

Oscillation properties of matter–wave bright solitons in harmonic potentials

Shu-Wen Guan(关淑文), Ling-Zheng Meng(孟令正), and Li-Chen Zhao(赵立臣)
Citation: Chin. Phys. B, 2022, 31 (**8**): 080506. DOI: 10.1088/1674-1056/ac4e0f
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Soliton fusion and fission for the high-order coupled nonlinear Schrödinger system in fiber lasers

Tian-Yi Wang(王天一), Qin Zhou(周勤), and Wen-Jun Liu(刘文军)
Chin. Phys. B, 2022, 31 (**2**): 020501. DOI: 10.1088/1674-1056/ac2d22

Effects of time delays in a mathematical bone model

Li-Fang Wang(王莉芳), Kang Qiu(仇康), Ya Jia(贾亚)
Chin. Phys. B, 2017, 26 (**3**): 030503. DOI: 10.1088/1674-1056/26/3/030503

Directed transport of coupled Brownian ratchets with time-delayed feedback

Gao Tian-Fu, Zheng Zhi-Gang, Chen Jin-Can
Chin. Phys. B, 2013, 22 (**8**): 080502. DOI: 10.1088/1674-1056/22/8/080502

Feshbach resonance management of vector solitons in two-component Bose–Einstein condensates

Wang Qiang, Wen Lin, Li Zai-Dong
Chin. Phys. B, 2012, 21 (**8**): 080501. DOI: 10.1088/1674-1056/21/8/080501

Spatiotemporal self-similar solutions for the nonautonomous (3+1)-dimensional cubic–quintic Gross–Pitaevskii equation

Dai Chao-Qing(戴朝卿), Chen Rui-Pin(陈瑞品), and Wang Yue-Yue(王悦悦)
Chin. Phys. B, 2012, 21 (**3**): 030508. DOI: 10.1088/1674-1056/21/3/030508

Oscillation properties of matter–wave bright solitons in harmonic potentials

Shu-Wen Guan(关淑文)[1], Ling-Zheng Meng(孟令正)[1], and Li-Chen Zhao(赵立臣)[1,2,3,†]

[1] *School of Physics, Northwest University, Xi'an* 710127, *China*
[2] *NSFC–SPTP Peng Huanwu Center for Fundamental Theory, Xi'an* 710127, *China*
[3] *Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xi'an* 710127, *China*

(Received 22 December 2021; revised manuscript received 15 January 2022; accepted manuscript online 24 January 2022)

We investigate the oscillation periods of bright soliton pair or vector bright soliton pair in harmonic potentials. We demonstrate that periods of low-speed solitons are greatly affected by the position shift during their collisions. The modified oscillation periods are described by defining a characterized speed, with the aid of asymptotic analysis on related exact analytic soliton solutions in integrable cases. The oscillation period can be used to distinguish the inter- and intra-species interactions between solitons. However, a bright soliton cannot oscillate in a harmonic trap, when it is coupled with a dark soliton (without any trapping potentials). Interestingly, it can oscillate in an anti-harmonic potential, and the oscillation behavior is explained by a quasi-particle theory. The modified period of two dark–bright solitons can be also described well by the characterized speed. These results address well the effects of position shift during soliton collision, which provides an important supplement for previous studies without considering phase shift effects.

Keywords: matter–wave bright soliton, harmonic potential, oscillation period

PACS: 05.45.Yv, 02.30.Ik, 42.65.Tg **DOI:** 10.1088/1674-1056/ac4e0f

1. Introduction

Bose–Einstein condensate (BEC) has become a good platform to investigate the motion of solitons, due to its high controllability on nonlinear interactions and external potentials.[1–8] The motion of solitons in external potentials can be used to discuss the dispersion relations and inertial mass of solitons,[9,10] which is very meaningful for soliton applications. There have been many discussions on motion of solitons in harmonic traps, due to that harmonic trap is usually used to confine BECs. Recent studies demonstrated that a bright soliton,[11–15] a dark soliton[16–18] or a dark–bright soliton[19,20] can oscillate periodically with distinct oscillating frequencies in an identical harmonic potential. For a bright soliton, the oscillating frequency is equal to that of the harmonic trap. For a dark solion, it is modified by a factor $\sqrt{2}$ induced by its striking dispersion relation.[17,18] For a dark–bright soliton, it can admit many possible oscillating frequencies due to the interplay between the bright soliton and dark soliton. The dispersion relation of one soliton can be used to explain those characters with a quasi-particle picture, which has deepen greatly our understanding on soliton's inertial mass.[9,10]

The motion of soliton pair were further discussed in a harmonic trap,[21–24] since multi-solitons can be prepared well in experiments and nonlinear interaction between solitons can also modify their oscillation behavior.[25] Quasi-particle model has been proposed to describe the motion of soliton pair well,[21,22] when the position shift induced by collision can be ignored. The position shift becomes larger with decreasing soliton's collision speed.[26,27] Then it is essential to discuss the effects of position shift on oscillation behavior of soliton pair.

In this paper, we report that the effects of position shift can be addressed well by defining a characterized speed, with the aid of asymptotic analysis on related exact analytic soliton solutions in integrable cases.[28] We investigate oscillation periods of scalar bright solitons, vector incoherent solitons,[29] bright–bright solitons,[30–32] and dark–bright solitons[19,33] in a harmonic trap or anti-harmonic trap. When the soliton velocities increase, the two-soliton oscillations tend to be consistent with the results in Refs. [21,22]. The oscillation period can be used to distinguish the collision induced by inter- and intra-species interactions. Especially, a bright soliton cannot oscillate in a harmonic trap, when it is coupled with a dark soliton (without any trapping potentials). But it can oscillate in an anti-harmonic potential, and the oscillation behavior is explained by a quasi-particle theory. These results would inspire some experiments to observe the modified oscillation periods of soliton pair.

The paper is organized as follows. In Section 2, we investigate the oscillation of a bright soliton pair with coherent collision in a harmonic trap. We define a characterized speed to describe the modified oscillation period theoretically, with the aid of asymptotic analysis on related exact analytic soliton solutions in integrable cases. In Section 3, we study the oscillation periods of vector soliton pairs with incoherent collision. In Section 4, we study the oscillation periods of vector soliton

†Corresponding author. E-mail: zhaolichen3@nwu.edu.cn
© 2022 Chinese Physical Society and IOP Publishing Ltd

pairs with both coherent and incoherent collision. The modified period differences are clarified for these three different types collisions, which can be used to distinguish the collision induced by inter- and intra-species interactions. In Section 5, we further investigate the oscillation of a dark–bright soliton in a harmonic trap or an anti-harmonic potential. The period of a dark–bright soliton pair is also obtained theoretically by the characterized speed. Finally, we make conclusion and discussion in Section 6.

