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Quantum fields presentation and generating functions of symplectic Schur functions and symplectic universal characters

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This paper is concerned with construction of quantum fields presentation and generating functions of symplectic Schur functions and symplectic universal characters. The boson–fermion correspondence for these symmetric functions have been presented. In virtue of quantum fields, we derive a series of infinite order nonlinear integrable equations, namely, universal character hierarchy, symplectic KP hierarchy and symplectic universal character hierarchy, respectively. In addition, the solutions of these integrable systems have been discussed.

Keywords: quantum fields, generating functions, integrable systems, symmetric functions, boson–fermion correspondence

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1. Introduction

Symmetric functions play a significant role in mathematical physics[1] especially in integrable systems.[2–6] The most elementary symmetric functions are Schur functions which are the solutions of differential equations in Kadomtsev–Petviashvili (KP) hierarchy.[7,8] Symplectic Schur function is the character of the irreducible highest weight representation of the symplectic group. Baker[9] presented the quantum fields presentation of the symmetric functions and studied the properties and applications of the vertex operators. Later, the generalization of symplectic Schur functions as well as the corresponding integrable systems have been well investigated.[10,11]

The universal character proposed by Koike[12] is the extension of the Schur function. Tsuda[13] developed a novel integrable system called universal character (UC) hierarchy with the property of the universal character, which is a generalization of the KP hierarchy. Then, Painlevé equations and their higher order analogues have been derived from KP hierarchy and UC hierarchy via reduced Virasoro constraints.[14] Later on, the universal characters with various systems including q-Painlevé systems, q-difference Painlevé equations, phase model and topological strings are well discussed.[15–19] Recently, Wang *et al.*[10] established a symplectic universal character (SUC) hierarchy which can be regarded as the generalization of SKP hierarchy and indicated the SUC hierarchy characterized by symplectic universal characters.

Vertex operators discovered in string theory are important research objects in mathematics.[20–22] Several symmetric functions can be realized by acting certain vertex operators on the constant function 1.[23–27] It has been shown that symmetric functions are closely connected to boson–fermion correspondence.[28] In Ref. [26], based upon the quantum fields of operators, the twisted quantum fields presentation of Hall–Littlewood polynomials has been derived and the authors extended a new deformed boson–fermion correspondence. Inspired by the connection between quantum fields and generating functions of the symmetric functions, in this paper, connecting with the charged free fermions, we will concentrate on the construction of quantum fields of symplectic Schur functions, universal characters and symplectic universal characters. Furthermore, the generating functions of the corresponding symmetric functions are obtained by the action of the vertex operators on the constant function 1. By means of the quantum fields of operators, we obtain a novel boson–fermion correspondence of these symmetric functions and a series of infinite order nonlinear integrable equations. Meanwhile, the solutions of these nonlinear integrable equations have been studied.

The present paper is built up as follows. In Section 2, we begin with a review of the elementary basics of symmetric functions involving Schur functions. Section 3 is devoted to construction of quantum fields and generating functions of universal characters. In Section 4, we derive the quantum fields presentation and generating functions of symplectic Schur function and symplectic universal character. Meanwhile, we present the corresponding integrable systems of these symmetric functions and develop their solutions. The last section is a summary and discussion.

2. An introductory discussion of symmetric functions

We retrospect some basic facts about symmetric functions including Schur functions $S_\lambda(\boldsymbol{x})$ and the action of operators on the ring of symmetric functions.

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2.1. Symmetric functions

The ring $\Lambda(\boldsymbol{x})$ is made up of symmetric functions in variables $\boldsymbol{x}=(x_1,x_2,\ldots)$. The rth elementary symmetric function e_r is the sum of all products of r distinct variables x_i, and it can be written as

$$e_r(\boldsymbol{x})=\sum_{i_1<i_2<\cdots<i_r<\infty} x_{i_1}x_{i_2}\ldots x_{i_r},\quad \text{for}\quad r\geq 1. \tag{1}$$

The rth complete symmetric function h_r is the sum of all monomials of total degree r in variables $x_1,x_2,\ldots$, so that

$$h_r(\boldsymbol{x})=\sum_{i_1\leq i_2\leq\ldots\leq i_r\leq\infty} x_{i_1}x_{i_2}\cdots x_{i_r},\quad \text{for}\quad r\geq 1. \tag{2}$$

The rth power sum p_r is

$$p_r(\boldsymbol{x})=\sum_i x_i^r. \tag{3}$$

The generating functions for the $e_r(\boldsymbol{x}),h_r(\boldsymbol{x})$, and $p_r(\boldsymbol{x})$ are given by

$$\begin{aligned} E(u) &= \sum_{k\geq 0} e_k(\boldsymbol{x})u^k=\prod_{i\geq 1}(1+x_iu),\\ H(u) &= \sum_{k\geq 0} h_k(\boldsymbol{x})u^k=\prod_{i\geq 1}\frac{1}{1-x_iu},\\ P(u) &= \sum_{k\geq 1} p_k(\boldsymbol{x})u^{k-1}=\frac{H'(u)}{H(u)}, \end{aligned} \tag{4}$$

with the convention $h_{-k}(\boldsymbol{x})=e_{-k}(\boldsymbol{x})=p_{-k}(\boldsymbol{x})=0$ for $k>0$ and $h_0=e_0=p_0=1$. The relations between the symmetric functions lead to

$$\begin{aligned} &H(u)E(-u)=1,\\ &H(u)=\exp\left(\sum_{n\geq 1}\frac{p_n}{n}u^n\right),\\ &E(u)=\exp\left(-\sum_{n\geq 1}\frac{(-1)^n p_n}{n}u^n\right). \end{aligned} \tag{5}$$

Let $\Lambda=\mathbb{C}[p_1,p_2,\ldots]$ and consider the boson Fock space $\mathcal{B}=\mathbb{C}[z,z^{-1}]\bigotimes\Lambda$. Note that we have the charge decomposition

$$\mathcal{B}=\bigoplus_{m\in\mathbb{Z}}\mathcal{B}^m,\ \text{ where }\ \mathcal{B}^m=z^m\cdot\mathbb{C}[p_1,p_2,\ldots]=z^m\Lambda. \tag{6}$$

The Heisenberg algebra is a complex Lie algebra generated by $\{a_n\}_{n\in\mathbb{Z}}$ and central element 1 with commutation relation

$$[a_m,a_n]=m\delta_{m,-n}\cdot 1. \tag{7}$$

Combine generators into a formal distribution $a(u)=\sum_{n\in\mathbb{Z}}a_nu^{-n-1}$, then the commutation relation can be written as

$$[a(u),a(v)]=\partial_v\delta(u,v), \tag{8}$$

where $\delta(u,v)=\sum_{\substack{k,m\in\mathbb{Z}\\ k+m=-1}}u^kv^m$ is the most famous formal delta-distribution.

The Heisenberg algebra has a natural action on the graded components $a_n:\mathcal{B}^m\to\mathcal{B}^m$ defined by the multiplication and differentiation operators:

$$a_{-n}=\frac{p_n}{n}\ (n>0),\ \ a_n=\frac{\partial}{\partial p_n}\ (n>0),\ \ a_0=m. \tag{9}$$

For each symmetric function $f\in\Lambda$, let $f^\perp:\Lambda\to\Lambda$ be the adjoint of multiplication by f

$$\langle f^\perp g,\omega\rangle=\langle g,f\omega\rangle,\quad g,f,\omega\in\Lambda. \tag{10}$$

Let us consider generating functions of the adjoint operators

$$E^\perp(u)=\sum_{k\geq 0}\frac{e_k^\perp}{u^k},\quad H^\perp(u)=\sum_{k\geq 0}\frac{h_k^\perp}{u^k}. \tag{11}$$

It is straightforward to show

$$\begin{aligned} E^\perp(u)&=\exp\left(-\sum_{k\geq 1}(-1)^k\frac{\partial}{\partial p_k}\frac{1}{u^k}\right),\\ H^\perp(u)&=\exp\left(\sum_{k\geq 1}\frac{\partial}{\partial p_k}\frac{1}{u^k}\right). \end{aligned} \tag{12}$$

Proposition 1 The generating functions satisfy the following relations (cf. Ref. [4]):

$$\begin{aligned} &\left(1-\frac{v}{u}\right)E^\perp(u)E(v)=E(v)E^\perp(u),\\ &\left(1-\frac{v}{u}\right)H^\perp(u)H(v)=H(v)H^\perp(u),\\ &H^\perp(u)E(v)=\left(1+\frac{v}{u}\right)E(v)H^\perp(u),\\ &E^\perp(u)H(v)=\left(1+\frac{v}{u}\right)H(v)E^\perp(u). \end{aligned} \tag{13}$$

2.2. Schur functions and KP hierarchy

For a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_l)$, the Schur function $S_\lambda(\boldsymbol{x})$ can be expressed as a polynomial in the complete symmetric functions h_r, the formula is

$$\begin{aligned} S_\lambda(\boldsymbol{x}) &= \det\left(h_{\lambda_i-i+j}(\boldsymbol{x})\right)_{1\leq i,j\leq l}\\ &= \prod_{1\leq i<j\leq l}(1-R_{\lambda_i\lambda_j})h_\lambda(\boldsymbol{x}), \end{aligned} \tag{14}$$

where $R_{\lambda_i\lambda_j}$ is a raising operator for partitions, so that for $i<j$,

$$\begin{aligned} &R_{\lambda_i\lambda_j}\cdot(\lambda_1,\ldots,\lambda_i,\ldots,\lambda_j,\ldots,\lambda_l)\\ &=(\lambda_1,\ldots,\lambda_i+1,\ldots,\lambda_j-1,\ldots,\lambda_l), \end{aligned} \tag{15}$$

and extend this to an action on the functions $h_\lambda=h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_l}$ by $R_{ij}h_\lambda=h_{R_{ij}\lambda}$.

