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Propagation and modulational instability of Rossby waves in stratified fluids

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Perturbation analysis and scale expansion are used to derive the (2+1)-dimensional coupled nonlinear Schrödinger (CNLS) equations that can describe interactions of two Rossby waves propagating in stratified fluids. The (2+1)-dimensional equations can reflect and describe the wave propagation more intuitively and accurately. The properties of the two waves in the process of propagation can be analyzed by the solution obtained from the equations using the Hirota bilinear method, and the influence factors of modulational instability are analyzed. The results suggest that, when two Rossby waves with slightly different wave numbers propagate in the stratified fluids, the intensity of bright soliton decreases with the increases of dark soliton coefficients. In addition, the size of modulational instable area is related to the amplitude and wave number in y direction.

Keywords: Rossby waves, Hirota bilinear method, modulational instability, stratified fluids

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1. Introduction

On the earth where people rely on, the area of ocean has exceeded the area of land. The exploration to the vast ocean has never been stopped. There are many phenomena in the ocean that are worth exploring and studying, such as nonlinear waves. Partial differential equations can be used to describe many nonlinear phenomena in atmosphere and ocean motion. These equations not only reveal the essence of the phenomena but also provide important theoretical basis and research value for nonlinear atmospheric and marine dynamics. Solitary waves are one kind of nonlinear waves, which were first discovered by Russell, a British scientist, and researchers draw a conclusion that the solitary waves can move forward continuously regardless of friction and dissipation. Rossby waves are one type of solitary waves, and they have stable isolated wave characteristics with large amplitudes. They are waves produced by horizontal disturbances of the atmosphere and ocean under the action of Rossby parameter β. It is the existence of Rossby waves that causes energy to propagate in the ocean and atmosphere. The origin of Rossby waves is atmospheric long waves. In 1939, Rossby first studied and analyzed the properties of atmospheric long waves in theory, and established the theory of atmospheric long waves. In order to commemorate him, researchers also call the atmospheric long waves the Rossby waves.

Rossby waves widely exist in space and earth fluid systems.[1] Their existence is related to many natural phenomena, from the great red spot in Jupiter's atmosphere to the swirling currents in the Gulf of Mexico. In addition to earth fluid systems, Rossby waves are also of great importance in marine atmospheric science. Researchers suggested that Rossby waves play an important role in the transfer of the energy, mass and momentum in the atmosphere and oceans, and to some extent determine the response of the oceans to atmospheric and other climate changes. Rossby waves spread the energy from east to west, it affects ocean surface water color and biological interactions, thus regulates the interannual characteristic behavior of the ocean, such as EI Niño.[2] In the research of Battisti,[3] non-equatorial Rossby waves were considered to be products of the ENSO (EI Niño/Southern Oscillation) rather than trigger mechanisms. In the study of the steady oscillation characteristics of sea surface temperatures (SST) variation,[4] the results show that the west boundary reflection of the Rossby waves excited by the wind stress caused by the interannual SST anomaly generates the eastward Kelvin waves, which eventually leads to the coupled instability of the eastern Pacific Ocean. Recently, Yang *et al.* studied the influence of the average airflow and topography in the barotropic atmosphere on the propagation of Rossby waves in the case of considering the vertical and zonal.[5] The discovery of Rossby waves in blood vessels has aroused extensive interest of researchers. Many researchers have put forward valuable theoretical basis for the problems in blood vessels by studying Rossby waves in blood vessels.[6]

As mentioned earlier, nonlinear waves can be described by many partial differential equations, so there are also many partial differential equations to describe Rossby waves, such as the KdV equation,[7] mKdV equation,[8] Boussinesq

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equation,[9] BO equation, ZK equation,[10] and the nonlinear Schrödinger (NLS) equation.[11] In recent decades, the NLS equation has been widely used in many fields of applied sciences, especially in deep water wave mechanics[12] and optical communication.[13] The form of the NLS equation has also been extended gradually, such as higher-order term and coupled equations, the NLS equation with higher-order term is also called the HNLS equation, coupled NLS equations are called the CNLS equation. Luo *et al.* used the HNLS equation to describe nonlinear modulated Rossby waves in geophysical fluids and discussed the effects of latitude and uniform basic background flow on the unstable growth rate and unstable area of uniform Rossby waves train.[14] They also used the CNLS equations to describe the propagation and interactions of two nonlinear Rossby waves in barotropic modes, then used these equations to study the collision interactions of two enveloping Rossby solitons.[15] Choy *et al.* regarded blood in blood vessels as incompressible fluids and obtained its governing equation using the perturbation method, the governing equation was variable-coefficient NLS equations.[16] Song *et al.* derived the NLS equation of weakly nonlinear deep-sea internal waves based on the basic equation of two-layered fluids, and numerically simulated the wave propagation in the deep-sea area of the South China Sea.[17] With further research, the equations can be used as a powerful tool to study in different fields. As for the methods of solving the equation, the Hirota bilinear method is used in this paper to solve the CNLS equations. In fact, there are many other methods to solve all kinds of equations, such as F-expansion,[18,19] $\frac{G}{G'}$ method,[20,21] Backlund transformation,[22,23] and Darboux transformation.[24,25] There are many different forms of the solutions, such as exact solution,[26,27] breather solution,[28,29] mixed-soliton solution,[30] and rogue wave solution.[31,32] Liu *et al.* derived the general periodic solution by the bilinear method and obtained rogue waves after further research.[33] Because the propagation of waves is not stable and fixed, there will be unstable areas in the process of propagation and the existence of these areas will cause uncertain impact on waves propagation, so the study of the modulational instability of waves is an important part.[34–37]

In real atmospheric and oceanic motion, the stratification of fluid density (the stratification effect) makes the problem more complicated. Due to stratified fluids being closer to real fluids, it is of great practical significance to study the evolution and development of nonlinear Rossby waves in it, which can better explain the fluctuation mechanism of some large-scale nonlinear waves in atmospheric and oceanic motions. In this paper, the stratified fluids are regarded as background, then the propagation and interactions between the two Rossby waves are investigated. This paper is organized as follows: In Section 2, based on the quasi-geostrophic vorticity equation of stratified fluids we derive the equations by taking the β effect into account and using scale analysis and perturbation expansion. In Section 3, the Hirota bilinear method is used to solve the equations and the properties of the solutions in the interaction process are analyzed. In Section 4, we analyze the modulational instability of CNLS equations. Finally, we draw conclusions in Section 5.

2. Derivation of the model

Start from the quasi-geostrophic vorticity equation of stratified fluids

$$\left(\frac{\partial}{\partial t}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial x}\right)\times\left[\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\varphi}{\partial z}\right)+\beta(y)\right]=0, \tag{1}$$

make it dimensionless, $\varphi=L_0U_0\varphi'$, $\beta=\frac{U_0}{L_0}\beta'$, $L_0(x,y)=L_0(x',y')$, $z=H_0z'$. Then, we can obtain

$$\left(\frac{\partial}{\partial t}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial x}\right)\times\left[\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi\right)+\beta(y)\right]=0, \tag{2}$$

where $K^2=\frac{L_0^2}{L_{00}^2}$, $L_{00}^2=\frac{N^2H_0^2}{f_0^2}$, L_0 is the spatial horizontal eigenvectors, U_0 is the characteristic quantity of velocity, H_0 is the characteristic quantity of liquid thickness, and the total flow function is $\psi=\bar{U}(y)+\varepsilon\varphi'(x,y,z,t)$. Next, perform some transformations as follows:

$$\begin{aligned}&T_1=\varepsilon t,\ \ T_2=\varepsilon^2 t;\ \ X_1=\varepsilon x,\ \ X_2=\varepsilon^2 x;\ \ Y=\varepsilon y;\ \ z=z,\\&\frac{\partial}{\partial t}\longrightarrow\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial T_1}+\varepsilon^2\frac{\partial}{\partial T_2},\\&\frac{\partial}{\partial x}\longrightarrow\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X_1}+\varepsilon^2\frac{\partial}{\partial X_2},\\&\frac{\partial}{\partial y}\longrightarrow\frac{\partial}{\partial y}+\varepsilon\frac{\partial}{\partial Y},\\&\frac{\partial}{\partial z}\longrightarrow\frac{\partial}{\partial z},\end{aligned} \tag{3}$$

and we can find an asymptotic solution to Eq. (1) by writing

$$\begin{aligned}\varphi=&\ \bar{U}y+\varepsilon\varphi^{(1)}(T_1,T_2,X_1,X_2,Y,x,y,z,t)\\&+\varepsilon^2\varphi^{(2)}(T_1,T_2,X_1,X_2,Y,x,y,z,t)\\&+\varepsilon^3\varphi^{(3)}(T_1,T_2,X_1,X_2,Y,x,y,z,t)+\cdots.\end{aligned} \tag{4}$$