2. The oscillation property of a bright soliton pair with coherent collision

Under the mean-field approximation, dynamics of a quasi-1D BEC can be governed by a nonlinear Schrödinger equation (NLSE) with dimensionless units,

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi + V(x)\psi, \tag{1}$$

where ψ represents the mean-field wave function of BEC, x and t denote the axial coordinate and time, the parameter g describes nonlinear interaction between atoms, $V(x) = \frac{1}{2}\Omega^2 x^2$ is the harmonic potential. When $g < 0$, the interaction is attractive and the bright soliton can exist.[15] In the case of $g = -1$ and $\Omega = 0$, it is well known that an exact bright soliton solution can be obtained by Darboux transformation (DT)[34,35]

$$\psi(x,t) = A\ \mathrm{sech}[A(x-x_0)]\ \exp\left\{\mathrm{i}[vx - \frac{1}{2}(v^2 - A^2)t]\right\},$$

where A and v denote the amplitude and velocity of a bright soliton, x_0 is the initial position of the soliton center. If $\Omega \neq 0$ and $x_0 \neq 0$, a bright soliton can oscillate periodically under the effect of an external potential, which is formally identical with a particle.[4] The oscillation period is $T_0 = 2\pi/\Omega$, when the external potential vary slowly in the soliton scale. This character can be also explained by the dispersion relation of one soliton (see Appendix A). For a bright soliton pair, two solitons will collide with each other in a harmonic trap. It is known that a sudden position shift emerges after the collision process,[36] which surely affects the oscillation period. Quasi-particle model has been proposed to describe the motion of soliton pair well, when the position shift induced by collision can be ignored.[21,22,36] Then we would like to discuss the cases for which the position shift effects can not be ignored. We consider a general case for which a bright soliton pair is located in harmonic potential and the potential varies slowly within the soliton size scale, as shown in Fig. 1(a). The initial state can be given as[37]

$$\psi_0 = A_1\ \mathrm{sech}[A_1(x-x_0)] + A_2\ \mathrm{sech}[A_2(x+x_0)]. \tag{2}$$

The initial position x_0 can be used to vary the collision speed conveniently. According to asymptotic analysis of the two-soliton solution,[38] we obtain the position shift caused by soliton collision

$$\Delta x_1 = \frac{1}{A_1}\ln\left[\frac{(A_1-A_2)^2 + (v_1-v_2)^2}{(A_1+A_2)^2 + (v_1-v_2)^2}\right],$$

where v_1 and v_2 denote the moving velocity of two solitons. We can see that the position shift increases with decreasing the relative speed. Therefore, we investigate the cases with the solitons' relative speed is small. As an example, we show one case numerically in Fig. 1(b), for which the time is normalized with the period T_0. One can see that the soliton oscillation period is much less than T_0 in this case , which means that it is indeed affected by the position shift effects.

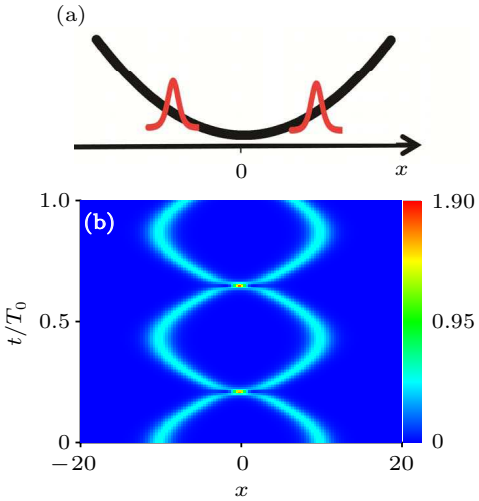


Fig. 1. (a) The profile schematic diagram of the initial scalar soliton pair in a harmonic potential. The two solitons are symmetrically placed at the position $\pm x_0$ ($x_0 = 10$). (b) Oscillation of the soliton pair in the harmonic potential. The oscillation period is obviously less than that of only one bright soliton $T_0 = 2\pi/\Omega$. The harmonic frequency $\Omega = 0.02$, and other parameters $A_1 = A_2 = \sqrt{2}/2$.

Next, we try to quantitatively explore the oscillation period modified by solitons' collision. The position shift of the collision depends on soliton velocities. Thus, it is necessary to calculate the velocities of the solitons when they collide at the bottom of harmonic potential. For a soliton centered at x_c, the soliton energy

$$E_b = \int \frac{1}{2}\left|\frac{\partial\psi}{\partial x}\right|^2 + \frac{1}{2}|\psi|^2\,\mathrm{d}x = A(A^2 + v^2),$$

the potential energy

$$E_p = \int \frac{1}{2}\Omega^2 x^2|\psi|^2\,\mathrm{d}x = \frac{1}{2}\Omega^2 x_c^2\int|\psi|^2\,\mathrm{d}x = A\,\Omega x_c^2$$

if the harmonic potential varies slowly within the soliton size scale. The soliton center position x_c is x_0 for the initial state. Based on the law of energy conservation, $E_{b0} + E_{p0} = E_b + E_p$ (the E_{b0} and E_{p0} are given by initial state), we can obtain the collision velocity as

$$v = \Omega x_0. \tag{3}$$

It indicates that the collision velocity is positively related to the solitons' initial position x_0, thus we can control the relative

speed by placing solitons at different initial positions. Near the collision location (the harmonic trap center), the collision process of the two solitons can be approximately described by the two-soliton solutions with no external potential. Then the position shift induced by collision can be given by asymptotic analysis of the two-soliton solution.[38] The position shift should induce a time duration modification ΔT_1, which can be described by

$$\Delta T_1 = \frac{\Delta x_1}{v_{\rm m}}, \tag{4}$$

where $v_{\rm m}$ is a characterized speed defined as $v_{\rm m} = (v_1 - v_2)/2$. Since there are two collisions in one period, the factor ΔT_1 should be calculated twice. Therefore, we obtain the modified soliton oscillation period

$$T = T_0 - 2\Delta T_1. \tag{5}$$

We find that numerical results agree well with our theoretical predictions when the relative speed is above the critical speed for tunneling behavior.[39] For example, we show the theoretical modification of the oscillation period vs different initial positions (red solid curves in Fig. 2) for the two bright solitons initially located symmetrically in the harmonic trap. The corresponding numerical simulation results are shown by the blue dashed curve in Fig. 2. It is seen that our theoretical modification Eq. (5) can describe well the oscillation period of the soliton pair in a harmonic potential.

