li's graph-theoretic method for Lyapunov functions.[39] Therefore, model (1)'s Lyapunov functions satisfy

$$
V=\sum_{k=1}^{n}c_kV_k(I_k,C_k,S_k)<0.
$$

Moreover, $\dot{V}_k(S_k,C_k,I_k)=0$ if and only if

$$
(S_k,C_k,I_k,T_k)=(S_k^*,C_k^*,I_k^*,T_k^*).
$$

By the LaSalle invariance principle, E^* is globally asymptotically stable.

4. Stochastic stabilization method on rumor propagation

Inspired by the favorable effects of white noise in epidemic system, we focus on the stochastic control strategy for rumor propagation in this section. The controller is made of the g-type white noise driven by Brownian motions. In particular, we design an intermittent time, aperiodic start and end points strategy to minimize the control cost and maximize the flexibility of control operation. Specifically, each time span $[t_k,t_{k+1})$ is divided into a work pattern $[t_k,t_k+w_k)$ and a rest pattern $[t_k+w_k,t_{k+1})$, where w_k denotes the k-th perturbed width which satisfies $0\le w_k\le t_{k+1}-t_k$. The start time t_k and the perturbed width w_k would be different, and the total perturbed time ratio w satisfies

$$
\lim_{n\to\infty}\frac{\sum_{j+1}^{n}w_j}{t_{n+1}-t_0}=w. \tag{7}
$$

Adding the g-type white noise to model (1), we get a new aperiodically intermittent stochastic perturbed system (AISPS) as follows:

$$
\mathrm{d}\begin{bmatrix}I_k(t)\\C_k(t)\\S_k(t)\\T_k(t)\end{bmatrix}=\begin{bmatrix}b_k-\mu_kI_k^*(t)-(\alpha_k+\beta_k+\gamma_k)k\Theta(t)I_k(t)\\\alpha_kk\Theta(t)I_k(t)-(\eta_k+\mu_k)C_k(t)\\\beta_kk\Theta(t)I_k(t)+(1-h_k)\eta_kC_k(t)-(\delta_k+\mu_k)S_k(t)\\\delta_kS_k(t)+\gamma_kk\Theta(t)I_k(t)+h_k\eta_kC_k(t)-\mu_kT_k(t)\end{bmatrix}\mathrm{d}t+g(t)\,\mathrm{d}B(t), \tag{8}
$$

where

$$
g(t)=\begin{cases}\sigma(I_k(t),C_k(t),S_k(t),T_k(t))^{\mathrm{T}}\,\mathrm{d}B(t), & t\in[t_j,t_j+w_j),\\ 0, & t\in[t_j+w_j,t_{j+1})\end{cases}
$$

for each $j\in N$. σ represents the aperiodically intermittent perturb intensity. In the rest of this section, a stochastic stabilization criterion on rumor spreading of model (1) is established by Itô formula, exponential martingale inequality, Borel-Cantelli's lemma and stochastic stabilization theory.

Theorem 4 If the aperiodically intermittent perturb intensity σ satisfies $\sigma^2>(\delta+\mu)(R_0-1)/w$, where $\delta+\mu=\max_{k=1,\dots,n}(\delta_k+\mu_k)$, the commenter $C_k(t)$ and the sharer $S_k(t)$ tend to zero exponentially for all $k\in N$, which means rumor will die out with probability one.

Proof Define $\mathfrak{L}(t)=\sum_{k=1}^{n}\frac{kp(k)}{\delta_k+\mu_k}\left(\frac{(1-h_k)\eta_k}{\eta_k+\mu_k}C_k(t)+S_k(t)\right)$, differentiation of which gives

$$
\mathrm{d}\mathfrak{L}(t)=F(t)\,\mathrm{d}t+G(t)\,\mathrm{d}B(t),
$$

where

$$
\begin{aligned}
F(t)&=\sum_{k=1}^{n}\frac{kp(k)}{\delta_k+\mu_k}\left(\frac{(1-h_k)\eta_k}{\eta_k+\mu_k}+\beta_k\right)k\Theta(t)I_k(t)-\sum_{k=1}^{n}kp(k)S_k(t),\\
G(t)&=\begin{cases}\sigma\sum_{k=1}^{n}\frac{kp(k)\sigma_k}{\delta_k+\mu_k}\left(\frac{(1-h_k)\eta_k}{\eta_k+\mu_k}C_k(t)+S_k(t)\right), & t\in[t_j,t_j+w_j),\\ 0, & t\in[t_j+w_j,t_{j+1}).\end{cases}
\end{aligned}
$$