In this form, ε is a small perturbation, $\bar{U}$ is the background flow. Substituting Eqs. (3) and (4) into Eq. (2), the equations satisfying $\varphi^{(1)}$, $\varphi^{(2)}$ and $\varphi^{(3)}$ can be obtained as follows:

$$L(\varphi^{(1)}) = \left(\frac{\partial}{\partial t}-\bar{U}\frac{\partial}{\partial x}\right)\left[\nabla^2\varphi^{(1)}+\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi^{(1)}\right)\right]+\beta'\frac{\partial}{\partial x}\varphi^{(1)}=0, \tag{5}$$

$$\begin{aligned}L(\varphi^{(2)}) =& -\left(\frac{\partial}{\partial T_1}-\bar{U}\frac{\partial}{\partial X_1}\right)\left[\nabla^2\varphi^{(1)}+\frac{\partial}{\partial z}(K^2\frac{\partial}{\partial z}\varphi^{(1)})\right]-2\left(\frac{\partial}{\partial t}-\bar{U}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x\partial X_1}+\frac{\partial^2}{\partial y\partial Y}\varphi^{(1)}\right)\\ &-\beta'\frac{\partial}{\partial X_1}\varphi^{(1)}-J[\varphi^{(1)},\nabla^2\varphi^{(1)}]-J\left[\varphi^{(1)},\frac{\partial}{\partial z}(K^2\frac{\partial}{\partial z}\varphi^{(1)})\right],\end{aligned} \tag{6}$$

$$\begin{aligned}L(\varphi^{(3)}) =& -\left(\frac{\partial}{\partial T_2}-\bar{U}\frac{\partial}{\partial X_2}\right)\left[\nabla^2\varphi^{(1)}+\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi^{(1)}\right)\right]-2\left(\frac{\partial}{\partial T_1}-\bar{U}\frac{\partial}{\partial X_1}\right)\left(\frac{\partial^2\varphi^{(1)}}{\partial x\partial X_1}+\frac{\partial^2\varphi^{(1)}}{\partial y\partial Y}\right)\\ &-\left(\frac{\partial}{\partial T_1}-\bar{U}\frac{\partial}{\partial X_1}\right)\left[\nabla^2\varphi^{(2)}+\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi^{(2)}\right)\right]-\left(\frac{\partial}{\partial t}-\bar{U}\frac{\partial}{\partial x}\right)\left(2\frac{\partial^2}{\partial x\partial X_2}+\frac{\partial^2}{\partial X_1^2}+\frac{\partial^2}{\partial Y^2}\right)\varphi^{(1)}\\ &-2\left(\frac{\partial}{\partial t}-\bar{U}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x\partial X_1}+\frac{\partial^2}{\partial y\partial Y}\right)\varphi^{(2)}-\frac{\partial}{\partial x}\varphi^{(1)}\frac{\partial}{\partial y}\nabla^2\varphi^{(2)}-\frac{\partial}{\partial x}\varphi^{(2)}\frac{\partial}{\partial y}\nabla^2\varphi^{(1)}+\frac{\partial}{\partial y}\varphi^{(1)}\frac{\partial}{\partial x}\nabla^2\varphi^{(2)}\\ &+\frac{\partial}{\partial y}\varphi^{(2)}\frac{\partial}{\partial x}\nabla^2\varphi^{(1)}-\frac{\partial}{\partial x}\varphi^{(1)}\frac{\partial}{\partial Y}\nabla^2\varphi^{(1)}+\frac{\partial}{\partial Y}\varphi^{(1)}\frac{\partial}{\partial x}\nabla^2\varphi^{(1)}-\frac{\partial}{\partial X_1}\varphi^{(1)}\frac{\partial}{\partial y}\nabla^2\varphi^{(1)}-\beta'\frac{\partial}{\partial X_2}\varphi^{(1)}\\ &-2\frac{\partial}{\partial x}\varphi^{(1)}\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x\partial X_1}+\frac{\partial^2}{\partial y\partial Y}\right)\varphi^{(1)}+2\frac{\partial}{\partial y}\varphi^{(1)}\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x\partial X_1}+\frac{\partial^{(2)}}{\partial y\partial Y}\right)\varphi^{(1)}+\frac{\partial}{\partial y}\varphi^{(1)}\frac{\partial}{\partial X_1}\nabla^2\varphi^{(1)}\\ &-\frac{\partial}{\partial x}\varphi^{(1)}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial z}(K^2\frac{\partial}{\partial z}\varphi^{(2)})\right]-\frac{\partial}{\partial x}\varphi^{(2)}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi^{(1)}\right)\right]-\frac{\partial}{\partial x}\varphi^{(1)}\frac{\partial}{\partial Y}\left[\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi^{(1)}\right)\right]\\ &+\frac{\partial}{\partial y}\varphi^{(1)}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi^{(2)}\right)\right]+\frac{\partial}{\partial y}\varphi^{(2)}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}(K^2\frac{\partial}{\partial z}\varphi^{(1)})\right]-\frac{\partial}{\partial X_1}\varphi^{(1)}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi^{(1)}\right)\right]\\ &+\frac{\partial}{\partial Y}\varphi^{(1)}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi^{(1)}\right)\right]+\frac{\partial}{\partial y}\varphi^{(1)}\frac{\partial}{\partial X_1}\left[\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\varphi^{(1)}\right)\right]-\beta'\frac{\partial}{\partial X_1}\varphi^{(2)}.\end{aligned} \tag{7}$$

Next assume

$$\begin{aligned}\varphi^{(1)} =& \sum_n A_n(T_1,T_2,X_1,X_2,Y)\phi_n(y)\\ &\times\exp[\mathrm{i}(k_nx+l_nz-\omega_nt)]+\text{c.c.},\end{aligned} \tag{8}$$

where $c.c.$ represents the conjugate of all the preceding terms. A_n is the amplitude as a function of the slow space-time variable, k_n and l_n are wave numbers in x and y, respectively, and the ω_n represents frequency. The differential operator is defined in the following form:

$$L\equiv\left(\frac{\partial}{\partial t}-\bar{U}\frac{\partial}{\partial x}\right)\left[\nabla^2+\frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\right)\right]+\beta'(y)\frac{\partial}{\partial x}, \tag{9}$$

then putting Eq. (8) into Eq. (9), the condition satisfying $\phi_n(y)$ can be obtained as follows:

$$\begin{aligned}&\phi_n''-\left(k_n^2+K^2l_n^2+\frac{\beta'}{\bar{U}+c_n}\right)\phi_n=0,\\ &c_n=\frac{\omega_n}{k_n},\quad \phi_n(0)=\phi_n(\pi)=0.\end{aligned} \tag{10}$$