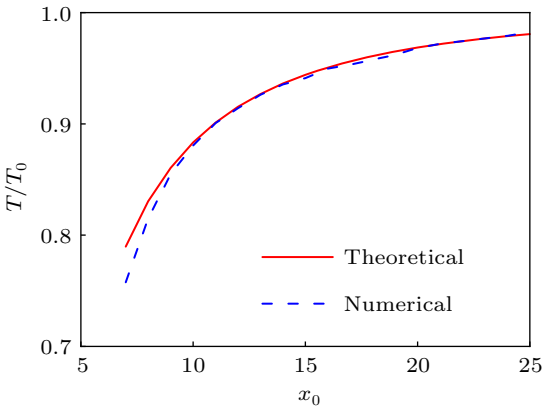


Fig. 2. The oscillation periods of soliton pairs modified by collision interactions. The red solid and the blue dashed curves are given by our analytical results Eq. (5) and the numerical simulations. The two curves fit well with each other in a wide range of soliton speed. But they show a slight deviation when soliton speed is very small due to the tunneling effect. The period tends to T_0 as the soliton speed increases.

It is known that the position shift tends to be zero with increasing the relative speeds between solitons, and the oscillation period will gradually tends to T_0.[21,22,36] It should be noted that the effects of position shift can be still seen after many oscillation periods in the harmonic trap for large relative speed cases.[22] Interestingly, our results suggest that the period modification can occur obviously in only one oscillation period, when the relative speed is small. If we further decrease the relative speed (decrease the $|x_0|$ value), the oscillation periods will deviate from our theoretical predictions (see Fig. 2). The deviation becomes much larger when the relative speed is lower. The deviation is induced the tunneling effect or resonant interaction between the two bright solitons,[39] which makes the quasi-particle picture can not hold anymore. The quantitative interpretation for the deviation is still need further discussion, which should be beyond the quasi-particle theory.

The effect of solitons collision in the identical component is demonstrated clearly in the above discussion. The collision can be seen as a coherent collision, due to that the interference pattern depends essentially on the relative phase between the two solitons. The collision between vector solitons such as bright–bright solitons, will involve both coherent and incoherent collision effects. Then we would like to discuss the oscillation behavior of bright–bright soliton pair in a harmonic trap. Recent studies on non-degenerate bright solitons[29,40,41] suggest that it is possible to investigate pure incoherent collision between solitons, for which the two solitons are placed in the two components separately (see Fig. 3(a)). The collision between solitons in different components are induced by inter-component interactions, and there is no interference pattern during the collision process. The relative phase plays no role in the collision process, therefore the collision can be seen as incoherent collision. Then we investigate the oscillation property of vector soliton pairs with pure incoherent collision in follows.

3. The oscillation property of vector soliton pairs with incoherent collision

We consider a two-component BEC trapped in general external potentials, whose dynamical equation can be written as the dimensionless coupled nonlinear Schrödinger equations (CNLSE)

$$\mathrm{i}\frac{\partial \psi_1}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi_1}{\partial x^2} + g(|\psi_1|^2 + |\psi_2|^2)\psi_1 + V_1(x)\psi_1, \tag{6}$$

$$\mathrm{i}\frac{\partial \psi_2}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi_2}{\partial x^2} + g(|\psi_1|^2 + |\psi_2|^2)\psi_2 + V_2(x)\psi_2, \tag{7}$$

where $g = -1$ denotes the nonlinear interaction strengths. $V_1(x)$ and $V_2(x)$ are the external potential for the two components separately. When $V_1(x) = V_2(x) = 0$, the CNLSE admits several vector soliton solutions, such as bright–bright solitons,[30–32] bright–dark solitons,[42] and the non-degenerate bright solitons.[29] These solutions can be used to give the initial conditions for observing soliton motion in weak external potentials. Firstly, we investigate the oscillation property of vector soliton pairs with pure incoherent collision in a harmonic trap, with the aid of non-degenerate bright solitons solution.[29] The external potential is also set as $V_1(x) = V_2(x) = (1/2)\Omega^2 x^2$, and the two soliton are placed

symmetrically, as shown in Fig. 3(a). The initial state can be given as

$$\psi_{10} = A_1 \ \mathrm{sech}[A_1\ (x - x_0)], \tag{8}$$

$$\psi_{20} = A_2 \ \mathrm{sech}[A_2\ (x + x_0)]. \tag{9}$$

For convenient comparison with the results for coherent collision case, we set the particle number of each soliton be identical with that of the scalar bright soliton pair, and the harmonic trap frequency is also chosen identically. The numerical evolution of two solitons are shown in Figs. 3(b) and 3(c).

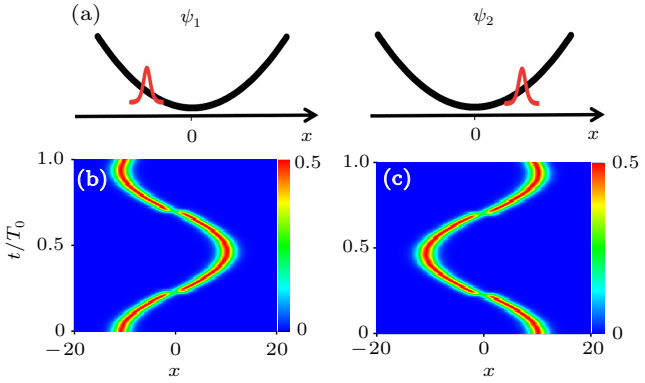


Fig. 3. (a) The profile schematic diagram of a vector soliton pair in a harmonic potential. A bright soliton is placed at x_0 in the first component and the other is placed at $-x_0$ in the second component. Panels (b) and (c) show the oscillation of the vector soliton pair. The incoherent collision effects also induce a position shift and make the oscillation of the vector soliton pair smaller than T_0. The initial position $x_0 = 10$ and other parameters are $A_1 = A_2 = \sqrt{2}/2$.