Note that $I_k(t) \le N_k(t) = \frac{b_k}{\mu_k}, C_k(t) \ge 0$ for all $k \in N$ and $t \ge t_0$. We use the similar argument as that in the proof of Theorem 2, where $\mathfrak{L}(t)$ satisfies

$$F(t) \le (R_0 - 1) \sum_{k=1}^{n} kp(k) S_k(t).$$

Combining with the structure of $\mathfrak{L}(t)$, we obtain

$$\frac{F(t)}{\mathfrak{L}(t)(t)} \le (\delta + \mu)(R_0 - 1),$$
$$\frac{G^2(t)}{\mathfrak{L}(t)^2(t)} = \begin{cases} \sigma^2, & t \in [t_j, t_j + w_j), \\ 0, & t \in [t_j + w_j, t_{j+1}). \end{cases}$$

Now we use Itô formula to calculate the logarithm of $\mathfrak{L}(t)(t)$

$$\begin{aligned} \log \mathfrak{L}(t)(t) &= \log \mathfrak{L}(t)(t_0) + \int_{t_0}^{t} \left(\frac{F(t)}{\mathfrak{L}(t)(t)} - \frac{G^2(t)}{2\mathfrak{L}(t)^2(t)} \right) \mathrm{d}t \\ &\quad + \int_{t_0}^{t} \frac{G(t)}{\mathfrak{L}(t)(t)} \mathrm{d}B(t) \\ &\le \log \mathfrak{L}(t)(t_0) + (\delta + \mu)(R_0 - 1)t \\ &\quad - \int_{t_0}^{t} \left(\frac{G^2(t)}{2\mathfrak{L}(t)^2(t)} \right) \mathrm{d}t + \int_{t_0}^{t} \frac{G(t)}{\mathfrak{L}(t)(t)} \mathrm{d}B(t). \end{aligned}$$

By exponential martingale inequality and Borel-Cantelli's lemma (see Mao,[40] Theorem 7.4 on P44 and Lemma 2.4 on P7), there exists an integer $s_0 = s_0(w)$ for almost all $\omega \in \Omega$ and any positive number T, ε, if $s > s_0$, the continuous martingale

$$\int_{t_0}^{t} \frac{G(t)}{\mathfrak{L}(t)(t)} \mathrm{d}B(t) \le \frac{2}{\varepsilon} \log(s) + \frac{\varepsilon}{2} \int_{t_0}^{t} \frac{G^2(t)}{\mathfrak{L}(t)^2(t)} \mathrm{d}t,$$

holds for all $t_0 \le t \le T$. It yields that

$$\begin{aligned} \log \mathfrak{L}(t)(t) &= \log \mathfrak{L}(t)(t_0) + \int_{t_0}^{t} \left(\frac{F(t)}{\mathfrak{L}(t)(t)} - \frac{G^2(t)}{2\mathfrak{L}(t)^2(t)} \right) \mathrm{d}t \\ &\quad + \int_{t_0}^{t} \frac{G(t)}{\mathfrak{L}(t)(t)} \mathrm{d}B(t) \\ &\le \log \mathfrak{L}(t)(t_0) + (\delta + \mu)(R_0 - 1)t \\ &\quad - \int_{t_0}^{t} \left(\frac{(1-\varepsilon)G^2(t)}{2\mathfrak{L}(t)^2(t)} \right) \mathrm{d}t + \frac{2}{\varepsilon} \log(s). \end{aligned}$$

Since $G(t)$ is a piecewise function, we need to discuss two different cases based on the region of t to calculate the above integral. Obviously, there exists an integer n such that $t \in [t_n, t_{n+1})$.