Let $R(u)$ act on the elements of the form z^mf, $f\in\Lambda$, $m\in\mathbb{Z}$, $R(u):\mathcal{B}\to\mathcal{B}$ is defined as[29]

$$R(u)\left(z^mf(\boldsymbol{x})\right)=z^{m+1}u^{m+1}f. \tag{16}$$

Then it leads to

$$R^{-1}(u)\left(z^m f(\boldsymbol{x})\right) = z^{m-1}u^{-m}f. \tag{17}$$

Operators $R^{\pm 1}(u)$ map the grading of the boson Fock space $\mathcal{B}^{(m)}$ into $\mathcal{B}^{(m\pm 1)}$.

Define the formal distributions of operators acting on the space $\mathcal{B}$ via the action $R^{\pm 1}(u)$ (cf. Ref. [30])

$$\begin{aligned}\Phi^+(u) &= u^{-1}R(u)H(u)E^\perp(-u),\\ \Phi^-(u) &= R^{-1}(u)E(-u)H^\perp(u).\end{aligned} \tag{18}$$

Notice that

$$\begin{aligned}\Phi^+(u)|_{\mathcal{B}^{(m)}} &= zu^m H(u)E^\perp(-u),\\ \Phi^-(u)|_{\mathcal{B}^{(m)}} &= z^{-1}u^{-m}E(-u)H^\perp(u).\end{aligned} \tag{19}$$

It is proved that Eq. (19) defines quantum fields $\Phi^\pm(u)$ of operators acting on $\mathcal{B}$.[30] The operators $\Phi^\pm$ can be expressed as

$$\Phi^\pm = \sum_{k\in\mathbb{Z}} \Phi^\pm_{k+\frac{1}{2}} u^{\pm k}, \tag{20}$$

where the operators $\Phi^\pm_{k+\frac{1}{2}}$ are called the charged free fermions which change the charge by ± 1.

Proposition 2 The formal distributions $\Phi^\pm(u)$ satisfy the algebraic relation of charged free fermions

$$\begin{aligned}&\Phi^\pm(u)\Phi^\pm(v) + \Phi^\pm(v)\Phi^\pm(u) = 0,\\ &\Phi^+(u)\Phi^-(v) + \Phi^-(v)\Phi^+(u) = \delta(u,v).\end{aligned} \tag{21}$$

Equivalently, Eq. (21) have the following expressions:

$$\begin{aligned}&\Phi^\pm_{k+\frac{1}{2}}\Phi^\pm_{l+\frac{1}{2}} + \Phi^\pm_{l+\frac{1}{2}}\Phi^\pm_{k+\frac{1}{2}} = 0,\\ &\Phi^+_{k+\frac{1}{2}}\Phi^-_{l+\frac{1}{2}} + \Phi^-_{l+\frac{1}{2}}\Phi^+_{k+\frac{1}{2}} = \delta_{k,l-1}.\end{aligned} \tag{22}$$

The following proposition describes the quantum fields presentation of Schur functions.

Proposition 3

$$\Phi^+(u_1)\cdots\Phi^+(u_l)(1) = z^l u_1^{l-1}\cdots u_l^0 G(u_1,\ldots,u_l),$$

where

$$G(u_1,\ldots,u_l) = \prod_{1\le i<j\le l}\left(1-\frac{u_j}{u_i}\right)\prod_{i=1}^{l}H(u_i). \tag{23}$$

It is widely known that $G(u_1,\ldots,u_l)$ as a formal power series is a generating function of Schur functions.[3,4] Therefore, for any partition $\lambda = (\lambda_1 \ge \cdots \ge \lambda_l \ge 0)$ the coefficient of $u_1^{\lambda_1}\cdots u_l^{\lambda_l}$ is properly a Schur function $G_{(\lambda_1,\ldots,\lambda_l)} = S_\lambda(\boldsymbol{x})$.

Proposition 4 (The classical boson–fermion correspondence) From formula (9), we easily get the bosonic form of the formal distributions $\Phi^\pm(u)$:

$$\begin{aligned}\Phi^+(u) &= u^{-1}R(u)\exp\left(\sum_{n\ge1}a_{-n}u^n\right)\exp\left(-\sum_{n\ge1}a_n\frac{1}{u^n}\right),\\ \Phi^-(u) &= R^{-1}(u)\exp\left(-\sum_{n\ge1}a_{-n}u^n\right)\exp\left(\sum_{n\ge1}a_n\frac{1}{u^n}\right).\end{aligned} \tag{24}$$

Definition 1 For an unknown function $\tau = \tau(x)$, the bilinear equation

$$\Omega(\tau\otimes\tau) = 0 \tag{25}$$

is called the KP hierarchy, where

$$\begin{aligned}\Omega &= \mathrm{Res}_{u=0}\left(\frac{1}{u}\Phi^+(u)\otimes\Phi^-(u)\right)\\ &= \sum_{k-l=-1}\Phi^+_{k+\frac{1}{2}}\otimes\Phi^-_{l+\frac{1}{2}}.\end{aligned} \tag{26}$$

Proposition 5 All the Schur functions $S_\lambda(\boldsymbol{x})$ are the solutions of the KP hierarchy.

3. Universal characters and UC hierarchy

In this section, let us construct quantum fields and generating functions of the universal characters $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ and define an integrable system called UC hierarchy which are different from that in Ref. [13].

For each pair of partitions

$$[\lambda,\mu] = [(\lambda_1,\lambda_2,\ldots,\lambda_l),(\mu_1,\mu_2,\ldots,\mu_m)],$$

the universal character $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ is defined as

$$S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = \det\left(h_{\mu_{m-i+1}+i-j}(\boldsymbol{y}),\ \begin{matrix}1\le i\le m\\ m+1\le i\le l+m\end{matrix}\ h_{\lambda_{i-m}-i+j}(\boldsymbol{x})\right)_{1\le i,j\le l+m}. \tag{27}$$

In Ref. [10], the operator $D_{\lambda_i\mu_a}$ acts on $h_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ by

$$D_{\lambda_i\mu_a}h_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = h_{[\lambda',\mu']}(\boldsymbol{x},\boldsymbol{y}), \tag{28}$$

where $[\lambda',\mu'] = [(\lambda_1,\ldots,\lambda_i-1,\ldots,\lambda_l),(\mu_1,\ldots,\mu_a-1,\ldots,\mu_m)]$ and $h_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = h_\lambda(\boldsymbol{x})h_\mu(\boldsymbol{y})$. Therefore, the universal character $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ can also be written as

$$\begin{aligned}S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = &\prod_{1\le i<j\le l}(1-R_{\lambda_i\lambda_j})\prod_{1\le a<b\le m}(1-R_{\mu_a\mu_b})\\ &\times\prod_{\substack{1\le i\le l\\ 1\le a\le m}}(1-D_{\lambda_i\mu_a})h_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}).\end{aligned} \tag{29}$$

Define the formal distribution $\Phi^{\pm}(u)$ and $\Psi^{\pm}(u)$ of operators acting on $\mathcal{B}$ as

$$\begin{aligned}
\Phi^{+}(u) &= u^{-1}R(u)H(u)E'^{\perp}(-1/u)E^{\perp}(-u),\\
\Phi^{-}(u) &= R^{-1}(u)E(-u)H'^{\perp}(1/u)H^{\perp}(u),\\
\Psi^{+}(u) &= u^{-1}R(u)H'(u)E^{\perp}(-1/u)E'^{\perp}(-u),\\
\Psi^{-}(u) &= R^{-1}(u)E'(-u)H^{\perp}(1/u)H'^{\perp}(u),
\end{aligned} \tag{30}$$

where $E'(u)$ and $H'(u)$ denote the generating functions for the $e_r(\boldsymbol{y})$, $h_r(\boldsymbol{y})$, $\boldsymbol{y}=(y_1,y_2,\ldots)$ and their adjoint operators hold for the variable $\boldsymbol{y}$.