It can be seen that the oscillation period is also less than T_0 in this case, which confirms incoherent collision also affect their oscillation period in the harmonic potential trap. We can see that the oscillation period modification induced by incoherent collision is a bit less than the ones induced by coherent collision, with keeping identical particle number and external potential. The difference will be discussed in details in the following section. Then, we try to quantitatively explain the modification on the oscillation period. In this case, the collision velocity $v = \Omega x_0$ is the same as a scalar bright soliton. According to asymptotic analysis, we obtain the position shift caused by collision $\Delta x_2 = (1/2A_1)\ln[((A_1 - A_2)^2 + (v_1 - v_2)^2)/((A_1 + A_2)^2 + (v_1 - v_2)^2)]$. Each collision will modify the oscillation period by

$$\Delta T_2 = \frac{\Delta x_2}{v_\mathrm{m}}, \tag{10}$$

where $v_\mathrm{m} = (v_1 - v_2)/2$ is the characterized speed. Then we obtain the modified period affected by the incoherent collision

$$T = T_0 - 2\Delta T_2. \tag{11}$$

We make numerical simulations with different velocities (blue dashed curve in Fig. 4) to verify this theoretical modification of the oscillation period (red solid curve in Fig. 4). One can see that the theoretical predictions Eq. (11) are in good agreement with these numerical results, even if the collision speed is very small. Since there is no coherent interaction between solitons, the tunneling behavior is absent or the resonant interaction is much weaker for the two solitons even if the relative velocity is small, which enables the theoretical results holds more widely than the one for coherent collision case (see Fig. 2).

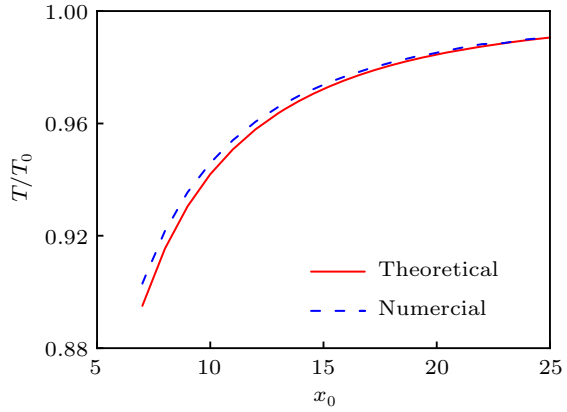


Fig. 4. The oscillation periods of vector soliton pairs modified by incoherent collision interactions. The red solid and the blue dashed curves are given by our analytical results Eq. (11) and the numerical simulations. The numerical results are in good agreement with the theoretical results, since the tunneling effect is absent.

4. The oscillation property of vector soliton pairs with both coherent and incoherent collision

We consider the case which involves both coherent and incoherent collision effects. We put two vector bright soliton pair in harmonic potential, shown in Fig. 5(a). The initial state is

$$\psi_{10} = A_{11}\,\mathrm{sech}[A_1(x + x_0)] + A_{21}\,\mathrm{sech}[A_2(x - x_0)], \tag{12}$$

$$\psi_{20} = A_{12}\,\mathrm{sech}[A_1(x + x_0)] + A_{22}\,\mathrm{sech}[A_2(x - x_0)]. \tag{13}$$

The parameters A_{1j} and A_{2j} $(j = 1, 2)$ are introduced to vary the relative particle population in the two components for the two solitons. The particle number of vector bright soliton pairs is also set to be identical with that of the scalar bright soliton pair, which inquires that $A_{j1}^2 + A_{j2}^2 = A_j^2$. We choose $A_{11} = A_{12} = A_1/\sqrt{2}$ and $A_{21} = A_{22} = A_2/\sqrt{2}$ to investigate the effects of coherent and incoherent collision effects, for which the particle populations in the two components are equal for the each vector soliton. We do numerical simulation to show the modification of oscillation period, shown in Figs. 5(b) and 5(c). One can see that the oscillation period is less than T_0. Then, the modified oscillation period can be derived as

$$T = T_0 - 2\Delta T_3, \tag{14}$$

where $\Delta T_3 = \Delta x_3/v_\mathrm{m}$, and the position shift after soltion collisions: $\Delta x_3 = (1/A_1)\ln[((A_1 - A_2)^2 + (v_1 - v_2)^2)/((A_1 + A_2)^2 + (v_1 - v_2)^2)]$. This theoretical result and numerical simulations are compared in Fig. 6. One can see that the theoretical predictions are also in good agreement with the numerical results, even if the collision speed is very small. The coher-

ent interaction for larger deviation in Fig. 2 could be inhibited greatly by the incoherent interaction between solitons.

We further compare the effects of these three different collisions on the periods of matter–wave bright solitons in a harmonic potential, the numerical results are shown in Fig. 7.

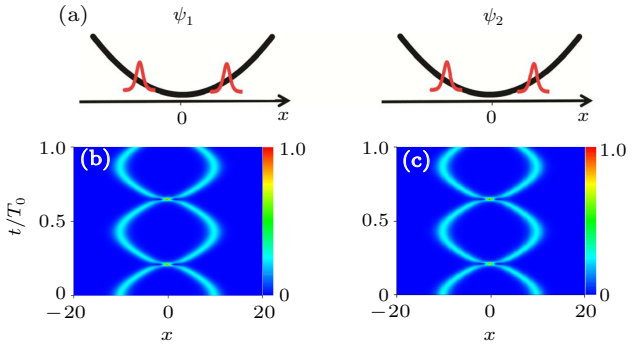


Fig. 5. (a) The profile schematic diagram of a vector bright–bright soliton pair in a harmonic potential. They are placed symmetrically at $\pm x_0$ positions. Panels (b) and (c) show the oscillation of the vector bright–bright soliton pair in the harmonic potentials of $\Omega = 0.02$. The oscillation period is also smaller than T_0. The initial position $x_0 = \pm 10$, and other parameters are $A_j = \sqrt{2}/2$.