Case 1 For all $\omega \in \Omega$, if $t_n \le t < t_n + w_n$ and $t_n > s_0$,

$$\begin{aligned} \log \mathfrak{L}(t) &\le \log \mathfrak{L}(t_0) + (\delta + \mu)(R_0 - 1)t \\ &\quad + \int_{t_0}^{t_0 + w_0} \left(-\frac{(1-\varepsilon)G^2(t)}{2\mathfrak{L}^2(t)} \right) \mathrm{d}t \\ &\quad + \int_{t_0 + w_0}^{t_1} \left(-\frac{(1-\varepsilon)G^2(t)}{2\mathfrak{L}^2(t)} \right) \mathrm{d}t \\ &\quad + \cdots + \int_{t_n}^{t_n + w_n} \left(-\frac{(1-\varepsilon)G^2(t)}{2\mathfrak{L}^2(t)} \right) \mathrm{d}t + \frac{2}{\varepsilon} \log(s) \\ &= \log \mathfrak{L}(t_0) + (\delta + \mu)(R_0 - 1)t \\ &\quad - \frac{1-\varepsilon}{2} \sum_{i=0}^{n} w_i \sigma^2 + \frac{2}{\varepsilon} \log(s). \end{aligned}$$

Case 2 For all $\omega \in \Omega$, if $t_n + w_n \le t < t_{n+1}$ and $t_n > s_0$,

$$\begin{aligned} \log \mathfrak{L}(t) &\le \log \mathfrak{L}(t_0) + (\delta + \mu)(R_0 - 1)t \\ &\quad + \int_{t_0}^{t_0 + w_0} \left(-\frac{(1-\varepsilon)G^2(t)}{2\mathfrak{L}^2(t)} \right) \mathrm{d}t \\ &\quad + \int_{t_0 + w_0}^{t_1} \left(-\frac{(1-\varepsilon)G^2(t)}{2\mathfrak{L}^2(t)} \right) \mathrm{d}t \\ &\quad + \cdots + \int_{t_n + w_n}^{t_{n+1}} \left(-\frac{(1-\varepsilon)G^2(t)}{2\mathfrak{L}^2(t)} \right) \mathrm{d}t + \frac{2}{\varepsilon} \log(s) \\ &= \log \mathfrak{L}(t_0) + (\delta + \mu)(R_0 - 1)t \\ &\quad - \frac{1-\varepsilon}{2} \sum_{i=0}^{n} w_i \sigma^2 + \frac{2}{\varepsilon} \log(s). \end{aligned}$$

Based on the above analysis, if $t_n \le t < t_{n+1}$ and $t_n > s_0$,

$$\begin{aligned} \log \mathfrak{L}(t) &\le \log \mathfrak{L}(t_0) + (\delta + \mu)(R_0 - 1)t \\ &\quad - \frac{1-\varepsilon}{2} \sum_{i=0}^{n} w_i \sigma^2 + \frac{2}{\varepsilon} \log(s) \end{aligned}$$

always holds for all $\omega \in \Omega$. Then we obtain

$$\begin{aligned} \frac{1}{t} \log \mathfrak{L}(t) &\le \frac{\log \mathfrak{L}(t_0) + \frac{2}{\varepsilon} \log(s)}{t_{n+1} - t_0} \\ &\quad - \frac{1-\varepsilon}{2} \frac{\sum_{i=0}^{n} w_i \sigma^2}{t_{n+1} - t_0} + (\delta + \mu)(R_0 - 1). \end{aligned}$$

Combining with Eq. (7), it follows that

$$\frac{1}{t} \limsup_{t \to \infty} \log \mathfrak{L}(t) \le -\frac{1-\varepsilon}{2} w\sigma^2 + (\delta + \mu)(R_0 - 1) \quad \text{a.s.}$$

Since ε is arbitrary, we set $\varepsilon \to 0$, then the following inequality holds

$$\frac{1}{t} \limsup_{t \to \infty} \log \mathfrak{L}(t) \le -\frac{1}{2} w\sigma^2 + (\delta + \mu)(R_0 - 1) \quad \text{a.s.}$$

If $\sigma^2 > \frac{2(\delta+\mu)(R_0-1)}{w}$, then we conclude

$$\frac{1}{t} \limsup_{t \to \infty} \log \mathfrak{L}(t) \le -\frac{1}{2} w\sigma^2 + (\eta + \mu)(R_0 - 1) < 0 \quad \text{a.s.},$$

which implies that $\lim_{t \to \infty} C_k = 0$ a.s., and $\lim_{t \to \infty} S_k = 0$ a.s., for all $k \in N$. The proof is completed.