The 0 and π are boundary conditions of fluids. Because the discussion is about two-wave situation, assume the form of $\varphi^{(1)}$ as

$$\begin{aligned}\varphi^{(1)} =& A_1\phi_1(y)\exp[\mathrm{i}(k_1x+l_1z-\omega_1t)]\\ &+A_2\phi_2(y)\exp[\mathrm{i}(k_2x+l_2z-\omega_2t)]+\text{c.c.},\end{aligned} \tag{11}$$

substituting Eq. (11) into Eq. (6), we can write $L(\varphi^{(2)})$ as

$$\begin{aligned}L(\varphi^{(2)}) =& \sum_{n=1}^{2}G_{1n}\exp[\mathrm{i}(k_nx+l_nz-\omega_nt)]\\ &-\sum_{n=1}^{2}\mathrm{i}g_{2n}A_n^2\exp[\mathrm{i}(k_nx+l_nz-\omega_nt)]\\ &-\mathrm{i}g_3A_1A_2\exp[\mathrm{i}(k_{12}x+l_{12}z-\omega_{12}t)]\\ &-\mathrm{i}g_4A_1A_2^*\exp[\mathrm{i}(\alpha_{12}x+\sigma_{12}z-\nu_{12}t)],\end{aligned} \tag{12}$$

where

$$\begin{aligned}\alpha_{12} =& k_1-k_2,\ \ \sigma_{12}=l_1-l_2,\ \ \upsilon_{12}=\omega_1-\omega_2,\\ k_{12} =& k_1+k_2,\ \ l_{12}=l_1+l_2,\ \ \omega_{12}=\omega_1+\omega_2,\\ G_{1n} =& [\phi_n''(y)-(k_n^2+K^2l_n^2)\phi_n(y)]\frac{\partial A_n}{\partial T_1}\\ &-\bar{U}[\phi_n''(y)-(k_n^2+K^2l_n^2)\phi_n(y)]\frac{\partial A_n}{\partial X_1}\\ &+[2(k_n\omega_n+\bar{U}k_n^2)\phi_n(y)+\beta'\phi_n(y)]\frac{\partial A_n}{\partial X_1}\\ &-2[\mathrm{i}\omega_n\phi_n'(y)+\bar{U}\,\mathrm{i}k_n\phi_n'(y)]\frac{\partial A_n}{\partial Y},\\ g_{2n} =& k_n\left(\phi_n\frac{\mathrm{d}}{\mathrm{d}x}-\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right)\frac{\mathrm{d}^2\phi_n}{\mathrm{d}y^2},\\ g_3 =& \left(k_1\phi_1\frac{\mathrm{d}}{\mathrm{d}y}-k_2\frac{\mathrm{d}\phi_1}{\mathrm{d}y}\right)\left[\frac{\mathrm{d}^2\phi_2}{\mathrm{d}y^2}-(k_2^2+K^2l_2^2)\phi_2\right]\\ &+\left(k_2\phi_2\frac{\mathrm{d}}{\mathrm{d}y}-k_1\frac{\mathrm{d}\phi_2}{\mathrm{d}y}\right)\left[\frac{\mathrm{d}^2\phi_1}{\mathrm{d}y^2}-(k_1^2+K^2l_1^2)\phi_1\right],\\ g_4 =& \left(k_1\phi_1\frac{\mathrm{d}}{\mathrm{d}y}+k_2\frac{\mathrm{d}\phi_1}{\mathrm{d}y}\right)\left[\frac{\mathrm{d}^2\phi_2}{\mathrm{d}y^2}-(k_2^2+K^2l_2^2)\phi_2\right]\\ &-\left(k_2\phi_2\frac{\mathrm{d}}{\mathrm{d}y}+k_1\frac{\mathrm{d}\phi_2}{\mathrm{d}y}\right)\left[\frac{\mathrm{d}^2\phi_1}{\mathrm{d}y^2}-(k_1^2+K^2l_1^2)\phi_1\right].\end{aligned} \tag{13}$$

The second, third, and fourth inhomogeneous terms in Eq. (12) can yield particular solutions in the following forms:

$$\begin{aligned}
\varphi_{2n}^{(2)} &= Y_{2n}A_n^2\exp[2\mathrm{i}(k_nx+l_nz-\omega_nt)]+\mathrm{c.c.},\\
\varphi_{3}^{(2)} &= Y_{3}A_1A_2\exp[\mathrm{i}(k_{12}x+l_{12}z-\omega_{12}t)]+\mathrm{c.c.},\\
\varphi_{4}^{(2)} &= Y_{4}A_1A_2^*\exp[\mathrm{i}(\alpha_{12}x+\sigma_{12}z-\nu_{12}t)]+\mathrm{c.c.},
\end{aligned}\tag{14}$$

and $\varphi_{2n}^{(2)}$, $\varphi_{3}^{(2)}$, $\varphi_{4}^{(2)}$ satisfy the following conditions:

$$\begin{aligned}
&\frac{\mathrm{d}^2Y_{2n}}{\mathrm{d}y^2}-\left[4(k_n^2+K^2l_n^2)+\frac{\beta' k_n}{\omega_n+\bar{U}k_n}\right]Y_{2n}\\
&=\sum_{n=1}^{2}\frac{g_{2n}}{2(\omega_n+\bar{U}k_n)},\quad Y_{2n}(0)=Y_{2n}(\pi)=0,\\
&\frac{\mathrm{d}^2Y_{3}}{\mathrm{d}y^2}-\left(k_{12}^2+K^2l_{12}^2+\frac{\beta' k_{12}}{\omega_{12}+\bar{U}k_{12}}\right)Y_{3}\\
&=\frac{g_3}{\omega_{12}+\bar{U}k_{12}},\quad Y_3(0)=Y_3(\pi)=0,\\
&\frac{\mathrm{d}^2Y_{4}}{\mathrm{d}y^2}-\left(\alpha_{12}^2+K^2\sigma_{12}^2+\frac{\beta'\alpha_{12}}{\nu_{12}+\bar{U}\alpha_{12}}\right)Y_{4}\\
&=\frac{g_4}{\nu_{12}+\bar{U}\alpha_{12}},\quad Y_4(0)=Y_4(\pi)=0.
\end{aligned}\tag{15}$$

The first inhomogeneous term in Eq. (12) yields the following special solution:

$$\varphi_{1n}^{(2)}=\phi_{1n}^{(2)}\exp[\mathrm{i}(k_nx+l_nz-\omega_nt)]+\mathrm{c.c.},\tag{16}$$

where $\phi_{1n}^{(2)}$ satisfies

$$\begin{aligned}
&\frac{\mathrm{d}^2\phi_{1n}^{(2)}}{\mathrm{d}y^2}-\left(k_n^2+K^2l_n^2+\frac{\beta' k_n}{\omega_n+\bar{U}k_n}\right)\phi_{1n}^{(2)}\\
&=-\frac{\mathrm{i}}{\omega_n+\bar{U}k_n}\left[\frac{\mathrm{d}^2\phi_{1n}}{\mathrm{d}y^2}-(k_n^2+K^2l_n^2)\phi_{1n}\right]\frac{\partial A_n}{\partial T_1}\\
&\quad+\frac{\mathrm{i}}{\omega_n+\bar{U}k_n}\left[\bar{U}\left(\frac{\mathrm{d}^2\phi_{1n}}{\mathrm{d}y^2}-(k_n^2+K^2l_n^2)\phi_{1n}\right)\right.\\
&\quad\left.-2(\omega_nk_n+\bar{U}k_n^2)\phi_{1n}-\beta'\phi_{1n}\right]\frac{\partial A_n}{\partial X_1}\\
&\quad+\frac{\mathrm{i}}{\omega_n+\bar{U}k_n}[2\mathrm{i}(\omega_n+\bar{U}k_n)\phi_{1n}']\frac{\partial A_n}{\partial Y}.
\end{aligned}\tag{17}$$

Multiplying the left-hand side of Eq. (17) by ϕ_n, integrating over y from 0 to π and using the boundary conditions show that the integration is equal to zero. Thus, the same operations are made to the right-hand side of Eq. (17) and we can obtain the consistent results. These lead to the solvable condition

$$\frac{\partial A_n}{\partial T_1}+C_{gn}\frac{\partial A_n}{\partial X_1}=0,\tag{18}$$

where

$$C_{gn}=C_n+\frac{2\int_0^\pi k_n^2\phi_n^2\mathrm{d}y}{\int_0^\pi\frac{\beta'}{(c_n+\bar{U})^2}\phi_n^2\mathrm{d}y}.\tag{19}$$

It is not difficult to find from Eq. (18) that the amplitude A_n spreads at the speed of C_{gn}, which means

$$A_n=A_n(X_1-C_{gn}T_1,X_2,T_2,Y),\tag{20}$$

further we can suppose

$$\phi_{1n}^{(2)}=\mathrm{i}Y_{1n}\frac{\partial A_n}{\partial X_1}.\tag{21}$$

Substituting Eq. (21) into Eq. (17) and considering the condition Eq. (18), we can obtain the following equation:

$$\begin{aligned}
&\frac{\mathrm{d}^2Y_{1n}}{\mathrm{d}y^2}-\left(k_n^2+K^2l_n^2+\frac{\beta' k_n}{\omega_n+\bar{U}k_n}\right)Y_{1n}\\
&=\frac{1}{\omega_n+\bar{U}k_n}\left\{(C_{gn}+\bar{U})\left[\frac{\mathrm{d}^2\phi_n}{\mathrm{d}y^2}-(k_n^2+K^2l_n^2)\phi_n\right]\right.\\
&\quad\left.-2(\omega_n+\bar{U}k_n)k_n\phi_n-\beta'\phi_n\right\}.
\end{aligned}\tag{22}$$