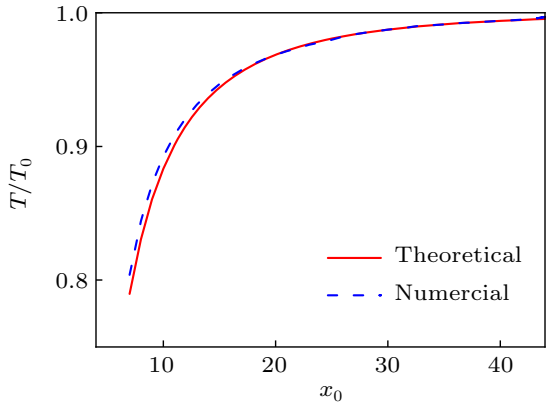


Fig. 6. The oscillation periods of vector bright–bright soliton pairs modified by both coherent and incoherent collisions. The red solid and the blue dashed curves are given by our theoretical results Eq. (14) and the numerical simulations.

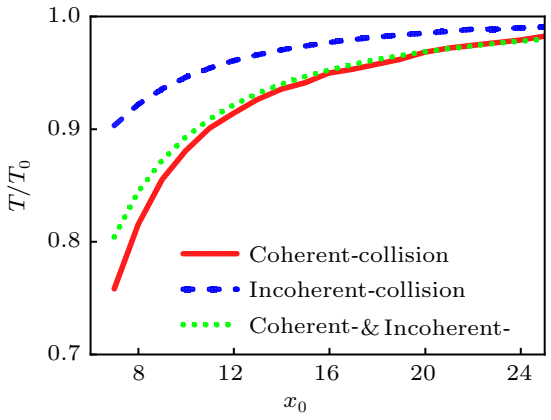


Fig. 7. Comparison of oscillation periods under three different collisions (obtained by numerical simulations). The red solid (blue dashed) curve denotes that the solitons collide with only coherent (incoherent) interactions. The green dotted curve indicates that both of them exist. The coherent (incoherent) collision interactions have the greatest (least) modification on the oscillation period. When both two collision effects exist, the modification is determined by the interplay between them.

One can see that coherent (incoherent) collision have the greatest (least) modification on the oscillation period of matter–wave bright solitons. When the two collision effects both exist, the modification is determined by the interplay between them, which depends on the relative population ratio of two components. With increasing collision speed, these three oscillation periods tend to be T_0. Based on these results, the oscillation period can be used to distinguish the coherent and incoherent collisions between solitons.

5. The oscillation property of dark–bright soliton pairs

The bright–dark solitons can also exist in the model Eq. (6) and Eq. (7) with attractive interaction between atoms. This could be used to investigate the effects of coupling with dark soliton on previously discussed bright soliton's oscillation in the harmonic trap. However, this type of vector soliton is unstable due to the modulational instability. Then we can set interaction between atoms to be repulsive, namely, $g = 1$, for which there are stable dark–bright solitons. When $V_1 = 0, V_2 = 0$ and $g = 1$, the CNLSE Eq. (6) and Eq. (7) admits one dark–bright soliton solution as follows:

$$\psi_1 = -\frac{\sqrt{c^2-(w^2+v^2)}}{(v+\mathrm{i}w)}\,\mathrm{sech}[w(x-vt)] \times \exp\left\{\mathrm{i}\left[vx-\left(\frac{1}{2}(v^2-w^2)+c^2\right)t\right]\right\}, \tag{15}$$

$$\psi_2 = \frac{c}{\sqrt{w^2+v^2}}[v-\mathrm{i}w\,\tanh[w(x-vt)]]\mathrm{e}^{-\mathrm{i}c^2t}, \tag{16}$$

where v is velocity of dark–bright solitons, w is the width and c is amplitude of plane wave background. The above solution becomes a scalar dark soliton, when $w^2 + v^2 = c^2$. The scalar dark soliton in a harmonic trap was shown to admit an oscillation frequency modified by a factor $\sqrt{2}$,[17] which was seen as Busch–Anglin effect. To observe the effects of dark soliton on bright soliton's motion, we would like to avoid the Bush–Anglin effect by setting $V_1(x) = \frac{1}{2}\Omega^2 x^2$ and $V_2(x) = 0$.

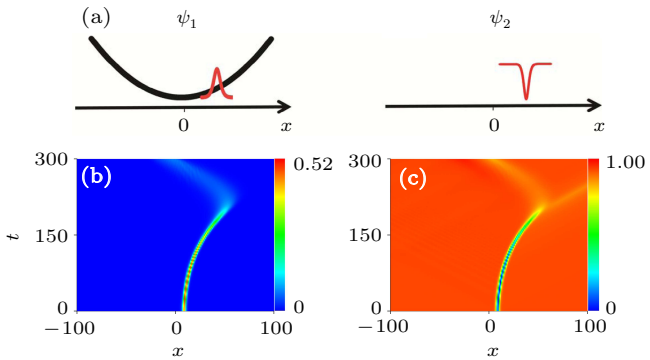


Fig. 8. (a) The profile schematic diagram of a dark–bright soliton when the harmonic potential is applied to the bright soliton component. Panels (b) and (c) show that the dark–bright soliton cannot oscillate in this case. Instead, it firstly moves outwards and then diffused. Other parameters are $\Omega = 0.02, x_0 = 10, w = \sqrt{2}/2$, and $v = 0$.

As shown in Fig. 8(a), we put one bright soliton at the position $x = x_0$ ($x_0 > 0$) of the harmonic potential and a dark soliton at the same position. The initial states are given by the above dark–bright soliton solution. The density evolution of the dark–bright soliton is shown in Figs. 8(b) and 8(c). The result shows that a bright soliton cannot oscillate in a harmonic trap, when it is coupled with a dark soliton (without any trapping potentials). Strikingly, both bright soliton and dark soliton accelerates in the opposite direction to the force exerted through V_1 and finally diffuses. This phenomenon is consistent with that the dark–bright soliton is driven out of the external trap.[19] In order to investigate a stable oscillation of dark–bright solitons, we change the harmonic potential to be an anti-harmonic potential

$$V_1(x) = -\frac{1}{2}\Omega^2 x^2$$

with keeping other conditions, which is shown in Fig. 9(a). The motion of dark–bright soliton is shown in Figs. 9(b) and 9(c), one can see that the dark–bright soliton can oscillate steadily with certain period in the anti-harmonic potential. Interestingly, the oscillation period T_0' is not $T_0 = 2\pi/\Omega$ anymore (in contrast to the scalar bright soliton and the bright–bright soliton), which can be explained by soliton's dispersion relation. The motion of the dark–bright soliton can be described well by a quasi-particle theory (see details in Appendix B), which is shown by white dashed curves in Figs. 9(b) and 9(c). Then we further discuss the effects of the position shift induced by solitons' collision on the oscillation behavior of the dark–bright solitons.