Remark 3 In the special case $t_{i+1} - t_i = T$, $w_i = \delta$, $i = 1, \ldots, n$, this method is equivalent to the periodically intermittent perturb method.[34] More specially, if $w = 1$, this method is a traditional stochastic stabilization method.[2]

Remark 4 The stochastic stabilization method provided in Theorem 2 can effectively suppress the rumor propagation if the the stochastic perturbation intensity σ is strong enough and the suitable perturb time ratio w. The extinction of rumor depends on the propagation force R_0. If $R_0 < 1$, the condition $2(\eta + \mu)(R_0 - 1)/w < 0 < \sigma^2$ is always satisfied, the rumor will automaticly die out over time. If $R_0 > 1$,the perturb intensity bigger than $\sqrt{2(\eta + \mu)(R_0 - 1)/w}$, then the rumor propagation can be suppressed.

5. Numerical examples

In this section, we conduct numerical simulations to illustrate the verification of the theoretical results and analyze how stochastic noises prevent the spreading process.

5.1. The simulation of rumor spreading process

Firstly, we discuss the effect of $\mu_k, \eta_k, \alpha_k, \beta_k, \delta_k$ and h_k on R_0. By using the partial rank correlation coefficients (PRCCs) method, the key parameters for rumor spreading can be determined. The input parameters μ_k, η_k, α_k, β_k, δ_k, and h_k are set to be uniformly distributed in Table 2, and we carry out 1000 runs to calculate multiple time points and plotted versus time. PRCCs plots of the basic reproduction number R_0 and the scatter plot of six parameters are shown in Fig. 2. The contact rate β and the guidance rate h are strongly positive correlated to R_0, and there is the negative correlation between emigration rate η and R_0. It implies that guidance measure can effectively control the spread of rumors, and we suggest the government or the official should focus on interventions measures on emigration falls from commentator to spreader.

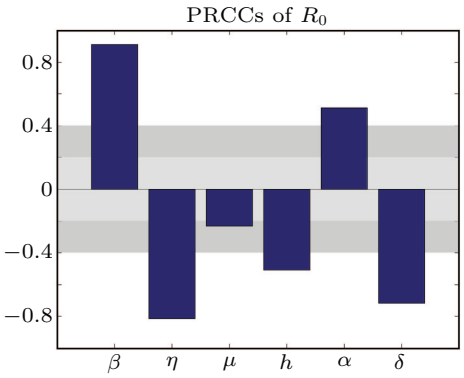


Fig. 2. PRCCs of R_0 of the ISCT model.

Table 3. PRCC values for R_0.

Parameter	Description	Range	PRCC	P value	Reference
μ_k	probability of an individual emigrating to heterogeneous network	$[10^{-5}, 0.3]$	–0.24144	1.3397×10^{-132}	[21]
α_k	probability of an ignoramus commenting the rumor	$[10^{-5}, 0.6]$	0.51939	0	[21,22]
β_k	probability of an ignoramus spreading the rumor	$[10^{-5}, 0.6]$	0.91084	0	[21,22]
η_k	probability of a commentator emigrating to other categories	$[10^{-5}, 0.5]$	–0.81388	0	[21,22]
h_k	guidance rate of the rumor	$[10^{-5}, 1]$	0.50635	0	see text
δ_k	probability of a spreader becoming to a stifler	$[10^{-5}, 0.3]$	–0.71871	0	[21,22]