From Eq. (16), we obtain

$$\begin{aligned}
\Phi^{+}(u)|_{\mathcal{B}^{(m)}} &= zu^{m}H(u)E'^{\perp}(-1/u)E^{\perp}(-u),\\
\Phi^{-}(u)|_{\mathcal{B}^{(m)}} &= z^{-1}u^{-m}E(-u)H'^{\perp}(1/u)H^{\perp}(u),\\
\Psi^{+}(u)|_{\mathcal{B}^{(m)}} &= zu^{m}H'(u)E^{\perp}(-1/u)E'^{\perp}(-u),\\
\Psi^{-}(u)|_{\mathcal{B}^{(m)}} &= z^{-1}u^{-m}E'(-u)H^{\perp}(1/u)H'^{\perp}(u).
\end{aligned} \tag{31}$$

Proposition 6 The following relations of quantum fields $\Phi^{\pm}(u)$ and $\Psi^{\pm}(u)$ hold:

$$[\Phi^{\pm}(u),\Phi^{\pm}(v)]_{+}=0,\quad [\Psi^{\pm}(u),\Psi^{\pm}(v)]_{+}=0, \tag{32}$$

$$[\Phi^{+}(u),\Phi^{-}(v)]_{+}=\delta(u,v),$$

$$[\Psi^{+}(u),\Psi^{-}(v)]_{+}=\delta(u,v), \tag{33}$$

$$v\Phi^{+}(u)\Psi^{+}(v)-u\Psi^{+}(v)\Phi^{+}(u)=0,$$

$$u\Phi^{+}(u)\Psi^{-}(v)-v\Psi^{-}(v)\Phi^{+}(u)=0, \tag{34}$$

$$u\Phi^{-}(u)\Psi^{+}(v)-v\Psi^{+}(v)\Phi^{-}(u)=0,$$

$$v\Phi^{-}(u)\Psi^{-}(v)-u\Psi^{-}(v)\Phi^{-}(u)=0, \tag{35}$$

where $[X,Y]_{+}=_{\text{def}}XY+YX$.

Proof By Proposition 1, for any $m\in\mathbb{Z}$ and $f\in\Lambda$, we have

$$\Phi^{+}(u)\Phi^{+}(v)(z^{m}f)=z^{m+2}(uv)^{m}(u-v)H(u)H(v)E'^{\perp}(-1/u)E^{\perp}(-u)E'^{\perp}\left(-\frac{1}{v}\right)E^{\perp}(-v)(f),$$

$$\Phi^{-}(u)\Phi^{-}(v)(z^{m}f)=z^{m-2}(uv)^{-m}(u-v)E(-u)E(-v)H'^{\perp}(1/u)H^{\perp}(u)H'^{\perp}\left(\frac{1}{v}\right)H^{\perp}(v)(f).$$

Exchanging the roles of u and v and adding the corresponding products in quantum fields, the first formula of Eq. (32) can be obtained.

For the first formula in Eq. (33), we can write

$$\Phi^{+}(u)\Phi^{-}(v)(z^{m}f)=\frac{1}{u-v}z^{m}\left(\frac{u}{v}\right)^{m}H(u)E(-v)E'^{\perp}(-1/u)E^{\perp}(-u)H'^{\perp}\left(\frac{1}{v}\right)H^{\perp}(v)(f),$$

$$\Phi^{-}(v)\Phi^{+}(u)(z^{m}f)=\frac{1}{v-u}z^{m}\left(\frac{u}{v}\right)^{m}H(u)E(-v)E'^{\perp}(-1/u)E^{\perp}(-u)H'^{\perp}\left(\frac{1}{v}\right)H^{\perp}(v)(f).$$

According to $\frac{1}{u-v}+\frac{1}{v-u}=\sum\limits_{n\in\mathbb{Z}}\frac{u^{n}}{v^{n+1}}=\delta(u,v)$ and $\delta(u,v)f(u)=\delta(u,v)f(v)$, we have

$$\left(\Phi^{+}(u)\Phi^{-}(v)+\Phi^{-}(v)\Phi^{+}(u)\right)(z^{m}f)=\delta(u,v)z^{m}\left(\frac{u}{v}\right)^{m}H(u)E(-v)E'^{\perp}(-1/u)E^{\perp}(-u)H'^{\perp}\left(\frac{1}{v}\right)H^{\perp}(v)(f)=\delta(u,v)z^{m}f.$$

Therefore, $\Phi^{+}(u)\Phi^{-}(v)+\Phi^{-}(v)\Phi^{+}(u)=\delta(u,v)$. The second formula in the second line can be derived using the similar procedure.

For the first formula in Eq. (34), by applying operator $\Phi^{+}(u)\Psi^{+}(v)$ and $\Psi^{+}(v)\Phi^{+}(u)$ to $z^{m}f$, it is simple to show

$$\Phi^{+}(u)\Psi^{+}(v)(z^{m}f)=z^{m+2}u^{m+1}v^{m}(1-uv)H(u)H'(v)E'^{\perp}(-1/u)E^{\perp}(-u)E'^{\perp}\left(-\frac{1}{v}\right)E^{\perp}(-v)(f),$$

$$\Psi^{+}(v)\Phi^{+}(u)(z^{m}f)=z^{m+2}u^{m}v^{m+1}(1-uv)H(u)H'(v)E'^{\perp}(-1/u)E^{\perp}(-u)E'^{\perp}\left(-\frac{1}{v}\right)E^{\perp}(-v)(f).$$

Multiply the first formula by v and the second formula by u, and then subtract to obtain $v\Phi^{+}(u)\Psi^{+}(v)-u\Psi^{+}(v)\Phi^{+}(u)=0$. The proof method of the rest of the proposition is similar and will not be given here.

Proposition 6 immediately indicates that the formal distributions $\Phi^{\pm}(u)$ and $\Psi^{\pm}(u)$ provide the quantum fields presentation of universal characters. Let

$$\begin{aligned}
&G(u_1,\ldots,u_l,v_1,\ldots,v_m)\\
&=\prod_{1\le i<j\le l}\left(1-\frac{u_j}{u_i}\right)\prod_{1\le a<b\le m}\left(1-\frac{v_b}{v_a}\right)\\
&\quad\times\prod_{\substack{1\le i\le l\\1\le a\le m}}(1-u_iv_a)\prod_{i=1}^{l}H(u_i)\prod_{a=1}^{m}H'(v_a).
\end{aligned} \tag{36}$$

It is expanded into rational function form

$$G(u_1,\dots,u_l,v_1,\dots,v_m)=\sum_{\substack{\lambda_1,\dots,\lambda_l\geq 0\\ \mu_1,\dots,\mu_m\geq 0}} G_{[\lambda,\mu]}u_1^{\lambda_1}\cdots u_l^{\lambda_l}v_1^{\mu_1}\cdots v_m^{\mu_m}.$$

Therefore, for any partition $\lambda=(\lambda_1\geq\cdots\geq\lambda_l\geq 0)$ and $\mu=(\mu_1\geq\cdots\geq\mu_m\geq 0)$ the coefficient of $u_1^{\lambda_1}\cdots u_l^{\lambda_l}v_1^{\mu_1}\cdots v_m^{\mu_m}$ is exactly universal characters $G_{[\lambda,\mu]}=S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$.

Proposition 7 (The quantum fields presentation of universal characters)

$$\begin{aligned}&\Phi^+(u_1)\cdots\Phi^+(u_l)\Psi^+(v_1)\cdots\Psi^+(v_m)(1)\\ &=z^{l+m}u_1^{m+l-1}\cdots u_l^m v_1^{m-1}\cdots v_m^0 G(u_1,\dots,u_l,v_1,\dots,v_m).\end{aligned}\tag{37}$$

Proof From the definitions of Eq. (30) and Proposition 1, we immediately discover

$$\begin{aligned}&\Phi^+(u_1)\cdots\Phi^+(u_l)\Psi^+(v_1)\cdots\Psi^+(v_m)(1)\\ &=z^{l+m}\prod_{i=1}^{l}u_i^{m+l-i}\prod_{a=1}^{m}v_a^{m-a}\prod_{1\leq i<j\leq l}\left(1-\frac{u_j}{u_i}\right)\\ &\times\prod_{1\leq a<b\leq m}\left(1-\frac{v_b}{v_a}\right)\prod_{\substack{1\leq i\leq l\\ 1\leq a\leq m}}(1-u_iv_a)\\ &\times\prod_{i=1}^{l}H(u_i)\prod_{a=1}^{m}H'(v_a)\prod_{i=1}^{l}E'^{\perp}(-1/u_i)E^{\perp}(-u_i)\\ &\times\prod_{a=1}^{m}E^{\perp}(-1/v_a)E'^{\perp}(-v_a)(1),\end{aligned}$$

which equals to $z^{l+m}u_1^{m+l-1}\cdots u_l^m v_1^{m-1}\cdots v_m^0 G(u_1,\dots,u_l,v_1,\dots,v_m)$ since $E^{\perp}(-u)(1)=1$ and $E'^{\perp}(-v)(1)=1$.