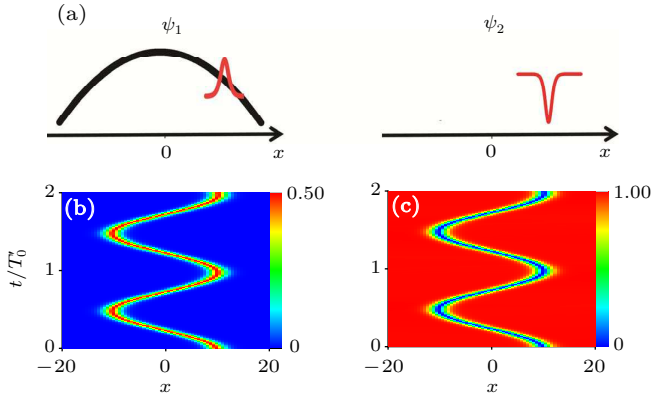


Fig. 9. (a) The profile schematic diagram of a dark–bright soliton in anti-harmonic potential. A bright soliton is placed at the position of $x_0 = 10$ in anti-harmonic potential and a dark soliton without trapping potential is placed at the same position. Panels (b) and (c) show that the dark–bright soliton can oscillate in an anti-harmonic potential of $\Omega = 0.02$. The moving trajectory of the soliton can be derived by a quasi-particle theory (white dashed curves). Other parameters are $w = \sqrt{2}/2$, and $v = 0$.

We put two identical bright solitons into the potential at the position $x = \pm x_0$ ($x_0 > 0$) in the first component and two identical dark solitons at the position $x = \pm x_0$ ($x_0 > 0$) in the second component without potential (Fig. 10(a)). The initial states are given as[43]

$$\begin{aligned}\psi_1 &= \sqrt{1-w^2}\,\{\mathrm{sech}[w\,(x-x_0)] + \mathrm{sech}[w\,(x+x_0)]\}\,,\\ \psi_2 &= \tanh[w(x-x_0)]\cdot\tanh[w(x+x_0)].\end{aligned} \tag{17}$$

The particle number of the bright soliton pair is the same with that of the scalar bright soliton pair in the above discussions, by choosing the parameter w value. The evolutions of two dark–bright solitons are shown in Figs. 10(b) and 10(c).

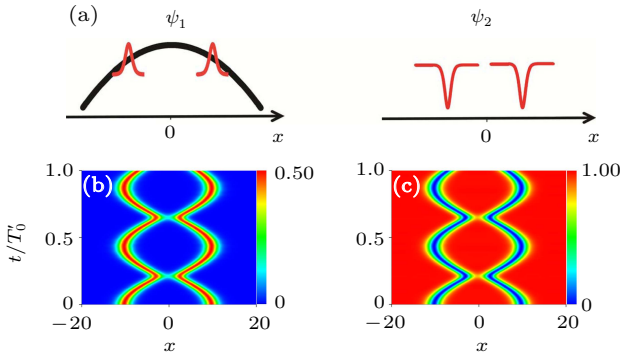


Fig. 10. (a) The profile schematic diagram of a dark–bright soliton pair in anti-harmonic potential ($\Omega = 0.02$). Two solitons are placed at $\pm x_0$. The potential is only applied to the bright solion component. Panels (b) and (c) demonstrate that the oscillation period of the dark–bright soliton pair is modified by their collision, and the period is smaller than T_0'. Other parameters are $x_0 = 10$, $w = \sqrt{2}/2$, and $v = 0$.

Time is normalized with the oscillation period T_0'. One can see that the oscillation period of the dark–bright solitons in the two components is less than T_0'. The period modification can be seen obviously in one oscillation period. The modified period induced by collision is

$$T = T_0' - 2\Delta T_4, \tag{18}$$

where $\Delta T_4 = \Delta x_4 / v_{\mathrm{m}}$, and the position shift is

$$\Delta x_4 = \frac{1}{2w_1}\ln\frac{|\Gamma|^2|\Delta_{12}|^4}{|\Xi_{12}|^2|\Lambda_{12}|^4},$$

where

$$\begin{aligned}|\Gamma|^2 &= -2v_1v_2 + v_2^2w_1^2 + (1+w_1w_2)^2 + v_1^2(v_2^2+w_2^2),\\ |\Xi_{12}|^2 &= -2v_1v_2 + v_2^2w_1^2 + (-1+w_1w_2)^2 + v_1^2(v_2^2+w_2^2),\\ |\Delta_{12}|^4 &= ((v_1-v_2)^2 + (w_1-w_2)^2)^2,\\ |\Lambda_{12}|^4 &= ((v_1-v_2)^2 + (w_1+w_2)^2)^2.\end{aligned}$$

Noticing that the velocity v_1 and v_2 are not $\pm\Omega x_0$ anymore, which should be derived from soliton's acceleration given by the quasi-particle theory.

The theoretically modified oscillation period (red solid curve in Fig. 11) is verified by numerical simulations (blue dashed curve in Fig. 11). As the collision speed increases, the oscillation period also tends to T_0'. Since the oscillation period of tunneling behavior is much larger than the collision duration for dark–bright solitons,[44] the theoretical results also

hold more widely than the ones for coherent collision between two bright solitons.

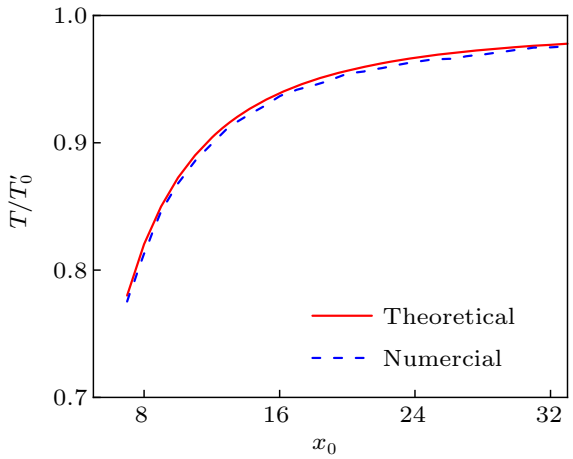


Fig. 11. The oscillation periods of vector dark–bright soliton pairs modified by their collisions.The red solid and the blue dashed curves are given by our theoretical results Eq. (18) and the numerical simulations.