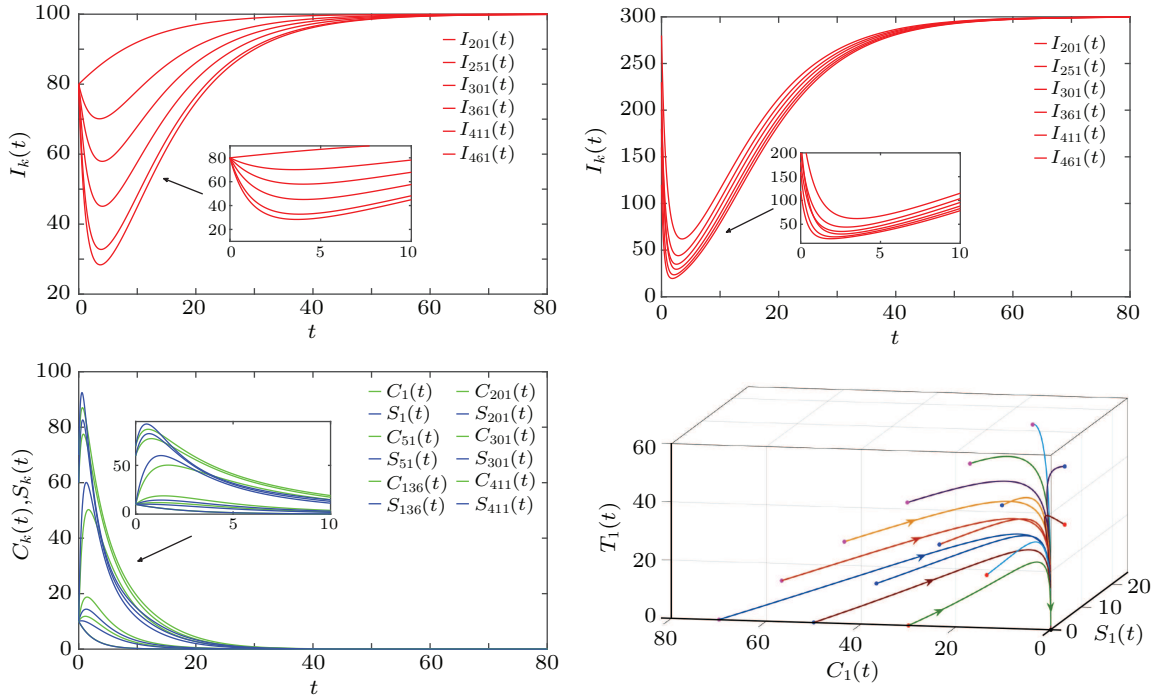


Fig. 3. The stability of rumor-free equilibrium E_0 with $R_0 < 1$.

Next, we concentrate on the test of global stability of E_0 and E^*. The heterogeneous complex network is generated by using the scale-free network with the degree distribution $p(k) = 2m^2k^{-3}$. We set the minimum degree of the network is 1, and the maximum degree is 500. Thus, the average degree of the network is $\langle k \rangle = 1.3688$ by calculation. System parameters are summarized in Table 4, which fully consider the node degree difference as a result of parameter values. The

initial values for $1 \le k \le 200$ are $I_k^0 = 80, C_k^0 = 10, S_k^0 = 10$, and $I_k^0 = 280, C_k^0 = 10, S_k^0 = 10$ for $201 \le k \le 500$. If we choose the parameter values in column 1-2, one can calculate that $R_0 = 0.1203 < 1$, $E_0 = (b_k/\mu_k, 0, 0)$, where $b_k/\mu_k = 100$ if $0 \le k \le 200$, and $b_k/\mu_k = 300$ if $201 \le k \le 500$. Consequently, E_0 is globally asymptotically stable by Theorem 1. To verify the global property, the trajectories of (I_k, C_k, S_k, T_k) start at different initial conditions. We observe these trajectories tend to information-elimination steady states E_0 under different initial points (see Fig. 3 and Table 5), which means E_0 is globally asymptotically stable if $R_0 < 1$.

Using the parameter values in the second column of Table 4, one can calculate that $R_0 = 1.4003$, $E_0 = (b_k/\mu_k, 0, 0)$, and $b_k/\mu_k = 100$. From Theorem 1, E_0 is unstable, and the information-prevailing equilibrium E^* is globally asymptotically stable. To verify it, the trajectories of (I_k, C_k, S_k, T_k) start at eighteen different initial conditions. We observe these trajectories tend to E^* under different initial points(see Fig. 4 and Table 6), which means E^* is globally asymptotically stable if $R_0 > 1$.

Table 4. Parameter values.

Parameter	Values in Fig. 3	Values in Fig. 3	Values in Figs. 4–7	Values in Figs. 4–7
k	1–200	201–500	1–200	201–500
b_k	10	30	10	10
μ_k	0.1	0.1	0.1	0.1
α_k	0.08	0.06	0.1	0.12
β_k	0.05	0.09	0.1	0.12
γ_k	0.06	0.07	0.06	0.06
η_k	0.3	0.2	0.25	0.28
h_k	0.5	0.6	0.5	0.5
δ_k	0.45	0.5	0.25	0.25