Corollary 1 The generating functions $G(u_1,\dots,u_l,v_1,\dots,v_m)$ of universal characters and $G(u_1,\dots,u_l)$ for Schur functions are related by

$$\begin{aligned}&G(u_1,\dots,u_l,v_1,\dots,v_m)\\ &=\prod_{1\leq a<b\leq m}\left(1-\frac{v_b}{v_a}\right)\prod_{\substack{1\leq i\leq l\\ 1\leq a\leq m}}(1-u_iv_a)\\ &\times\prod_{a=1}^{m}H'(v_a)G(u_1,\dots,u_l).\end{aligned}\tag{38}$$

Proposition 8 (The boson–fermion correspondence for universal characters)

(a) From the formula (9), we easily get the bosonic form of the fields $\Phi^{\pm}(u)$ and $\Psi^{\pm}(u)$:

$$\begin{aligned}\Phi^+(u)&=u^{-1}R(u)\exp\left(\sum_{n\geq 1}a_{-n}u^n\right)\exp\left(-\sum_{n\geq 1}a'_nu^n\right)\\ &\times\exp\left(-\sum_{n\geq 1}a_n\frac{1}{u^n}\right),\\ \Phi^-(u)&=R^{-1}(u)\exp\left(-\sum_{n\geq 1}a_{-n}u^n\right)\exp\left(\sum_{n\geq 1}a'_nu^n\right)\\ &\times\exp\left(\sum_{n\geq 1}a_n\frac{1}{u^n}\right),\\ \Psi^+(u)&=u^{-1}R(u)\exp\left(\sum_{n\geq 1}a'_{-n}u^n\right)\exp\left(-\sum_{n\geq 1}a_nu^n\right)\\ &\times\exp\left(-\sum_{n\geq 1}a'_n\frac{1}{u^n}\right),\\ \Psi^-(u)&=R^{-1}(u)\exp\left(-\sum_{n\geq 1}a'_{-n}u^n\right)\exp\left(\sum_{n\geq 1}a_nu^n\right)\\ &\times\exp\left(\sum_{n\geq 1}a'_n\frac{1}{u^n}\right),\end{aligned}\tag{39}$$

where a'_{-n} and a'_n denote the multiplication and differentiation operators about the variable $\boldsymbol{y}$.

(b) Let $\Phi^{\pm}(u)=\sum_{k\in\mathbb{Z}}\Phi^{\pm}_{k+\frac{1}{2}}u^{\pm k}$ and $\Psi^{\pm}(u)=\sum_{m\in\mathbb{Z}}\Psi^{\pm}_{m+\frac{1}{2}}u^{\pm m}$ satisfy the relations (32)–(35), and let

$$\begin{aligned}&\Phi^+(u)_+=\sum_{k\geq 1}\Phi^+_{k+\frac{1}{2}}u^k,\quad \Phi^+(u)_-=\sum_{k\leq 0}\Phi^+_{k+\frac{1}{2}}u^k,\\ &\Psi^+(u)_+=\sum_{m\geq 1}\Psi^+_{m+\frac{1}{2}}u^m,\quad \Psi^+(u)_-=\sum_{m\leq 0}\Psi^+_{m+\frac{1}{2}}u^m.\end{aligned}\tag{40}$$

Then the relations (32)–(35) have the following expressions:

$$\begin{aligned}&\Phi^{\pm}_{k+\frac{1}{2}}\Phi^{\pm}_{l+\frac{1}{2}}+\Phi^{\pm}_{l+\frac{1}{2}}\Phi^{\pm}_{k+\frac{1}{2}}=0,\\ &\Psi^{\pm}_{m+\frac{1}{2}}\Psi^{\pm}_{n+\frac{1}{2}}+\Psi^{\pm}_{n+\frac{1}{2}}\Psi^{\pm}_{m+\frac{1}{2}}=0,\\ &\Phi^+_{k+\frac{1}{2}}\Phi^-_{l+\frac{1}{2}}+\Phi^-_{l+\frac{1}{2}}\Phi^+_{k+\frac{1}{2}}=\delta_{k,l-1},\\ &\Psi^+_{m+\frac{1}{2}}\Psi^-_{n+\frac{1}{2}}+\Psi^-_{n+\frac{1}{2}}\Psi^+_{m+\frac{1}{2}}=\delta_{m,n-1},\\ &\Phi^+_{k+\frac{1}{2}}\Psi^+_{m+\frac{1}{2}}-\Psi^+_{m+\frac{3}{2}}\Phi^+_{k-\frac{1}{2}}=0,\\ &\Phi^+_{k+\frac{1}{2}}\Psi^-_{n+\frac{1}{2}}-\Psi^-_{n+\frac{3}{2}}\Phi^+_{k+\frac{3}{2}}=0,\\ &\Phi^-_{l+\frac{1}{2}}\Psi^+_{m+\frac{1}{2}}-\Psi^+_{m-\frac{1}{2}}\Phi^-_{l-\frac{1}{2}}=0,\\ &\Phi^-_{l+\frac{1}{2}}\Psi^-_{n+\frac{1}{2}}-\Psi^-_{n-\frac{1}{2}}\Phi^-_{l+\frac{3}{2}}=0,\end{aligned}\tag{41}$$

and the coefficients of the formal distribution $a(u)=:\Phi^+(u)\cdot\Phi^-(u):=\Phi^+(u)_+\Phi^-(u)-\Phi^-(u)\Phi^+(u)_-$, $a'(u)=:\Psi+(u)\cdot\Psi^-(u):=\Psi^+(u)_+\Psi^-(u)-\Psi^-(u)\Psi^+(u)_-$ satisfy the relations of the Heisenberg algebra $[a_k,a_l]=k\delta_{k,-l}\cdot 1$ and $[a'_m,a'_n]=m\delta_{m,-n}\cdot 1$, respectively.

(c) Let $\Phi^{\pm}(u)=\sum_{k\in\mathbb{Z}}\Phi^{\pm}_{k+\frac{1}{2}}u^{\pm k}$ and $\Psi^{\pm}(u)=\sum_{m\in\mathbb{Z}}\Psi^{\pm}_{m+\frac{1}{2}}u^{\pm m}$ satisfy the relations (32)–(35). For any $\beta\in\mathbb{C}$, define

$$\begin{aligned}L^{\beta}(u)&=\sum_{k\in\mathbb{Z}}L_ku^{-k-2}=\beta:\partial\Phi^+(u)\Phi^-(u):\\ &\quad+(1-\beta):\Phi^+(u)\partial\Phi^-(u):,\\ L'^{\beta}(u)&=\sum_{l\in\mathbb{Z}}L'_lu^{-l-2}=\beta:\partial\Psi^+(u)\Psi^-(u):\\ &\quad+(1-\beta):\Psi^+(u)\partial\Psi^-(u):,\end{aligned}\tag{42}$$

the coefficients of the formal distribution $L^{\beta}(u)$ and $L'^{\beta}(u)$ satisfy the relations of the Virasoro algebra

$$[L^{\beta}(u),L^{\beta}(v)]=\partial_vL(v)\delta(u,v)+2L(v)\partial_v\delta(u,v)$$

$$+\frac{c_\beta}{12}\partial_v^3\delta(u,v),$$
$$[L'^{\beta}(u),L'^{\beta}(v)] = \partial_v L'(v)\delta(u,v)+2L'(v)\partial_v\delta(u,v) +\frac{c_\beta}{12}\partial_v^3\delta(u,v), \tag{43}$$

where $c_\beta=-12\beta^2+12\beta-2$ is central charge.

Definition 2 For an unknown function $\tau=\tau(x,y)$, the system of bilinear relations

$$\Omega(\tau\otimes\tau)=0 \tag{44}$$

is called the UC hierarchy, which is an extension of the KP hierarchy, where

$$\begin{aligned}\Omega &= \mathrm{Res}_{u=0}\left(\frac{1}{u}\Phi^+(u)\otimes\Phi^-(u)\right)\\ &= \mathrm{Res}_{u=0}\left(\frac{1}{u}\Psi^+(u)\otimes\Psi^-(u)\right)\\ &= \sum_{k-l=-1}\Phi^+_{k+\frac12}\otimes\Phi^-_{l+\frac12}\\ &= \sum_{m-n=-1}\Psi^+_{m+\frac12}\otimes\Psi^-_{n+\frac12}.\end{aligned} \tag{45}$$

Proposition 9 All the universal characters $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ are solutions of the UC hierarchy.

Proof It is easy to see that $\tau=1$ is a solution.