Up to now, the solution to Eq. (12) can be written as

$$\begin{aligned}
\varphi^{(2)} &= \varphi_{11}^{(2)}+\varphi_{12}^{(2)}+\varphi_{21}^{(2)}+\varphi_{22}^{(2)}+\varphi_{3}^{(2)}+\varphi_{4}^{(2)}\\
&\quad+\xi(y,T_1,X_1,Y),
\end{aligned}\tag{23}$$

where $\xi(y,T_1,X_1,Y)$ is the homogeneous solution to Eq. (12), which represents the regional flow correction caused by the existence of finite-amplitude wave and can be expressed later. Substituting $\varphi^{(1)}$ and $\varphi^{(2)}$ into the right-hand side of Eq. (7), we can obtain all the inhomogeneous terms. There are terms that are independent of x, z and t. Thus, considering the form of the linear operation on the left-hand side, it is obvious that these terms must disappear equally, resulting in the condition related to the correction to the average flow to the wave amplitude, implying

$$\begin{aligned}
\left(\frac{\partial}{\partial T_1}-\bar{U}\frac{\partial}{\partial X_1}\right)\frac{\partial^2\xi}{\partial y^2}+\beta'\frac{\partial\xi}{\partial X_1} &= \sum_{n=1}^{2}\left\{\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left[k_n\left(Y_{1n}\frac{\mathrm{d}\phi_n}{\mathrm{d}y}-\phi_n\frac{\mathrm{d}Y_{1n}}{\mathrm{d}y}\right)\right]+4k_n^2\phi_n\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right\}\frac{\partial}{\partial X_1}|A_n|^2\\
&\quad+\sum_{n=1}^{2}\left\{\left(\phi_n\frac{\mathrm{d}}{\mathrm{d}y}-\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2\phi_n}{\mathrm{d}y^2}+k_n^2\phi_n\right)-\left(k_n\phi_n'''Y_{1n}-k_n\phi_n\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}^2Y_{1n}}{\mathrm{d}y^2}\right)\right\}\frac{\partial|A_n|^2}{\partial X_1}\\
&\quad+\sum_{n=1}^{2}\left\{2\mathrm{i}k_n\left[(k_n^2+K^2l_n^2)\phi_n^2-\left(\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\frac{\mathrm{d}\phi_n}{\mathrm{d}y}+2\phi_n\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right)\right]\right\}\frac{\partial}{\partial Y}|A_n|^2.
\end{aligned}\tag{24}$$

There are other kinds of inhomogeneous terms on the right-hand side of Eq. (7) that are proportional to $\exp[-\mathrm{i}(k_n x + l_n z - \omega_n t)]$. In order to obtain solutions that correspond to these conditions, some solvable conditions must be set up. Multiplying both sides of Eq. (7) by $\phi_n \exp[-\mathrm{i}(k_n x + l_n z - \omega_n t)]$ and integrating over x from 0 to $\frac{2\pi}{k_n}$, over t from 0 to $\frac{2\pi}{\omega_n}$, over z from 0 to $\frac{2\pi}{l_n}$ and over y from 0 to π, the integration of the left-hand side is zero, so the integration of the right-hand side is also zero. Considering some particular items, we can obtain the solvable conditions as follows:

$$\begin{aligned}
&\left(\frac{\partial}{\partial T_1} + C_{g1}\frac{\partial}{\partial X_2}\right)A_1 \\
&-\mathrm{i}\alpha_1\frac{\partial^2 A_1}{\partial X_1^2} - \mathrm{i}(\rho_1|A_1|^2 + \gamma_{12}|A_2|^2 + \lambda_1)A_1 - \mathrm{i}\eta_1\frac{\partial^2 A_1}{\partial Y^2} = 0, \\
&\left(\frac{\partial}{\partial T_1} + C_{g2}\frac{\partial}{\partial X_2}\right)A_2 - \mathrm{i}\alpha_2\frac{\partial^2 A_2}{\partial X_1^2} \\
&-\mathrm{i}(\rho_2|A_2|^2 + \gamma_{21}|A_1|^2 + \lambda_2)A_2 - \mathrm{i}\eta_2\frac{\partial^2 A_2}{\partial Y^2} = 0,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
&\alpha_n = \frac{I_{n0}}{II_n}, \quad \eta_n = \frac{I_{0n}}{II_n}, \quad \rho_n = \frac{I_{nn}}{II_n}, \\
&\gamma_{12} = \frac{I_{12}}{II_1}, \quad \lambda_n = \frac{I_{n3}}{II_n},
\end{aligned} \tag{26}$$

while

$$\begin{aligned}
&II_n = -\int_0^\pi \frac{\phi_n}{\bar{U}+C_n} f_n \mathrm{d}y, \quad I_{n0} = \int_0^\pi \frac{\phi_n}{\bar{U}+C_n} f_{n0}\mathrm{d}y, \\
&I_{0n} = -\int_0^\pi \frac{\phi_n}{\bar{U}+C_n} h_0 \mathrm{d}y, \quad I_{12} = -\int_0^\pi \frac{\phi_1}{\bar{U}+C_1} f_{12}\mathrm{d}y, \\
&I_{21} = -\int_0^\pi \frac{\phi_2}{\bar{U}+C_2} f_{21}\mathrm{d}y, \quad I_{n3} = -\int_0^\pi \frac{\phi_n}{\bar{U}+C_n} f_{n3}\mathrm{d}y,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
f_n &= -\left[\frac{\mathrm{d}^2\phi_\mathrm{n}}{\mathrm{d}y^2} - (k_n^2 + K^2 l_n^2)\phi_n\right], \\
f_{n0} &= -\bar{U}(k_n^2 + K^2 l_n^2)Y_{1n} - 2k_n(\omega_n + \bar{U}k_n)Y_{1n} + \mathrm{i}(\omega_n + \bar{U}k_n)\phi_n + \bar{U}\frac{\mathrm{d}^2 Y_{1n}}{\mathrm{d}y^2} - \beta' Y_{1n} + 2\bar{U}k_n\phi_n, \\
h_0 &= (\omega_n + \bar{U}k_n)\phi_n, \\
f_{nn} &= -\left(2k_n Y_{2n}\frac{\mathrm{d}}{\mathrm{d}y} + k_n\frac{\mathrm{d}Y_{2n}}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2\phi_\mathrm{n}}{\mathrm{d}y^2} - k_n^2\phi_n\right) + \left(k_n\phi_n\frac{\mathrm{d}}{\mathrm{d}y} + 2k_n\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2\mathrm{Y}_{2n}}{\mathrm{d}y^2} - 4k_n^2 Y_{2n}\right) - 3K^2 l_n^2 k_n\left(\phi_n\frac{\mathrm{d}Y_{2n}}{\mathrm{d}y} + 2\phi_n'' Y_{2n}\right), \\
f_{12} &= -\left(k_{12}Y_3\frac{\mathrm{d}}{\mathrm{d}y} + k_2\frac{\mathrm{d}Y_3}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2\phi_2}{\mathrm{d}y^2} - k_2^2\phi_2\right) + \left(k_2\phi_2\frac{\mathrm{d}}{\mathrm{d}y} + k_{12}\frac{\mathrm{d}\phi_2}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2 Y_3}{\mathrm{d}y^2} - k_{12}^2 Y_3\right) \\
&\quad -\left(k_2\phi_2\frac{\mathrm{d}}{\mathrm{d}y} - \alpha_{12}\frac{\mathrm{d}\phi_2}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2 Y_4}{\mathrm{d}y^2} - \alpha_{12}^2 Y_4\right) - \left(\alpha_{12}Y_4\frac{\mathrm{d}}{\mathrm{d}y} - k_2\frac{\mathrm{d}Y_4}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2\phi_2}{\mathrm{d}y^2} - k_2^2\phi_2\right) \\
&\quad -K^2\left(k_2\phi_2\frac{\mathrm{d}}{\mathrm{d}y} - \alpha_{12}\frac{\mathrm{d}\phi_2}{\mathrm{d}y}\right)(l_2^2 Y_4 - \sigma_{12}^2 Y_4) + K^2\left(k_2\phi_2\frac{\mathrm{d}}{\mathrm{d}y} + k_{12}\frac{\mathrm{d}\phi_2}{\mathrm{d}y}\right)(l_2^2 Y_3 - l_{12}^2 Y_3), \\
f_{21} &= -\left(k_{12}Y_3\frac{\mathrm{d}}{\mathrm{d}y} + k_1\frac{\mathrm{d}Y_3}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2\phi_1}{\mathrm{d}y^2} - k_1^2\phi_1\right) - \left(k_1\phi_1\frac{\mathrm{d}}{\mathrm{d}y} + \alpha_{12}\frac{\mathrm{d}\phi_1}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2 Y_4}{\mathrm{d}y^2} - \alpha_{12}^2 Y_4\right) \\
&\quad +\left(k_1\phi_1\frac{\mathrm{d}}{\mathrm{d}y} + k_{12}\frac{\mathrm{d}\phi_1}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2 Y_3}{\mathrm{d}y^2} - k_{12}^2 Y_3\right) + \left(\alpha_{12}Y_4\frac{\mathrm{d}}{\mathrm{d}y} + k_1\frac{\mathrm{d}Y_4}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2\phi_1}{\mathrm{d}y^2} - k_1^2\phi_1\right) \\
&\quad -K^2\left(k_1\phi_1\frac{\mathrm{d}}{\mathrm{d}y} + \alpha_{12}\frac{\mathrm{d}\phi_1}{\mathrm{d}y}\right)(l_1^2 Y_4 - \sigma_{12}^2 Y_4) + K^2\left(k_1\phi_1\frac{\mathrm{d}}{\mathrm{d}y} + k_{12}\frac{\mathrm{d}\phi_1}{\mathrm{d}y}\right)(l_1^2 Y_3 - l_{12}^2 Y_3), \\
f_{n3} &= -k_n\left\{\phi_n\frac{\partial^3\xi}{\partial y^3} - \left[\frac{\mathrm{d}^2\phi_n}{\mathrm{d}y^2} - (k_n^2 + K^2 l_n^2)\phi_n\right]\frac{\partial\xi}{\partial y}\right\}.
\end{aligned} \tag{28}$$