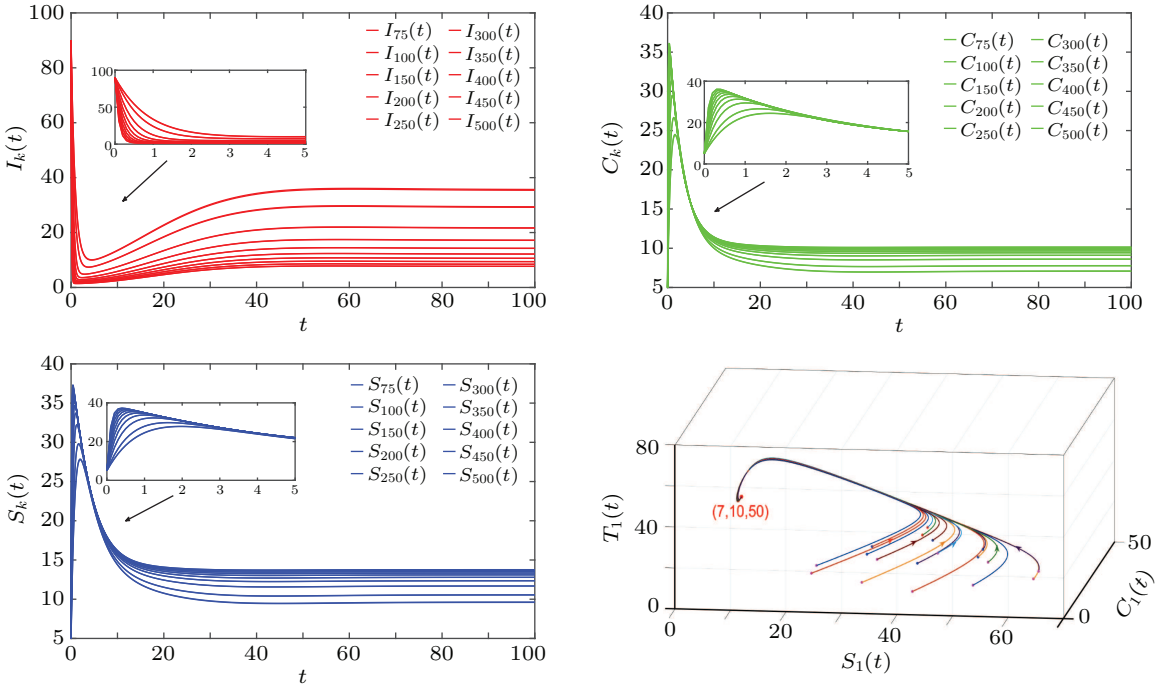


Fig. 4. The stability of rumor-free equilibrium E_0 with $R_0 > 1$.

Table 5. Numerical values at different time node degree under $R_0 < 1$.

t	(C_{50}, S_{50}, T_{50})	$(C_{100}, S_{100}, T_{100})$	$(C_{150}, S_{150}, T_{150})$	$(C_{200}, S_{200}, T_{200})$	$(C_{250}, S_{250}, T_{250})$	$(C_{300}, S_{300}, T_{300})$
0	(60,5,10)	(50,10,20)	(40,15,30)	(30,20,40)	(20,25,50)	(40,5,10)
50	(36,8.8,21)	(30,10,30)	(24,11,39)	(18,13,48)	(12,14,57)	(31,5,9)
100	(0.2,0.2,18)	(0.2,0.19,19)	(0.2,0.19,21)	(0.2,0.18,22)	(0.2,0.17,23)	(0.3,0.19,20)
125	(0.01,0.01,6.7)	(0.01,0.01,7.1)	(0.01,0.01,7.3)	(0.01,0.01,8.1)	(0.01,0.01,8.2)	(0.01,0.01,8.4)
150	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)
160	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)

Table 6. Numerical values at different time node degree under $R_0 > 1$.

t	(C_{50},S_{50},T_{50})	$(C_{100},S_{100},T_{100})$	$(C_{150},S_{150},T_{150})$	$(C_{200},S_{200},T_{200})$	$(C_{250},S_{250},T_{250})$	$(C_{300},S_{300},T_{300})$
0	(20,50,0)	(25,60,0)	(30,60,0)	(30,50,0)	(40,40,0)	(35,30,0)
50	(27,54,11)	(26,59,9)	(29,58,8)	(35,50,9)	(41,43,9)	(41,39,11)
100	(18,36,42)	(17,38,41)	(18,37,40)	(21,35,40)	(24,33,39)	(24,32,40)
125	(7,10,57)	(7.1,10.6,57)	(8,11,58)	(7.3,11.2,60.7)	(7.9,11,63.2)	(9,14,68)
150	(6.7,10,47.3)	(7,10,49)	(7.1,10.6,49)	(7.9,11,55)	(8,11,56)	(8.4,12.6,59)
160	(6.7,10,47.3)	(7,10,49)	(7.1,10.6,49)	(7.9,11,55)	(8,11,56)	(8.4,12.6,59)

5.2. Effectiveness of the proposed stochastic stabilization method

To stabilize the rumor, we choose different control intensities σ by adjusting control time to compare the stabilization effects (see Figs. 5–7). We fix parameter values in the second column of Table 4. If $\sigma = 0$, then rumor will propagate as shown in Fig. 3. In order to check the effectiveness of the proposed stabilization strategy, we choose three groups of perturb intensity and the perturb time ratio to show the power of the proposed stochastic stabilization.