Suppose that $\tau=\tau(x,y)$ is a solution of the UC hierarchy, using the relations (41) and $\Phi^+_{l-\frac12}\Phi^+_{l-\frac12}=0$, we have

$$\begin{aligned}&\left(\sum_{k-l=-1}\Phi^+_{k+\frac12}\otimes\Phi^-_{l+\frac12}\right)\Phi^+_{p+\frac12}\otimes\Phi^+_{p+\frac12}\\ =&\sum_{k-l=-1}\Phi^+_{k+\frac12}\Phi^+_{p+\frac12}\otimes\Phi^-_{l+\frac12}\Phi^+_{p+\frac12}\\ =&\sum_{k-l=-1}-\Phi^+_{p+\frac12}\Phi^+_{k+\frac12}\otimes\delta_{p,l-1}\\ &+\sum_{k-l=-1}\Phi^+_{p+\frac12}\Phi^+_{k+\frac12}\otimes\Phi^+_{p+\frac12}\Phi^-_{l+\frac12}\\ =&\Phi^+_{p+\frac12}\otimes\Phi^+_{p+\frac12}\left(\sum_{k-l=-1}\Phi^+_{k+\frac12}\otimes\Phi^-_{l+\frac12}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{k-l=-1}\Phi^+_{k+\frac12}(\Phi^+_{p+\frac12}(\tau))\otimes\Phi^-_{l+\frac12}(\Phi^+_{p+\frac12}(\tau))\\ =&\Phi^+_{p+\frac12}\otimes\Phi^+_{p+\frac12}\left(\sum_{k-l=-1}\Phi^+_{k+\frac12}(\tau)\otimes\Phi^-_{l+\frac12}(\tau)\right)=0.\end{aligned}$$

Moreover, we get

$$\begin{aligned}&\left(\sum_{m-n=-1}\Psi^+_{m+\frac12}\otimes\Psi^-_{n+\frac12}\right)\Phi^+_{p+\frac12}\otimes\Phi^+_{p+\frac12}\\ =&\Phi^+_{p+\frac32}\otimes\Phi^+_{p-\frac12}\left(\sum_{m-n=-1}\Psi^+_{m+\frac12}\otimes\Psi^-_{n+\frac12}\right).\end{aligned}$$

Then we have

$$\begin{aligned}&\sum_{m-n=-1}\Psi^+_{m+\frac12}(\Phi^+_{p+\frac12}(\tau))\otimes\Psi^-_{n+\frac12}(\Phi^+_{p+\frac12}(\tau))\\ =&\Phi^+_{p+\frac32}\otimes\Phi^+_{p-\frac12}\left(\sum_{m-n=-1}\Psi^+_{m+\frac12}(\tau)\otimes\Psi^-_{n+\frac12}(\tau)\right)=0.\end{aligned}$$

This shows that $\Phi^+_{p+\frac12}(\tau)$ is a solution of the UC hierarchy. It should be noticed that $\Psi^+_{p+\frac12}(\tau)$ is also a solution of the UC hierarchy. Hence, universal characters $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ are solutions of the UC hierarchy.

Corollary 2 The Schur function $S_\lambda(\boldsymbol{x})$ is the reduction of the universal character $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ when the partition $\mu=\emptyset$.

4. Symplectic Schur functions and generalized symplectic Schur functions

Firstly, in terms of multiplying $E^\perp(-\frac{1}{u})$ or $(1-u^2)H^\perp(\frac{1}{u})$ by the vertex operators of Schur functions $S_\lambda(\boldsymbol{x})$, we analyze the quantum fields presentation of symplectic Schur functions $S^{\mathrm{Sp}}_\lambda(\boldsymbol{x})$. Then we define SKP hierarchy whose tau functions can be obtained from symplectic Schur functions. In accordance with the same idea, we give the generalization for symplectic universal characters.

4.1. Symplectic Schur functions and SKP hierarchy

For a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_l)$, the symplectic Schur function $S^{\mathrm{Sp}}_\lambda(\boldsymbol{x})$ is a polynomial in $\mathbb{C}[\boldsymbol{x}]$:

$$S^{\mathrm{Sp}}_\lambda(\boldsymbol{x})=\frac12\det\left(h_{\lambda_i-i+j}(\boldsymbol{x})+h_{\lambda_i-i-j+2}(\boldsymbol{x})\right)_{1\le i,j\le l}. \tag{46}$$

Let $L_{\lambda_i\lambda_j}$ (Ref. [9]) be a lowering operator for partitions so that for $i<j$,

$$\begin{aligned}&L_{\lambda_i\lambda_j}\cdot(\lambda_1,\ldots,\lambda_i,\ldots,\lambda_j,\ldots,\lambda_l)\\ =&(\lambda_1,\ldots,\lambda_i-1,\ldots,\lambda_j-1,\ldots,\lambda_l).\end{aligned} \tag{47}$$

Then the following identity holds:

$$\begin{aligned}S^{\mathrm{Sp}}_\lambda(\boldsymbol{x}) &= \prod_{1\le i<j\le l}(1-L_{\lambda_i\lambda_j})S_\lambda(\boldsymbol{x})\\ &= \prod_{1\le i<j\le l}(1-L_{\lambda_i\lambda_j})(1-R_{\lambda_i\lambda_j})h_\lambda(\boldsymbol{x}).\end{aligned} \tag{48}$$

Let us set

$$\begin{aligned}\Phi^{\mathrm{Sp},+}(u)&=u^{-1}R(u)H(u)E^\perp(-u)E^\perp(-1/u),\\ \Phi^{\mathrm{Sp},-}(u)&=(1-u^2)R^{-1}(u)E(-u)H^\perp(u)H^\perp(1/u).\end{aligned} \tag{49}$$

The quantum fields are

$$\begin{aligned}\Phi^{\mathrm{Sp},+}(u)|_{\mathcal{B}^{(m)}}&=zu^mH(u)E^\perp(-u)E^\perp(-1/u),\\ \Phi^{\mathrm{Sp},-}(u)|_{\mathcal{B}^{(m)}}&=z^{-1}u^{-m}(1-u^2)E(-u)H^\perp(u)H^\perp(1/u).\end{aligned} \tag{50}$$

Proposition 10 It can be checked that $\Phi^{\mathrm{Sp},+}(u)$, $\Phi^{\mathrm{Sp},-}(u)$ satisfy the following relations:

$$\begin{aligned}&\Phi^{\mathrm{Sp},\pm}(u)\Phi^{\mathrm{Sp},\pm}(v)+\Phi^{\mathrm{Sp},\pm}(v)\Phi^{\mathrm{Sp},\pm}(u)=0,\\ &\Phi^{\mathrm{Sp},+}(u)\Phi^{\mathrm{Sp},-}(v)+\Phi^{\mathrm{Sp},-}(v)\Phi^{\mathrm{Sp},+}(u)=\delta(u,v).\end{aligned} \tag{51}$$

Proposition 11 (The quantum fields presentation of symplectic Schur functions)

$$\begin{aligned}&\Phi^{\mathrm{Sp},+}(u_1)\cdots\Phi^{\mathrm{Sp},+}(u_l)(1)\\ &= z^l u_1^{l-1}\cdots u_l^0 \prod_{1\le i<j\le l}\left(1-\frac{u_j}{u_i}\right)\prod_{1\le i<j\le l}(1-u_iu_j)\prod_{i=1}^{l}H(u_i)\\ &= z^l u_1^{l-1}\cdots u_l^0 G^{\mathrm{Sp}}(u_1,\ldots,u_l).\end{aligned}\tag{52}$$

Proof From Eq. (49) and Proposition 1, we have

$$\begin{aligned}&\Phi^{\mathrm{Sp},+}(u_1)\cdots\Phi^{\mathrm{Sp},+}(u_l)(1)\\ &= z^l\prod_{1\le i<j\le l}\left(1-\frac{u_j}{u_i}\right)\prod_{1\le i<j\le l}(1-u_iu_j)\\ &\quad\times\prod_{i=1}^{l}u_i^{l-i}H(u_i)E^{\perp}(-u_i)E^{\perp}(-1/u_i)(1),\end{aligned}$$

which simplifies to $z^l u_1^{l-1}\cdots u_l^0 G^{\mathrm{Sp}}(u_1,\ldots,u_l)$ because of $E^{\perp}(u)(1)=1$.

The quantum fields presentation of the symplectic Schur functions implies that $G^{\mathrm{Sp}}(u_1,\ldots,u_l)$ is a generating function for symplectic Schur functions. For any partition $\lambda=(\lambda_1\ge\ldots\ge\lambda_l\ge 0)$, the function $G^{\mathrm{Sp}}(u_1,\ldots,u_l)$ is expanded into a series of rational functions

$$G^{\mathrm{Sp}}(u_1,\ldots,u_l)=\sum_{\lambda_1,\ldots,\lambda_l\ge 0}G^{\mathrm{Sp}}_{\lambda}u_1^{\lambda_1}\ldots u_l^{\lambda_l},\quad |u_1|<\ldots<|u_l|.$$

Therefore, the coefficient of $u_1^{\lambda_1}\cdots u_l^{\lambda_l}$ is properly a symplectic Schur function: $G^{\mathrm{Sp}}_{(\lambda_1,\ldots,\lambda_l)}=S^{\mathrm{Sp}}_{\lambda}(\boldsymbol{x})$.