After a series of treatments, the solutions of A_1 and A_2 are obtained. Equation (25) includes coupled equations of A_1 and A_2, which describe the interactions between the two waves. The two equations can be further simplified after introducing the transforms by Jeffry and Kawahara,

$$T = T_2, X = \frac{1}{\varepsilon}(X_2 - C_{g1}T_2) = X_1 - C_{g1}T_1. \tag{29}$$

Using Eq. (28), Eq. (24) changes to

$$\begin{aligned}
\frac{\partial}{\partial X}[(C_{g1} + \bar{U})\frac{\partial^2\xi}{\partial y^2} - \beta'\xi] &= \sum_{n=1}^{2}\left\{\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left[k_n\left(\phi_n\frac{\mathrm{d}Y_{1n}}{\mathrm{d}y} - Y_{1n}\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right)\right] - 4k_n^2\phi_n\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right\}\frac{\partial}{\partial X}|A_n|^2 \\
&\quad + \sum_{n=1}^{2}\left\{-\left(\phi_n\frac{\mathrm{d}}{\mathrm{d}y} - \frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2\phi_n}{\mathrm{d}y^2} + k_n^2\phi_n\right) + k_n\phi_n''' Y_{1n} - k_n\phi_n\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}^2 Y_{1n}}{\mathrm{d}y^2}\right\}\frac{\partial}{\partial X}|A_n|^2,
\end{aligned}$$

$$2\mathrm{i}k_n\left[2\phi_n\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}\phi_n}{\mathrm{d}y}-\left(\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right)^2+(k_n^2+K^2l_n^2)\phi_n^2\right]\frac{\partial}{\partial Y}|A_n|^2=0. \tag{30}$$

The above equations have a solution in the form

$$\xi=\sum_{n=1}^{2}H_n(y)|A_n|^2, \tag{31}$$

where $H_n(y)$ satisfies

$$\begin{aligned}&(C_{g1}+\bar{U})\frac{\mathrm{d}^2H_n(y)}{\mathrm{d}y^2}-\beta' H_n\\ =\;& k_n\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\phi_n\frac{\mathrm{d}Y_{1n}}{\mathrm{d}y}-Y_{1n}\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right)\\ &-4k_n^2\phi_n\frac{\mathrm{d}\phi_n}{\mathrm{d}y}-\left(\phi_n\frac{\mathrm{d}}{\mathrm{d}y}-\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}^2\phi_n}{\mathrm{d}y^2}+k_n^2\phi_n\right)\\ &+k_n\phi_n'''Y_{1n}-k_n\phi_n\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}^2Y_{1n}}{\mathrm{d}y^2}.\end{aligned} \tag{32}$$

Substituting Eqs. (29) and (31) into Eq. (25) yields

$$\begin{aligned}&\mathrm{i}\frac{\partial A_1}{\partial T}+\alpha_1\frac{\partial^2A_1}{\partial X^2}+\eta_1\frac{\partial^2A_1}{\partial Y^2}+(\sigma_1|A_1|^2+\tau_{12}|A_2|^2)A_1=0,\\ &\mathrm{i}\frac{\partial A_2}{\partial T}-\mathrm{i}\delta\frac{\partial A_2}{\partial X}+\alpha_2\frac{\partial^2A_2}{\partial X^2}+\eta_2\frac{\partial^2A_2}{\partial Y^2}\\ &+(\sigma_2|A_2|^2+\tau_{21}|A_1|^2)A_2=0,\end{aligned} \tag{33}$$

where

$$\begin{aligned}\delta &= \frac{1}{\varepsilon}(C_{g1}-C_{g2}),\\ \sigma_n &= \rho_n-\frac{1}{II_n}\int_0^{\pi}\frac{k_n\phi_n}{\bar{u}+C_n}\\ &\quad\times\left[\phi_n\frac{\mathrm{d}^3H_n}{\mathrm{d}y^3}-\left(\frac{\mathrm{d}^2\phi_n}{\mathrm{d}y^2}-k_n^2\phi_n\right)\frac{\mathrm{d}H_n}{\mathrm{d}y}\right]\mathrm{d}y,\\ \tau_{21} &= \gamma_{21}-\frac{1}{II_2}\int_0^{\pi}\frac{k_2\phi_2}{\bar{u}+C_2}\\ &\quad\times\left[\phi_2\frac{\mathrm{d}^3H_1}{\mathrm{d}y^3}-\left(\frac{\mathrm{d}^2\phi_2}{\mathrm{d}y^2}-k_2^2\phi_2\right)\frac{\mathrm{d}H_1}{\mathrm{d}y}\right]\mathrm{d}y,\\ \tau_{12} &= \gamma_{12}-\frac{1}{II_1}\int_0^{\pi}\frac{k_1\phi_1}{\bar{u}+C_1}\\ &\quad\times\left[\phi_1\frac{\mathrm{d}^3H_2}{\mathrm{d}y^3}-\left(\frac{\mathrm{d}^2\phi_1}{\mathrm{d}y^2}-k_1^2\phi_1\right)\frac{\mathrm{d}H_2}{\mathrm{d}y}\right]\mathrm{d}y.\end{aligned} \tag{34}$$

Equation (33) includes the coupled nonlinear Schrödinger equations. In these equations, α_n is dispersion coefficient, σ_n are Landau constants, τ_{12} and τ_{21} are the wave-to-wave interaction coefficients. Assuming δ to be $o(1)$, this is according to the physical requirements for the interactions of two waves at longer timescales T_2. Introducing a transformation

$$A_2=B\exp\left[\mathrm{i}\frac{\delta}{2\alpha_2}\left(X+\frac{\delta}{2}T\right)\right]. \tag{35}$$

Using the above transformation, Eq. (33) reduces to

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha_1\frac{\partial^2A}{\partial X^2}+\eta_1\frac{\partial^2A}{\partial Y^2}+(\sigma_1|A|^2+\tau_{12}|B|^2)A=0,$$

$$\mathrm{i}\frac{\partial B}{\partial T}+\alpha_2\frac{\partial^2B}{\partial X^2}+\eta_2\frac{\partial^2B}{\partial Y^2}+(\sigma_2|B|^2+\tau_{21}|A|^2)B=0. \tag{36}$$

The above equations are the CNLS equations. The usual standard NLS equation is in the form

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha_1\frac{\partial^2A}{\partial X^2}+\sigma_1|A|^2A=0.$$

Compared to the normal NLS equation, Eq. (36) is different in that it is coupled and (2+1)-dimensional. Eqution (36) is called the CNLS equations. The standard NLS is able to be used to describe the spread of Rossby waves, and it can only describe the propagation of a single wave in the x direction of space and in time t. However, with the addition of y in the space direction, the CNLS equations can describe wave propagation more specifically than the standard NLS equation. Moreover, the CNLS equations are not limited to describe the propagation of a single wave, instead they can also describe more phenomena, such as wave–wave interactions.