6. Discussion and conclusion

In summary, we investigated the oscillation period of matter–wave bright soliton pair in harmonic traps. The results show that position shift caused by collisions can change the oscillation period of solitons significantly when collision speed is low. With the aid of a characterized speed, we theoretically describe the modified oscillation periods induced by coherent collision, incoherent collision, and both of them. Moreover, we investigated the oscillation behavior of a bright soliton, when it is coupled with a dark soliton. Interestingly, the bright soliton can oscillate in an anti-harmonic potential rather than a harmonic trap. The motion of them can be described well by the dispersion relation of a dark–bright soliton. The position shift induced by collision are also discussed for the oscillation of dark–bright solitons. These results would deepen our understanding on the motion of bright solitons in harmonic traps, and could inspire some experiments to observe the modified oscillation periods of soliton pair.[36] The dispersion relation of solitons discussed here could be extended to other solitons of nonlinear systems.[45–50]

The above discussions focus on the modification of oscillation period when two solitons are placed symmetrically in a harmonic potential. It is still necessary to discuss the cases for which the two solitons are placed asymmetrically. However, our numerical simulation results show that solitons oscillations are no longer periodic in asymmetric cases. This is partly due to that complex energy exchange between solitons occurs during the collision interaction, which results in different positions for each collision.

Appendix A: The dispersion relation for a bright soliton

We also consider the scalar nonlinear Schrödinger equation,

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - |\psi|^2\psi. \tag{A1}$$

It is well known that it admits a bright soliton solution, which can be written as:

$$\psi = A\ \mathrm{sech}[A(x-vt)]\exp\left\{\mathrm{i}[vx - \frac{1}{2}(v^2-A^2)t]\right\}, \tag{A2}$$

where A and v denote the soliton amplitude and soliton velocity respectively. To discuss the dispersion relation of one bright soliton, we firstly calculate the soliton energy, which can be derived as

$$\begin{aligned} E_\mathrm{s} &= \int_{-\infty}^{\infty}\psi^*\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi - \frac{1}{2}|\psi|^4\,\mathrm{d}x \\ &= Av^2 - \frac{A^3}{3}. \end{aligned} \tag{A3}$$

From the relations between moving velocity and canonical momentum of the bright soliton $v = \mathrm{d}E_\mathrm{s}/\mathrm{d}P_\mathrm{c}$, we can calculate the canonical momentum with

$$P_\mathrm{c} = \int_0^v \frac{1}{v}\frac{\mathrm{d}E_\mathrm{s}}{\mathrm{d}v}\,\mathrm{d}v = 2Av.$$

Then we obtain the dispersion relation for the bright soliton.

If we add a constant force F on the bright soliton atoms, a linear external potential $-Fx$ will emerge. Then the sum of soliton energy and its external potential energy will be conserved, namely, $E_\mathrm{s} + (-2Fx_\mathrm{c})A = Av^2 - A^3/3 + (-2Fx_\mathrm{c})A =$ constant. Deviating t to both hands, we obtain $\mathrm{d}v/\mathrm{d}t = F$, which is identical with each single atom under the action of the external force. This can be understood directly from the inertial mass of bright soliton $M^* = 2(\mathrm{d}E_\mathrm{s})/(\mathrm{d}(v^2)) = 2A$, which is identical to the particle number multiplying with the scaling atom mass 1. The effective force felt by the soliton is $2AF$. The acceleration of bright soliton in this case does not depend on the soliton amplitude A and its moving velocity, in sharp contrast to the dark soliton.

Appendix B: The trajectory of dark–bright soliton in anti-harmonic potential

Soliton's dispersion relation can be used to describe the motion of dark–bright soliton.[9,10,19] Therefore, we try to deduce the trajectory of dark–bright soliton (Eq. (17)) through energy conservation. The energy of the dark–bright soliton is

$$\begin{aligned} E_\mathrm{db} = \int &\frac{1}{2}|\psi_{1x}|^2 + \frac{1}{2}|\psi_{2x}|^2 + \frac{1}{2}|\psi_1|^2 \\ &+ \frac{1}{2}(|\psi_2|-1)^2 + |\psi_1|^2(|\psi_2|-1)\,\mathrm{d}x \end{aligned}$$

$$= \frac{1}{2}w(3-3v^2+w^2). \tag{B1}$$

Noting that the dark soliton energy form was firstly suggested in Ref. [43], and was derived explicitly in Ref. [51]. The soliton initial energy $E_{\rm db0}$ can be obtained according to the initial conditions. It is worth noting that the number of bright soliton particles must be conserved in this case

$$N_{\rm B} = \frac{2w(1-v^2-w^2)}{v^2+w^2} \equiv \text{constant}, \tag{B2}$$

which can be determined by initial soliton width w_0 and velocity v_0. Based on the condition Eq. (B2), we replace soliton width w with velocity v and $N_{\rm b}$ for simplicity.

Since the bright soliton is in an anti-harmonic potential and the potential varies slowly within the soliton size scale, the soliton potential energy can be written as

$$\begin{aligned} E_{\rm p} &= \int \frac{1}{2}\Omega^2 x^2 |\psi_1|^2 {\rm d}x = \frac{1}{2}\Omega^2 x_{\rm c}^2 \int |\psi_1|^2 {\rm d}x \\ &= \frac{1}{w}(w^2-1)\Omega^2 x_{\rm c}^2, \end{aligned} \tag{B3}$$

where $x_{\rm c}$ is the position of the soliton center. According to the conservation of dark–bright soliton energy:

$$E_{\rm db0} + E_{\rm p0} = E_{\rm db} + E_{\rm p}. \tag{B4}$$

Deviating t to both hands, we can obtain the acceleration of dark–bright soliton in anti-harmonic potential, when the soliton speed $v \to 0$. Thus, we obtain the theoretical trajectory of a dark–bright soliton which is shown in Figs. 9(b) and 9(c) as white dashed curves.

Acknowledgements

Project supported by the National Natural Science Foundation of China (Grant Nos. 12022513, 11775176, 11947301, and 12047502) and the Major Basic Research Program of the Natural Science of Foundation of Shaanxi Province, China (Grant Nos. 2018KJXX-094 and 2017KCT-12).