Corollary 3 The generating functions $G^{\mathrm{Sp}}(u_1,\ldots,u_l)$ of symplectic Schur functions and $G(u_1,\ldots,u_l)$ for Schur functions are related by

$$G^{\mathrm{Sp}}(u_1,\ldots,u_l)=\prod_{1\le i<j\le l}(1-u_iu_j)\,G(u_1,\ldots,u_l).\tag{53}$$

Proposition 12 (The boson–fermion correspondence for symplectic Schur functions)

(a) From the formula (9), we easily get the bosonic form of the fields $\Phi^{\mathrm{Sp},\pm}(u)$:

$$\begin{aligned}\Phi^{\mathrm{Sp},+}(u) &= u^{-1}R(u)\exp\left(\sum_{n\ge1}a_{-n}u^n\right)\exp\left(-\sum_{n\ge1}a_n\frac{1}{u^n}\right)\\ &\quad\times\exp\left(-\sum_{n\ge1}a_nu^n\right),\\ \Phi^{\mathrm{Sp},-}(u) &= (1-u^2)R^{-1}(u)\exp\left(-\sum_{n\ge1}a_{-n}u^n\right)\\ &\quad\times\exp\left(\sum_{n\ge1}a_n\frac{1}{u^n}\right)\exp\left(\sum_{n\ge1}a_nu^n\right).\end{aligned}\tag{54}$$

(b) Let $\Phi^{\mathrm{Sp},\pm}(u)=\sum_{k\in\mathbb{Z}}\Phi^{\mathrm{Sp},\pm}_{k+\frac12}u^{\pm k}$ satisfy the relations (51), and let

$$\Phi^{\mathrm{Sp},+}(u)_+=\sum_{k\ge1}\Phi^{\mathrm{Sp},+}_{k+\frac12}u^k,$$

$$\Phi^{\mathrm{Sp},+}(u)_-=\sum_{k\le0}\Phi^{\mathrm{Sp},+}_{k+\frac12}u^k.\tag{55}$$

Then the relations (51) have the following expressions:

$$\begin{aligned}&\Phi^{\mathrm{Sp},\pm}_{k+\frac12}\Phi^{\mathrm{Sp},\pm}_{l+\frac12}+\Phi^{\mathrm{Sp},\pm}_{l+\frac12}\Phi^{\mathrm{Sp},\pm}_{k+\frac12}=0,\\ &\Phi^{\mathrm{Sp},+}_{k+\frac12}\Phi^{\mathrm{Sp},-}_{l+\frac12}+\Phi^{\mathrm{Sp},-}_{l+\frac12}\Phi^{\mathrm{Sp},+}_{k+\frac12}=\delta_{k,l-1},\end{aligned}\tag{56}$$

and the coefficients of the formal distribution $a(u)=:\Phi^{\mathrm{Sp},+}(u)\cdot\Phi^{\mathrm{Sp},-}(u):=\Phi^{\mathrm{Sp},+}(u)_+\Phi^{\mathrm{Sp},-}(u)-\Phi^{\mathrm{Sp},-}(u)\Phi^{\mathrm{Sp},+}(u)_-$ satisfy the relations (7) of the Heisenberg algebra.

(c) Let $\Phi^{\mathrm{Sp},\pm}(u)=\sum_{k\in\mathbb{Z}}\Phi^{\mathrm{Sp},\pm}_{k+\frac12}u^{\pm k}$ satisfy the relations (51). For any $\beta\in\mathbb{C}$ the coefficients of the formal distribution $L^{\beta}(u)=\sum_{k\in\mathbb{Z}}L_ku^{-k-2}=\beta:\partial\Phi^{\mathrm{Sp},+}(u)\Phi^{\mathrm{Sp},-}(u):+(1-\beta):\Phi^{\mathrm{Sp},+}(u)\partial\Phi^{\mathrm{Sp},-}(u):$ satisfy the relations of the Virasoro algebra:

$$\begin{aligned}[L^{\beta}(u),L^{\beta}(v)] &= \partial_vL(v)\delta(u,v)+2L(v)\partial_v\delta(u,v)\\ &\quad+\frac{c_{\beta}}{12}\partial_v^3\delta(u,v),\end{aligned}\tag{57}$$

where $c_{\beta}=-12\beta^2+12\beta-2$ is central charge.

Definition 3 For an unknown function $\tau=\tau(x)$, the bilinear equation

$$\Omega(\tau\otimes\tau)=0\tag{58}$$

is called the SKP hierarchy, where

$$\begin{aligned}\Omega &= \mathrm{Res}_{u=0}\left(\frac{1}{u}\Phi^{\mathrm{Sp},+}(u)\otimes\Phi^{\mathrm{Sp},-}(u)\right)\\ &= \sum_{k-l=-1}\Phi^{\mathrm{Sp},+}_{k+\frac12}\otimes\Phi^{\mathrm{Sp},-}_{l+\frac12}.\end{aligned}\tag{59}$$

Proposition 13 All the symplectic Schur functions $S^{\mathrm{Sp}}_{\lambda}(\boldsymbol{x})$ are solutions of the SKP hierarchy.

4.2. Symplectic universal characters and SUC hierarchy

In this part, we transfer our research goal to the symplectic universal characters. We will give the quantum fields presentation and the boson–fermion correspondence for the symplectic universal characters. Then, we will construct a modified type of the system of bilinear equations and discuss the solutions of this system.

For each pair of partitions

$$[\lambda,\mu]=[(\lambda_1,\lambda_2,\ldots,\lambda_l),(\mu_1,\mu_2,\ldots,\mu_m)],$$

the symplectic universal character $S^{\mathrm{Sp}}_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ is defined as[10]

$$\begin{aligned}S^{\mathrm{Sp}}_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) &= \prod_{1\le i<j\le l}(1-L_{\lambda_i\lambda_j})\\ &\quad\times\prod_{1\le a<b\le m}(1-L_{\mu_a\mu_b})S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}),\end{aligned}\tag{60}$$

where $L_{\lambda_i\lambda_j}$ and $L_{\mu_a\mu_b}$ are lowering operators.

We set

$$\begin{aligned}
\Phi^{\mathrm{Sp},+}(u) &= u^{-1}R(u)H(u)E'^{\perp}(-1/u)E^{\perp}(-u)E^{\perp}(-1/u),\\
\Phi^{\mathrm{Sp},-}(u) &= (1-u^2)R^{-1}(u)E(-u)H'^{\perp}(1/u)H^{\perp}(u)H^{\perp}(1/u),\\
\Psi^{\mathrm{Sp},+}(u) &= u^{-1}R(u)H'(u)E^{\perp}(-1/u)E'^{\perp}(-u)E'^{\perp}(-1/u),\\
\Psi^{\mathrm{Sp},-}(u) &= (1-u^2)R^{-1}(u)E'(-u)H^{\perp}(1/u)\\
&\quad\times H'^{\perp}(u)H'^{\perp}(1/u). \qquad (61)
\end{aligned}$$

Observe that

$$\begin{aligned}
\Phi^{\mathrm{Sp},+}(u)|_{\mathcal{B}^{(m)}} &= zu^mH(u)E'^{\perp}(-1/u)E^{\perp}(-u)E^{\perp}(-1/u),\\
\Phi^{\mathrm{Sp},-}(u)|_{\mathcal{B}^{(m)}} &= z^{-1}u^{-m}(1-u^2)E(-u)H'^{\perp}(1/u)\\
&\quad\times H^{\perp}(u)H^{\perp}(1/u),\\
\Psi^{\mathrm{Sp},+}(u)|_{\mathcal{B}^{(m)}} &= zu^mH'(u)E^{\perp}(-1/u)\\
&\quad\times E'^{\perp}(-u)E'^{\perp}(-1/u),\\
\Psi^{\mathrm{Sp},-}(u)|_{\mathcal{B}^{(m)}} &= z^{-1}u^{-m}(1-u^2)E'(-u)H^{\perp}(1/u)\\
&\quad\times H'^{\perp}(u)H'^{\perp}(1/u). \qquad (62)
\end{aligned}$$

Proposition 14 The following relations of charged free fermions about quantum fields $\Phi^{\mathrm{Sp},\pm}(u)$ and $\Psi^{\mathrm{Sp},\pm}(u)$ hold:

$$\begin{aligned}
&[\Phi^{\mathrm{Sp},\pm}(u),\Phi^{\mathrm{Sp},\pm}(v)]_+=0,\\
&[\Phi^{\mathrm{Sp},+}(u),\Phi^{\mathrm{Sp},-}(v)]_+=\delta(u,v),\\
&[\Psi^{\mathrm{Sp},\pm}(u),\Psi^{\mathrm{Sp},\pm}(v)]_+=0,\\
&[\Psi^{\mathrm{Sp},+}(u),\Psi^{\mathrm{Sp},-}(v)]_+=\delta(u,v),\\
&v\Phi^{\mathrm{Sp},+}(u)\Psi^{\mathrm{Sp},+}(v)-u\Psi^{\mathrm{Sp},+}(v)\Phi^{\mathrm{Sp},+}(u)=0,\\
&u\Phi^{\mathrm{Sp},+}(u)\Psi^{\mathrm{Sp},-}(v)-v\Psi^{\mathrm{Sp},-}(v)\Phi^{\mathrm{Sp},+}(u)=0,\\
&u\Phi^{\mathrm{Sp},-}(u)\Psi^{\mathrm{Sp},+}(v)-v\Psi^{\mathrm{Sp},+}(v)\Phi^{\mathrm{Sp},-}(u)=0,\\
&v\Phi^{\mathrm{Sp},-}(u)\Psi^{\mathrm{Sp},-}(v)-u\Psi^{\mathrm{Sp},-}(v)\Phi^{\mathrm{Sp},-}(u)=0. \qquad (63)
\end{aligned}$$