3. Mixed soliton solution of coupled nonlinear Schrödinger equation

In the above section, a dynamical model describing the interaction of two waves was obtained. Now, we discuss the problem what forms and properties appear in the soliton solutions in the process of wave–wave interactions. In this section, the Hirota bilinear method is used to solve the mixed soliton solutions to Eq. (36) and to obtain the figures of solutions. According to Eq. (36), it is obvious that in the two components A and B, one component contains bright soliton and the other contains dark soliton. Above all, Hirota's bilinear operators D_z and D_t are defined as

$$\begin{aligned}D_z^pD_t^q(a\cdot b) &= \left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right)^p\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^q\\ &\quad\times a(z,t)b(z',t')|_{z=z',t=t'},\end{aligned} \tag{37}$$

then we can perform some transform to Eq. (36). Setting $\alpha_1=\alpha_2$, $\eta_1=\eta_2$, $\sigma_1=\tau_{21}$, $\sigma_2=\tau_{12}$, $T=t$, $X=x$, $Y=y$ then Eq. (36) changes to

$$\begin{aligned}&\mathrm{i}\frac{\partial A}{\partial t}+\alpha_1\frac{\partial^2A}{\partial x^2}+\eta_1\frac{\partial^2A}{\partial y^2}+(\sigma_1|A|^2+\tau_{12}|B|^2)A=0,\\ &\mathrm{i}\frac{\partial B}{\partial t}+\alpha_1\frac{\partial^2B}{\partial x^2}+\eta_1\frac{\partial^2B}{\partial y^2}+(\sigma_1|A|^2+\tau_{12}|B|^2)B=0.\end{aligned} \tag{38}$$

Assume the solution to Eq. (38) as

$$A=\frac{g}{f},\quad B=\frac{h}{f}, \tag{39}$$

where g and h are arbitrary complex functions of x, y, and t, while f is a real function. The bilinear form for Eq. (38) is

$$\begin{aligned}&(\mathrm{i}D_t+\alpha_1 D_x^2+\eta_1 D_y^2-\lambda)(g\cdot f)=0,\\&(\mathrm{i}D_t+\alpha_1 D_x^2+\eta_1 D_y^2-\lambda)(h\cdot f)=0,\\&(\alpha_1 D_x^2+\eta_1 D_y^2-\lambda)(f\cdot f)=\sigma_1|g|^2+\tau_{12}|h|^2.\end{aligned}\tag{40}$$

Expanding g, h and f formally as power series expansions in terms of a small arbitrary real parameter χ,

$$\begin{aligned}&g=\chi g_1+\chi^3 g_3+\chi^5 g_5+\cdots,\\&h=h_0(1+\chi^2 h_2+\chi^4 h_4+\cdots),\\&f=1+\chi^2 f_2+\chi^4 f_4+\cdots.\end{aligned}\tag{41}$$

3.1. Mixed (bright-dark) one-soliton solution

Restricting the power series expansion Eq. (41) as

$$\begin{aligned}&g=\chi g_1,\\&h=h_0(1+\chi^2 h_2),\\&f=1+\chi^2 f_2.\end{aligned}\tag{42}$$

Assuming $A=\frac{g}{f}$, $B=\frac{h}{f}$, for the linear partial differential equations obtained by recursive solution, the mixed one-soliton solution can be explicitly written as

$$A=A_1\,\mathrm{e}^{\mathrm{i}\theta_{1\mathrm{I}}}\,\mathrm{sech}\left(\theta_{1\mathrm{R}}+\frac{R}{2}\right),$$

$$B=\frac{c_1\,\mathrm{e}^{\mathrm{i}\xi_1}(1+b\mathrm{e}^{\theta_1+\theta_1^*+R})}{1+\mathrm{e}^{\theta_1+\theta_1^*+R}},\tag{43}$$

where

$$\begin{aligned}b&=\frac{\alpha_1(k_1+k_1^*)(il_1-k_1)+\eta_1(k_2+k_2^*)(il_2-k_2)}{\alpha_1(k_1+k_1^*)(il_1+k_1^*)+\eta_1(k_2+k_2^*)(il_2+k_2^*)},\\e^R&=\frac{\sigma_1|a_1|^2}{[\alpha_1(k_1+k_1^*)^2+\eta_1(k_2+k_2^*)^2]}\left[2+\tau_{12}|c_1|^2\right.\\&\quad\left.\times\frac{\alpha_1(k_1+k_1^*)^2+\eta_1(k_2+k_2^*)^2}{|\alpha_1(k_1+k_1^*)(k_1^*+\mathrm{i}l_1)+\eta_1(k_2+k_2^*)(k_2^*+\mathrm{i}l_2)|^2}\right]^{-1},\end{aligned}\tag{44}$$

$$\begin{aligned}\theta_1&=\theta_{1\mathrm{R}}+\mathrm{i}\theta_{1\mathrm{I}},\quad\theta_{1\mathrm{R}}=k_{1\mathrm{R}}x+k_{2\mathrm{R}}y,\\\theta_{1\mathrm{I}}&=(\alpha_1k_1^2+\eta_1k_2^2-\lambda)t,\quad A_1=\frac{a_1}{2}\mathrm{e}^{i\theta_{1\mathrm{I}}},\\\xi_1&=l_1x+l_2y-(\alpha_1l_1^2+\eta_1l_2^2+\lambda)t,\\c_1&=c_{1\mathrm{R}}+\mathrm{i}c_{1\mathrm{I}},\quad k_1=k_{1\mathrm{R}}+\mathrm{i}k_{1\mathrm{I}},\\a_1&=a_{1\mathrm{R}}+\mathrm{i}a_{1\mathrm{I}},\quad\lambda=-\tau_{12}|c_1|^2.\end{aligned}\tag{45}$$

Here a_1, k_1, k_2, c_1 are arbitrary complex parameters, while l_1 and l_2 are real parameters. In the above equations and in the following sections, subscribers R and I denote the real and imaginary parts, respectively. Figure 1 shows the mixed one-soliton solution.