References

[1] Davis K B, Mewes M O, Andrews M R, van Druten N J, Durfee D S, Kurn D M and Ketterle W 1995 *Phys. Rev. Lett.* **75** 3969
[2] Bradley C C, Sackett C A, Tollett J J and Hulet R G 1995 *Phys. Rev. Lett.* **75** 1687
[3] Cornell E A and Wieman C E 2002 *Rev. Mod. Phys.* **74** 875
[4] De Moura M A 1994 *J. Phys. A: Math. Gen.* **27** 7157
[5] Liu X X, Zhang X F and Zhang P 2010 *Chin. Phys. Lett.* **27** 070306
[6] Zhao L C, Yang Z Y, Zhang T and Shi K J 2009 *Chin. Phys. Lett.* **26** 120301
[7] Li Y E and Xue J K 2016 *Chin. Phys. Lett.* **33** 100502
[8] Guo H, Qiu X, Ma Y, Jiang H F and Zhang X F 2021 *Chin. Phys. B* **30** 060310
[9] Shamailov S S and Brand J 2018 *SciPost Phys.* **4** 018
[10] Zhao L C, Wang W, Tang Q, Yang Z Y, Yang W L and Liu J 2020 *Phys. Rev. A* **101** 043621
[11] Pitaevskii L P 1961 *Sov. Phys. JETP* **13** 451
[12] Tsuzuki T 1971 *J. Low Temp. Phys.* **4** 441
[13] Strecker K E, Partridge G B, Truscott A G and Hulet R G 2002 *Nature* **417** 150
[14] Khaykovich L, Schreck F, Ferrari G, Bourdel T, Cubizolles J, Carr L D, Castin Y and Salomon C 2002 *Science* **296** 1290
[15] Serkin V N and Belyaeva T L 1972 *Sov. Phys. JETP* **34** 62
[16] Burger S, Bongs K, Dettmer S, Ertmer W, Sengstock K, Sanpera A, Shlyapnikov G V and Lewenstein M 1999 *Phys. Rev. Lett.* **83** 5198
[17] Busch Th and Anglin J R 2000 *Phys. Rev. Lett.* **84** 2298
[18] Brazhnyi V A and Konotop V V 2003 *Phys. Rev. A* **68** 043613
[19] Busch Th and Anglin J R 2001 *Phys. Rev. Lett.* **87** 010401
[20] Becker C, Stellmer S, Soltan-Panahi P, Dörscher S, Baumert M, Richter E M, Kronjäger J, Bongs K and Sengstock K 2008 *Nature Phys.* **4** 496
[21] Martin A D, Adams C S and Gardiner S A 2007 *Phys. Rev. Lett.* **98** 020402
[22] Martin A D, Adams C S and Gardiner S A 2008 *Phys. Rev. A* **77** 013620
[23] Tenorio C H, Vargas E V, Serkin V N, Agüero Granados M, Belyaeva T L, Peña Moreno R and Morales Lara L 2005 *Quantum Electronics* **35** 778
[24] Tenorio C H, Vargas E V, Serkin V N, Granados M Agüero, Belyaeva T L, Moreno R Peña and Lara L Morales 2005 *Quantum Electronics* **35** 929 -937
[25] Kevrekidis P G, Frantzeskakis D J and Carretero-González R 2008 *Emergent Nonlinear Phenomena in Bose–Einstein Condensates: Theory and Experiment* (Berlin: Springer)
[26] Zakharov V E and Shabat A B 1973 *Sov. Phys. JETP* **64** 1627
[27] Islam M N and Soccolich C E 1991 *Opt. Lett.* **16** 1490
[28] Zakharov V E and Shabat A B 1971 *Sov. Phys. JETP* **61** 118
[29] Qin Y H, Zhao L C and Ling L 2019 *Phys. Rev. E* **100** 022212
[30] Zhang X F, Hu X H, Liu X X and Liu W M 2009 *Phys. Rev. A* **79** 033630
[31] Perez-Garcia V M and Beitia J B 2005 *Phys. Rev. A* **72** 033620
[32] Adhikari S K 2005 *Phys. Rev. A* **346** 179
[33] Alotaibi M O D and Carr L D 2017 *Phys. Rev. A* **96** 013601
[34] Matveev V B 1991 *Solitons* (Berlin: Springer-Verlag).
[35] Ling L, Zhao L C and Guo B 2015 *Nonlinearity* **28** 3243
[36] Nguyen J H V, Dyke P, Luo D, Malomed B A and Hulet R G 2014 *Nat. Phys.* **10** 918
[37] Kivshar Y S and Agrawal G 2003 *Optical solitons: from fibers to photonic crystals* (San Diego: Academic Press)
[38] Faddeev L D and Takhtajan L A 2007 *Hamiltonian methods in the theory of solitons* (Berlin, Heidelberg: Springer-Verlag)
[39] Zhao L C, Ling L, Yang Z Y and Yang W L 2017 *Nonlinear Dyn.* **88** 2957
[40] Stalin S, Ramakrishnan R, Senthilvelan M and Lakshmanan M 2019 *Phys. Rev. Lett.* **122** 043901
[41] Zhao L C, Yang Z Y and Yang W L 2019 *Chin. Phys. B* **28** 010501
[42] Nistazakis H E, Frantzeskakis D J, Kevrekidis P G, Malomed B A and Carretero-González R 2008 *Phys. Rev. A* **77** 033612
[43] Kivshar Y S and Królikowski W 1995 *Opt. Commun.* **114** 353
[44] Qin Y H, Wu Y, Zhao L C and Yang Z Y 2020 *Chin. Phys. B* **29** 020303
[45] Zhang Y S and He J S 2019 *Chin. Phys. Lett.* **36** 030201
[46] Kang Z Z and Xia T C 2019 *Chin. Phys. Lett.* **36** 110201
[47] Wang B, Zhang Z and Li B 2020 *Chin. Phys. Lett.* **37** 030501
[48] Yang Y, Tian Z H and Jing J L 2020 *Chin. Phys. Lett.* **37** 048501
[49] Wang W, Yao R X and Lou S Y 2020 *Chin. Phys. Lett.* **37** 100501
[50] Jia C Y and Liang Z X 2020 *Chin. Phys. Lett.* **37** 040502
[51] Zhao L C, Qin Y H, Wang W L and Yang Z Y 2020 *Chin. Phys. Lett.* **37** 050502