Firstly, assuming that the perturb intensity $\sigma = 0.8$ and the perturb time ratio $w = 0.2$, one can calculate that $\sigma^2 = 0.64$, and $2(\delta+\mu)(R_0-1)/w = 1.4$. By Theorem 2, the perturb intensity is not enough to control rumor spreading, so the rumor will propagate in Fig. 5.

Secondly, we fix the perturb intensity $\sigma = 0.8$, but increase the perturb time ratio $w = 0.2$ to 0.5. By calculation, the stabilization condition can be satisfied. The simulation result verifies our theoretical proof, and we observe that $(I_k(t),C_k(t),S_k(t),T_k(t))$ converges to E_0 in Fig. 6.

Finally, we increase the perturb time ratio w to 0.8 in Fig. 7, and observe that $(I_k(t),C_k(t),S_k(t)\ ,T_k(t))$ converges to E_0 faster than that for $w = 0.2$ (Fig. 5) and $w = 0.2$ (Fig. 6). So we can conclude that the proposed stochastic stabilization method can control the rumor propagation, and the bigger we apply the control intensity, the better control effect we have. Through above analysis, the optimal control strategy proposed in this paper is effective.

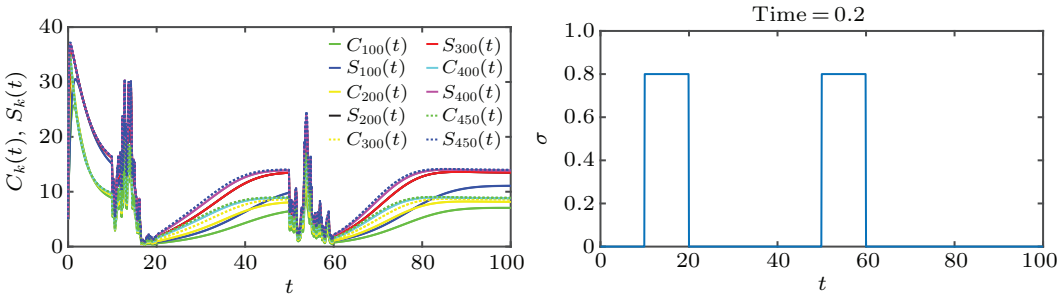


Fig. 5. Control ratio time $w = 0.2$.

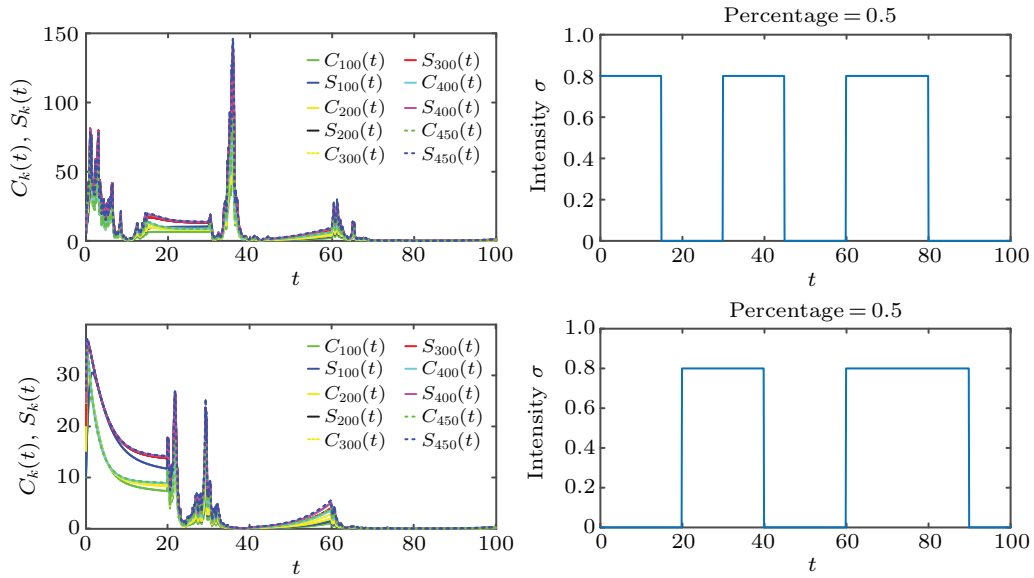


Fig. 6. Influence of the control ratio time $w = 0.5$ on convergence under two different intermittent strategies.