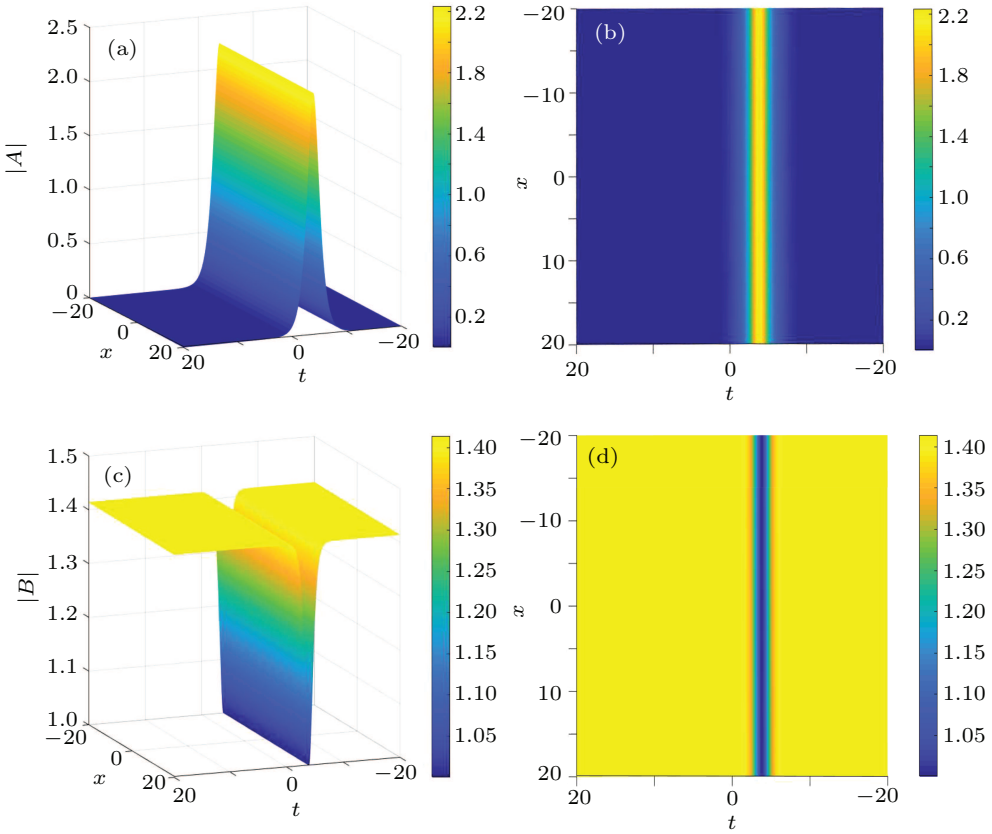


Fig. 1. The mixed one-soliton solution with $k_1=a_1=c_1=1+\mathrm{i}$, $k_2=1-\mathrm{i}$, $\alpha_1=\eta_1=\sigma_1=\tau_{12}=l_1=l_2=1$, $\lambda=-2$, $y=5$. (a) The soliton solution A. (b) The planform of solution A. (c) The soliton solution B. (d) The planform of solution B.

3.2. Mixed (bright-dark) two-soliton solution

The mixed two-soliton solution can be obtained by terminating the power series expansion (41) as

$$\begin{aligned} g &= \chi g_1 + \chi^3 g_3, \\ h &= h_0(1+\chi^2 h_2 + \chi^4 h_4), \\ f &= 1+\chi^2 f_2 + \chi^4 f_4. \end{aligned} \tag{46}$$

After solving the resulting bilinear equations recursively, the mixed two-soliton solution is obtained as follows:

$$\begin{aligned} A &= \frac{1}{f}(a_1 \mathrm{e}^{\theta_1} + a_2 \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_1+\theta_1^*+\theta_2+\delta_{11}} + \mathrm{e}^{\theta_1+\theta_2^*+\theta_2+\delta_{21}}), \\ B &= \frac{1}{f}[c_1 \mathrm{e}^{i\xi_1}(1+\mathrm{e}^{\theta_1+\theta_1^*+Q_{11}} + \mathrm{e}^{\theta_1+\theta_2^*+Q_{12}} + \mathrm{e}^{\theta_2+\theta_1^*+Q_{21}} \\ &\quad + \mathrm{e}^{\theta_2+\theta_2^*+Q_{22}} + \mathrm{e}^{\theta_1+\theta_1^*+\theta_2+\theta_2^*+Q_3})], \end{aligned} \tag{47}$$

where

$$\begin{aligned} f &= 1+\mathrm{e}^{\theta_1+\theta_1^*+R_1} + \mathrm{e}^{\theta_1+\theta_2^*+\delta_0} + \mathrm{e}^{\theta_2+\theta_1^*+\delta_0^*} \\ &\quad + \mathrm{e}^{\theta_2+\theta_2^*+R_2} + \mathrm{e}^{\theta_1+\theta_1^*+\theta_2+\theta_2^*+R_3}, \end{aligned} \tag{48}$$

and

$$\begin{aligned} \theta_1 &= k_1 x + k_2 y + \mathrm{i}(\alpha_1 k_1^2 + \eta_1 k_2^2 - \lambda)t, \\ \theta_2 &= k_3 x + k_4 y + \mathrm{i}(\alpha_1 k_3^2 + \eta_1 k_4^2 - \lambda)t. \end{aligned} \tag{49}$$

The mixed-two soliton solution are given in Fig. 2.

The above figures are the 3D mixed one-soliton solution and the mixed two-soliton solution when the parameter x and y are fixed. It is not difficult to find from these figures that bright soliton exists in the component A and dark soliton exists in the component B. Meanwhile, we can clearly see the collision interactions of Rossby solitons from Fig. 2. When the two solitons collide, obvious peak oscillation is caused, and then the solitons change the original trajectory. It is worth noting that the trajectories of the two solitons after the collision do not cross. Next, the problem about the influence of dark soliton parameters on the intensity of bright and dark solitons is discussed. Taking the mixed one-soliton solution as an example, when fixing other parameters, the relationship of $|A|^2$ and $|B|^2$ to t can be obtained. The diagrams are shown in Figs. 3 and 4.

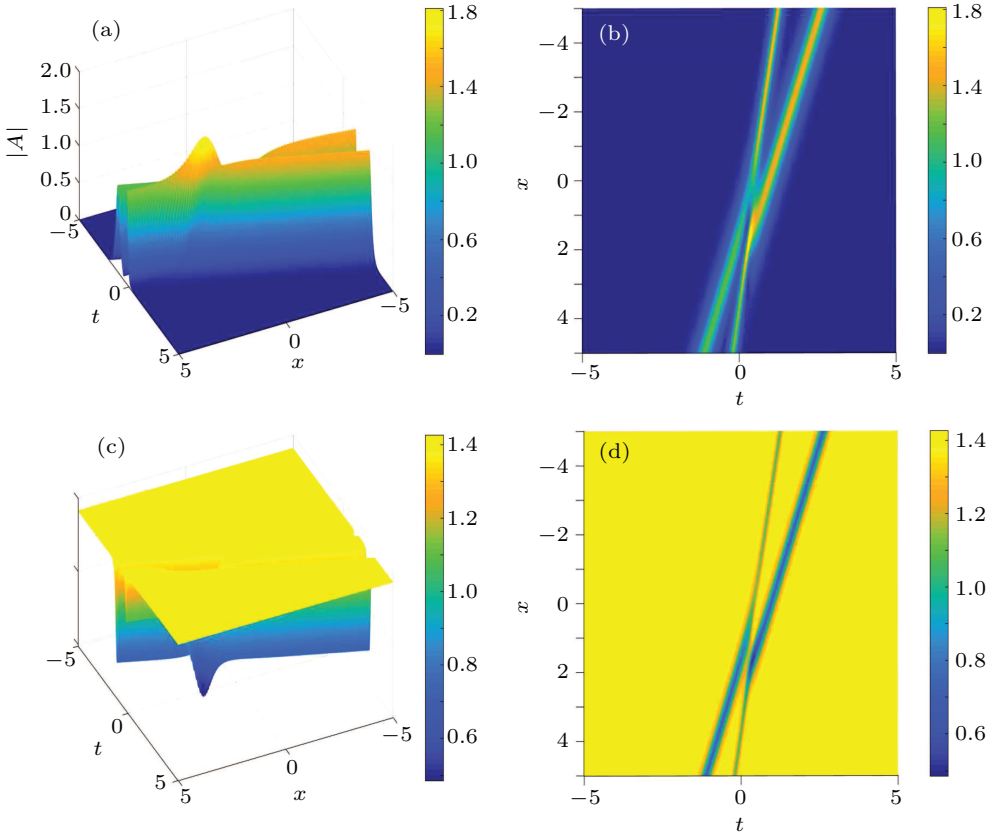


Fig. 2. The mixed two-soliton solution with $k_1 = 2-\mathrm{i}$, $k_2 = -1+\mathrm{i}$, $k_3 = -2-2\mathrm{i}$, $k_4 = 1+2\mathrm{i}$, $c_1 = a_1 = 1+\mathrm{i}$, $a_2 = 3-\mathrm{i}$, $\alpha_1 = \eta_1 = \sigma_1 = \tau_{12} = l_1 = 1$, $l_2 = 2, y = 5$. (a) The soliton solution A. (b) The planform of solution A. (c) The soliton solution B. (d) The planform of solution B.