Proposition 14 immediately indicates that the operators $\Phi^{\mathrm{Sp},\pm}(u),\Psi^{\mathrm{Sp},\pm}(u)$ provide the quantum fields presentation of symplectic universal characters. Let

$$\begin{aligned}
&G^{\mathrm{Sp}}(u_1,\dots,u_l,v_1,\dots,v_m)\\
&=\prod_{1\le i<j\le l}\left(1-\frac{u_j}{u_i}\right)(1-u_iu_j)\prod_{1\le a<b\le m}\left(1-\frac{v_b}{v_a}\right)(1-v_av_b)\\
&\quad\times\prod_{\substack{1\le i\le l\\1\le a\le m}}(1-u_iv_a)\prod_{i=1}^{l}H(u_i)\prod_{a=1}^{m}H'(v_a). \qquad (64)
\end{aligned}$$

It is expanded into rational function form

$$\begin{aligned}
&G^{\mathrm{Sp}}(u_1,\dots,u_l,v_1,\dots,v_m)\\
&=\sum_{\substack{\lambda_1,\dots,\lambda_l\ge0\\\mu_1,\dots,\mu_m\ge0}}\prod_{i<j}(1-u_iu_j)\prod_{a<b}(1-v_av_b)\\
&\quad\times\prod_{\substack{1\le i\le l\\1\le a\le m}}(1-u_iv_a)\prod_{i<j}\left(1-\frac{u_j}{u_i}\right)\prod_{a<b}\left(1-\frac{v_b}{v_a}\right)\cdot\\
&\quad h_{\lambda_1}(\boldsymbol{x})\cdots h_{\lambda_l}(\boldsymbol{x})h_{\mu_1}(\boldsymbol{y})\cdots h_{\mu_m}(\boldsymbol{y})u_1^{\lambda_1}\cdots u_l^{\lambda_l}v_1^{\mu_1}\cdots v_m^{\mu_m}\\
&=\prod_{i<j}(1-L_{\lambda_i\lambda_j})\prod_{a<b}(1-L_{\mu_a\mu_b})\\
&\quad\times\prod_{i<j}(1-R_{\lambda_i\lambda_j})\prod_{a<b}(1-R_{\mu_a\mu_b})\prod_{\substack{1\le i\le l\\1\le a\le m}}(1-D_{\lambda_i\mu_a})\cdot\\
&\quad h_{\lambda_1}(\boldsymbol{x})\cdots h_{\lambda_l}(\boldsymbol{x})h_{\mu_1}(\boldsymbol{y})\cdots h_{\mu_m}(\boldsymbol{y})u_1^{\lambda_1}\cdots u_l^{\lambda_l}v_1^{\mu_1}\cdots v_m^{\mu_m}\\
&=\sum_{\substack{\lambda_1,\dots,\lambda_l\ge0\\\mu_1,\dots,\mu_m\ge0}}G^{\mathrm{Sp}}_{[\lambda,\mu]}u_1^{\lambda_1}\cdots u_l^{\lambda_l}v_1^{\mu_1}\cdots v_m^{\mu_m}. \qquad (65)
\end{aligned}$$

Therefore, for any partition $\lambda=(\lambda_1\ge\cdots\ge\lambda_l\ge0)$ and $\mu=(\mu_1\ge\cdots\ge\mu_m\ge0)$ the coefficient of $u_1^{\lambda_1}\cdots u_l^{\lambda_l}v_1^{\mu_1}\cdots v_m^{\mu_m}$ is exactly symplectic universal characters: $G^{\mathrm{Sp}}_{[\lambda,\mu]}=S^{\mathrm{Sp}}_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$.

Proposition 15 (The quantum fields presentation of symplectic universal characters)

$$\begin{aligned}
&\Phi^{\mathrm{Sp},+}(u_1)\cdots\Phi^{\mathrm{Sp},+}(u_l)\Psi^{\mathrm{Sp},+}(v_1)\cdots\Psi^{\mathrm{Sp},+}(v_m)(1)\\
&=z^{l+m}u_1^{m+l-1}\cdots u_l^mv_1^{m-1}\cdots v_m^0\\
&\quad\times G^{\mathrm{Sp}}(u_1,\dots,u_l,v_1,\dots,v_m). \qquad (66)
\end{aligned}$$

Proof From Proposition 1 and in view of $E^{\perp}(-u)(1)=1$, we obtain

$$\begin{aligned}
&\Phi^{\mathrm{Sp},+}(u_1)\cdots\Phi^{\mathrm{Sp},+}(u_l)\Psi^{\mathrm{Sp},+}(v_1)\cdots\Psi^{\mathrm{Sp},+}(v_m)(1)\\
&=z^{l+m}\prod_{i=1}^{l}u_i^{m+l-i}\prod_{a=1}^{m}v_a^{m-a}\\
&\quad\times\prod_{1\le i<j\le l}\left(1-\frac{u_j}{u_i}\right)\prod_{1\le i<j\le l}(1-u_iu_j)\prod_{1\le a<b\le m}\left(1-\frac{v_b}{v_a}\right)\\
&\quad\times\prod_{1\le a<b\le m}(1-v_av_b)\prod_{\substack{1\le i\le l\\1\le a\le m}}(1-u_iv_a)\prod_{i=1}^{l}H(u_i)\\
&\quad\times\prod_{a=1}^{m}H'(v_a)\prod_{i=1}^{l}E'^{\perp}(-1/u_i)E^{\perp}(-u_i)E^{\perp}(-1/u_i),\\
&\quad\prod_{a=1}^{m}E^{\perp}(-1/v_a)E'^{\perp}(-v_a)E'^{\perp}(-1/v_a)(1)\\
&=z^{l+m}u_1^{m+l-1}\cdots u_l^mv_1^{m-1}\cdots v_m^0\\
&\quad\times G^{\mathrm{Sp}}(u_1,\dots,u_l,v_1,\dots,v_m). \qquad (67)
\end{aligned}$$

Corollary 4 The generating functions

$$G^{\mathrm{Sp}}(u_1,\dots,u_l,v_1,\dots,v_m)$$

of symplectic universal characters and $G(u_1,\dots,u_l,v_1,\dots,v_m)$ of universal characters are related by

$$\begin{aligned}
&G^{\mathrm{Sp}}(u_1,\dots,u_l,v_1,\dots,v_m)\\
&=\prod_{1\le i<j\le l}(1-u_iu_j)\prod_{1\le a<b\le m}(1-v_av_b)\\
&\quad\times G(u_1,\dots,u_l,v_1,\dots,v_m). \qquad (68)
\end{aligned}$$

Proposition 16 (The boson–fermion correspondence for symplectic universal characters)

(a) From the formula (2), we easily get the bosonic form of the fields $\Phi^{\mathrm{Sp},\pm}(u)$ and $\Psi^{\mathrm{Sp},\pm}(u)$:

$$\begin{aligned}
\Phi^{\mathrm{Sp},+}(u) &= u^{-1}R(u)\exp\Big(\sum_{n\geq1}a_{-n}u^n\Big)\exp\Big(-\sum_{n\geq1}a'_nu^n\Big)\exp\Big(-\sum_{n\geq1}a_n\frac{1}{u^n}\Big)\exp\Big(-\sum_{n\geq1}a_nu^n\Big),\\
\Phi^{\mathrm{Sp},-}(u) &= (1-u^2)R^{-1}(u)\exp\Big(-\sum_{n\geq1}a_{-n}u^n\Big)\exp\Big(\sum_{n\geq1}a'_nu^n\Big)\exp\Big(\sum_{n\geq1}a_n\frac{1}{u^n}\Big)\exp\Big(\sum_{n\geq1}a_nu^n\Big),\\
\Psi^{\mathrm{Sp},+}(u) &= u^{-1}R(u)\exp\Big(\sum_{n\geq1}a'_{-n}u^n\Big)\exp\Big(-\sum_{n\geq1}a_nu^n\Big)\exp\Big(-\sum_{n\geq1}a'_n\frac{1}{u^n}\Big)\exp\Big(-\sum_{n\geq1}a'_nu^n\Big),\\
\Psi^{\mathrm{Sp},-}(u) &= (1-u^2)R^{-1}(u)\exp\Big(-\sum_{n\geq1}a'_{-n}u^n\Big)\exp\Big(\sum_{n\geq1}a_nu^n\Big)\exp\Big(\sum_{n\geq1}a'_n\frac{1}{u^n}\Big)\exp\Big(\sum_{n\geq1}a'_nu^n\Big).
\end{aligned}\tag{69}$$