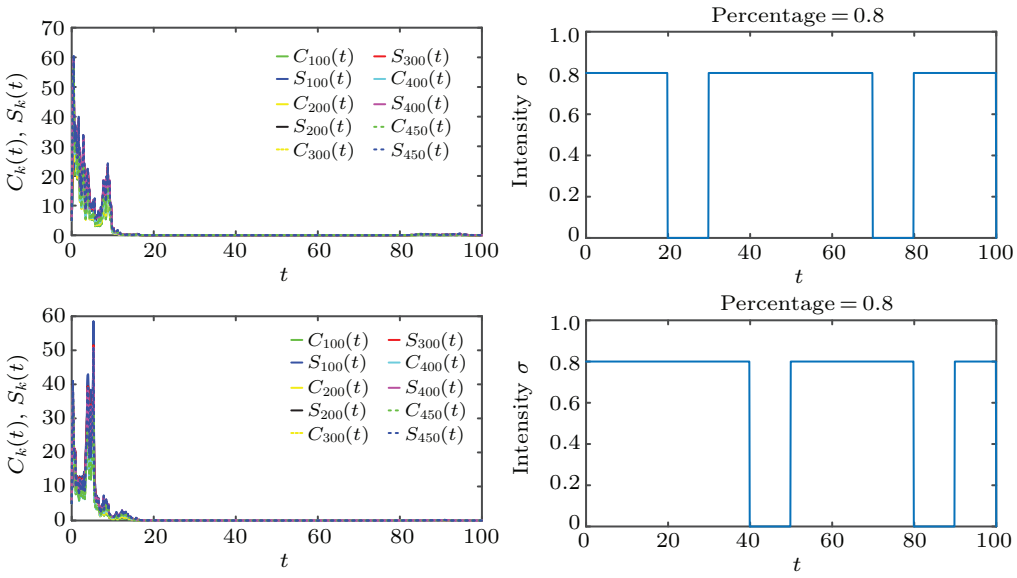


Fig. 7. Influence of the control ratio time $w = 0.8$ on convergence under two different intermittent strategies.

Remark 5 Figures 3 and 4 show the global stability of E_0 and E^*. In actual situation, rumor management department can predict the rumor propagation trend by calculating R_0: If $R_0 < 1$, there is no need to add any extra control. If $R_0 > 1$, control strategy need to be carried out. Figure 5 shows the power of stochastic control strategy under different control intensity. In actual situation, rumor management department can block the information propagation channel randomly. For example, a rumor is expected to spread widely in Weibo, and rumor management department can randomly prevent some rumor spreader to spreading rumor, instead of prevent all spreaders. The speed of the control is decided by the control intensity σ and the control time ratio τ.

6. Conclusions

In this paper, we propose a new ICST rumor propagation model that considers the social network connection heterogeneity and the guidance mechanism. According to the node degree, system parameters permit to be different, which is a generalization of Ref. [18]. Using the constructed mean field theory, we obtain the threshold R_0 and equilibria. Global stability of the rumor-free equilibrium and rumor-prevailing equilibrium are also be proved by Lyapunov functions under $R_0 \leq 0$ and $R_0 > 1$. Our studies show that the rumor threshold depends on the social network density and the guidance rate, and the positive guidance can reduce the rumor propagation force.

For rumor prevailing situation, we design an aperiodic intermittent stochastic controller to suppress rumor transmission. The expression of the minimum control intensity is calculated by Ito formula and exponential martingale inequality, which is inversely proportional to control time. It is highly recommended that control time and control intensity can be adjusted more flexibly to satisfy different control requirements. Our stochastic control method provides a new theoretical strategy for rumor propagation.

The simulation results show that rumor will naturally die out under $R_0 \leq 0$, regardless of initial points. When the threshold $R_0 > 1$, the rumor will prevalent in nature. But the rumor spreader can be stabilized to zero after an aperiodic intermittent noise controller, the greater the control intensity, the better the stabilization effect.

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