(b) Let $\Phi^{\mathrm{Sp},\pm}(u)=\sum_{k\in\mathbb{Z}}\Phi^{\mathrm{Sp},\pm}_{k+\frac12}u^{\pm k}$ and $\Psi^{\mathrm{Sp},\pm}(u)=\sum_{m\in\mathbb{Z}}\Psi^{\mathrm{Sp},\pm}_{m+\frac12}u^{\pm m}$ satisfy the relations (63), and let

$$\begin{aligned}
\Phi^{\mathrm{Sp},+}(u)_+ &= \sum_{k\geq1}\Phi^{\mathrm{Sp},+}_{k+\frac12}u^k,\\
\Phi^{\mathrm{Sp},+}(u)_- &= \sum_{k\leq0}\Phi^{\mathrm{Sp},+}_{k+\frac12}u^k,\\
\Psi^{\mathrm{Sp},+}(u)_+ &= \sum_{m\geq1}\Psi^{\mathrm{Sp},+}_{m+\frac12}u^m,\\
\Psi^{\mathrm{Sp},+}(u)_- &= \sum_{m\leq0}\Psi^{\mathrm{Sp},+}_{m+\frac12}u^m.
\end{aligned}\tag{70}$$

Then the relations (63) have the following expressions:

$$\begin{aligned}
&\Phi^{\mathrm{Sp},\pm}_{k+\frac12}\Phi^{\mathrm{Sp},\pm}_{l+\frac12}+\Phi^{\mathrm{Sp},\pm}_{l+\frac12}\Phi^{\mathrm{Sp},\pm}_{k+\frac12}=0,\\
&\Phi^{\mathrm{Sp},+}_{k+\frac12}\Phi^{\mathrm{Sp},-}_{l+\frac12}+\Phi^{\mathrm{Sp},-}_{l+\frac12}\Phi^{\mathrm{Sp},+}_{k+\frac12}=\delta_{k,l-1},\\
&\Psi^{\mathrm{Sp},\pm}_{m+\frac12}\Psi^{\mathrm{Sp},\pm}_{n+\frac12}+\Psi^{\mathrm{Sp},\pm}_{n+\frac12}\Psi^{\mathrm{Sp},\pm}_{m+\frac12}=0,\\
&\Psi^{\mathrm{Sp},+}_{m+\frac12}\Psi^{\mathrm{Sp},-}_{n+\frac12}+\Psi^{\mathrm{Sp},-}_{n+\frac12}\Psi^{\mathrm{Sp},+}_{m+\frac12}=\delta_{m,n-1},\\
&\Phi^{\mathrm{Sp},+}_{k+\frac12}\Psi^{\mathrm{Sp},+}_{m+\frac12}-\Psi^{\mathrm{Sp},+}_{m+\frac32}\Phi^{\mathrm{Sp},+}_{k-\frac12}=0,\\
&\Phi^{\mathrm{Sp},+}_{k+\frac12}\Psi^{\mathrm{Sp},-}_{n+\frac12}-\Psi^{\mathrm{Sp},-}_{n+\frac32}\Phi^{\mathrm{Sp},+}_{k+\frac32}=0,\\
&\Phi^{\mathrm{Sp},-}_{l+\frac12}\Psi^{\mathrm{Sp},+}_{m+\frac12}-\Psi^{\mathrm{Sp},+}_{m-\frac12}\Phi^{\mathrm{Sp},-}_{l-\frac12}=0,\\
&\Phi^{\mathrm{Sp},-}_{l+\frac12}\Psi^{\mathrm{Sp},-}_{n+\frac12}-\Psi^{\mathrm{Sp},-}_{n-\frac12}\Phi^{\mathrm{Sp},-}_{l+\frac32}=0,
\end{aligned}\tag{71}$$

and the coefficients of the formal distribution

$$\begin{aligned}
a(u) &= :\Phi^{\mathrm{Sp},+}(u)\cdot\Phi^{\mathrm{Sp},-}(u):\\
&= \Phi^{\mathrm{Sp},+}(u)_+\Phi^{\mathrm{Sp},-}(u)-\Phi^{\mathrm{Sp},-}(u)\Phi^{\mathrm{Sp},+}(u)_-,\\
a'(u) &= :\Psi^{\mathrm{Sp},+}(u)\cdot\Psi^{\mathrm{Sp},-}(u):\\
&= \Psi^{\mathrm{Sp},+}(u)_+\Psi^{\mathrm{Sp},-}(u)-\Psi^{\mathrm{Sp},-}(u)\Psi^{\mathrm{Sp},+}(u)_-
\end{aligned}$$

satisfy the relations of the Heisenberg algebra $[a_k,a_l]=k\delta_{k,-l}\cdot1$ and $[a'_m,a'_n]=m\delta_{m,-n}\cdot1$, respectively.

(c) Let $\Phi^{\mathrm{Sp},\pm}(u)=\sum_{k\in\mathbb{Z}}\Phi^{\mathrm{Sp},\pm}_{k+\frac12}u^{\pm k}$ and $\Psi^{\mathrm{Sp},\pm}(u)=\sum_{m\in\mathbb{Z}}\Psi^{\mathrm{Sp},\pm}_{m+\frac12}u^{\pm m}$ satisfy the relations (63). For any $\beta\in\mathbb{C}$, define

$$\begin{aligned}
L^\beta(u) &= \sum_{k\in\mathbb{Z}}L_ku^{-k-2}=\beta:\partial\Phi^{\mathrm{Sp},+}(u)\Phi^{\mathrm{Sp},-}(u):\\
&\quad+(1-\beta):\Phi^{\mathrm{Sp},+}(u)\partial\Phi^{\mathrm{Sp},-}(u):,\\
L'^\beta(u) &= \sum_{m\in\mathbb{Z}}L'_mu^{-m-2}=\beta:\partial\Psi^{\mathrm{Sp},+}(u)\Psi^{\mathrm{Sp},-}(u):\\
&\quad+(1-\beta):\Psi^{\mathrm{Sp},+}(u)\partial\Psi^{\mathrm{Sp},-}(u):,
\end{aligned}\tag{72}$$

the coefficients of the formal distribution $L^\beta(u)$ and $L'^\beta(u)$ satisfy the relations of the Virasoro algebra:

$$\begin{aligned}
[L^\beta(u),L^\beta(v)] &= \partial_vL(v)\delta(u,v)+2L(v)\partial_v\delta(u,v)\\
&\quad+\frac{c_\beta}{12}\partial_v^3\delta(u,v),\\
[L'^\beta(u),L'^\beta(v)] &= \partial_vL'(v)\delta(u,v)+2L'(v)\partial_v\delta(u,v)\\
&\quad+\frac{c_\beta}{12}\partial_v^3\delta(u,v),
\end{aligned}\tag{73}$$

where $c_\beta=-12\beta^2+12\beta-2$ is central charge.

Definition 4 For an unknown function $\tau=\tau(x,y)$, the system of bilinear relations

$$\Omega(\tau\otimes\tau)=0\tag{74}$$

is called the SUC hierarchy, which is an extension of the SKP hierarchy, where

$$\begin{aligned}
\Omega &= \mathrm{Res}_{u=0}\Big(\frac1u\Phi^{\mathrm{Sp},+}(u)\otimes\Phi^{\mathrm{Sp},-}(u)\Big)\\
&= \mathrm{Res}_{u=0}\Big(\frac1u\Psi^{\mathrm{Sp},+}(u)\otimes\Psi^{\mathrm{Sp},-}(u)\Big)\\
&= \sum_{k-l=-1}\Phi^{\mathrm{Sp},+}_{k+\frac12}\otimes\Phi^{\mathrm{Sp},-}_{l+\frac12}\\
&= \sum_{m-n=-1}\Psi^{\mathrm{Sp},+}_{m+\frac12}\otimes\Psi^{\mathrm{Sp},-}_{n+\frac12}.
\end{aligned}\tag{75}$$

Proposition 17 All the symplectic universal characters $S^{\mathrm{Sp}}_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ are solutions of the SUC hierarchy.

Proof The proof of this proposition is similar to Proposition 9 and it will not be included here.

Corollary 5 The symplectic Schur function $S^{\mathrm{Sp}}_\lambda(\boldsymbol{x})$ is the reduction of the symplectic universal character $S^{\mathrm{Sp}}_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ when the partition $\mu=\emptyset$.

5. Summary and discussion

In this paper, we have discussed quantum fields presentation and generating functions of the universal characters, symplectic Schur functions and symplectic universal characters. It should be pointed out that the generating functions play a key

role in constructing the vertex operators of symmetric functions. Furthermore, based upon the boson–fermion correspondence for the symmetric functions, we have derived infinite order nonlinear partial differential equations, namely UC hierarchy, SKP hierarchy and SUC hierarchy. Moreover, corresponding solutions of these equations have also been derived. It is worthwhile to mention that finding the exact solutions of these partial differential equations is difficult. Therefore, the relation of quantum fields operators may shed light on the solutions of these equations, which will be studied in the near future.

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