It can be clearly seen from Figs. 3 and 4 that the larger the $|c_1|^2$ is, the smaller the intensity of the bright soliton is. As the intensity of the bright soliton decreases, the intensity of the dark soliton gradually increases. Therefore, the energy is determined in the propagation process of two Rossby waves with different wave numbers.

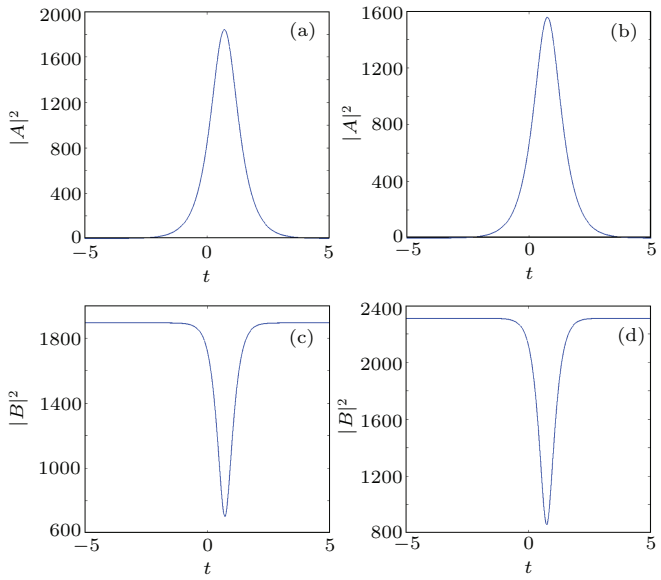


Fig. 3. The relation between the intensity of soliton and time t with $k_1=2+\mathrm{i}, a_1=k_2=1-\mathrm{i}, \alpha_1=\eta_1=\sigma_1=\tau_{12}=l_1=l_2=1=x=y$. (a) Mixed one-soliton solution with $c_1=1+\mathrm{i}$. (b) Mixed one-soliton solution with $c_1=1.2+\mathrm{i}$. (c) Mixed one-soliton solution with $c_1=1+\mathrm{i}$. (d) Mixed one-soliton solution with $c_1=1.2+\mathrm{i}$.

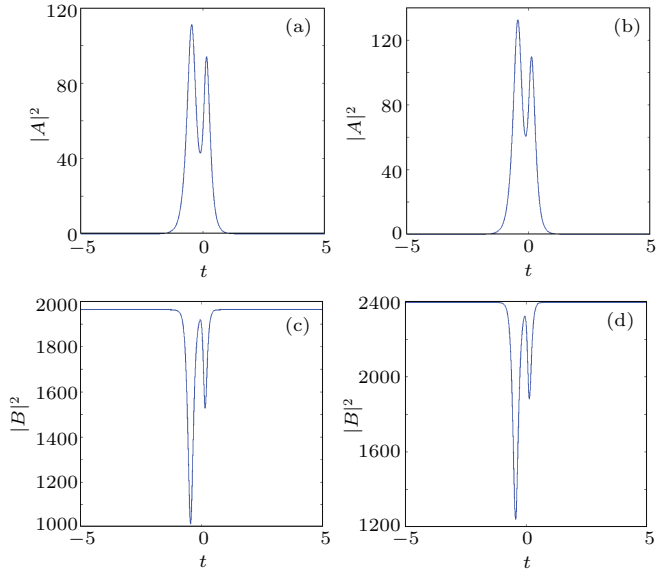


Fig. 4. The relation between the intensity of soliton and time t with $k_1=2-\mathrm{i}, k_2=-1+\mathrm{i}, k_3=-2-2\mathrm{i}, k_4=1+2\mathrm{i}, a_1=1+\mathrm{i}, a_2=3-\mathrm{i}$, $l_1=1$, $l_2=2$, $\alpha_1=\eta_1=\sigma_1=\tau_{12}=1=x=y$. (a) Mixed two-soliton solution with $c_1=1+\mathrm{i}$. (b) Mixed two-soliton solution with $c_1=1.2+\mathrm{i}$. (c) Mixed two-soliton solution with $c_1=1+\mathrm{i}$. (d) Mixed two-soliton solution with $c_1=1.2+\mathrm{i}$.

4. Modulational instability of the CNLS equation

Modulational instability is the famous phenomenon in the nonlinear propagation. It leads to the instability of Rossby waves. The study of modulational instability began in the 1960s. It was the scientist Benjamin Feir who discovered this phenomenon in the study of deep water waves, so the later researchers also called modulational instability Benjamin Feir instability. The study of modulational instability is of great significance in many aspects. In this section, the question about the modulational instability of uniform Rossby waves will be discussed. For the obtained CNLS Eq. (36), the plane wave solution is considered as follows:

$$\begin{cases} A=(A_0+\varphi_1)\exp[\mathrm{i}(\sigma_1A_0^2+\tau_{12}B_0^2)T], \\ B=(B_0+\varphi_2)\exp[\mathrm{i}(\sigma_1A_0^2+\tau_{12}B_0^2)T], \end{cases} \tag{50}$$

where φ_1 and φ_2 are small perturbations and they can be expressed as

$$\begin{cases} \varphi_1=\varphi_1(X,Y,T)=U_1+\mathrm{i}P_1, \\ \varphi_2=\varphi_2(X,Y,T)=U_2+\mathrm{i}P_2. \end{cases} \tag{51}$$

In the above $U_j=\mu_j\cos(mX+nY+\Omega T)$, $P_j=\rho_j\sin(mX+nY+\Omega T)$, $j=1,2$. Here m and n represent wave numbers, Ω represents frequency. Substituting A and B into Eq. (36) and linearizing, we can obtain the following equations:

$$\begin{aligned} &\mathrm{i}\frac{\partial\varphi_1}{\partial T}+\alpha_1\frac{\partial^2\varphi_1}{\partial X^2}+\eta_1\frac{\partial^2\varphi_1}{\partial Y^2}+\sigma_1[A_0^2(\varphi_1^*+\varphi_1)] \\ &+\tau_{12}[A_0B_0(\varphi_2^*+\varphi_2)]=0, \\ &\mathrm{i}\frac{\partial\varphi_2}{\partial T}+\alpha_1\frac{\partial^2\varphi_2}{\partial X^2}+\eta_1\frac{\partial^2\varphi_2}{\partial Y^2}+\sigma_1[A_0B_0(\varphi_1^*+\varphi_1)] \\ &+\tau_{12}[B_0^2(\varphi_2^*+\varphi_2)]=0. \end{aligned} \tag{52}$$

Using Eq. (51), the above formulas can be further arranged as

$$\begin{aligned} &\mathrm{i}\frac{\partial U_1}{\partial T}-\frac{\partial P_1}{\partial T}+\alpha_1\frac{\partial^2U_1}{\partial X^2}+\mathrm{i}\alpha_1\frac{\partial^2P_1}{\partial X^2}+\eta_1\frac{\partial^2U_1}{\partial Y^2} \\ &+\mathrm{i}\eta_1\frac{\partial^2P_1}{\partial Y^2}+2U_1\sigma_1A_0^2+2U_2\tau_{12}A_0B_0=0, \\ &\mathrm{i}\frac{\partial U_2}{\partial T}-\frac{\partial P_2}{\partial T}+\alpha_1\frac{\partial^2U_2}{\partial X^2}+\mathrm{i}\alpha_1\frac{\partial^2P_2}{\partial X^2}+\eta_1\frac{\partial^2U_2}{\partial Y^2} \\ &+\mathrm{i}\eta_1\frac{\partial^2P_2}{\partial Y^2}+2U_1\sigma_1A_0B_0+2U_2\tau_{12}B_0^2=0. \end{aligned} \tag{53}$$

Separating the real and imaginary parts of the above expressions we obtain

$$\begin{cases} \dfrac{\partial U_1}{\partial T}+\alpha_1\dfrac{\partial^2P_1}{\partial X^2}+\eta_1\dfrac{\partial^2P_1}{\partial Y^2}=0, \\ \dfrac{\partial P_1}{\partial T}-\alpha_1\dfrac{\partial^2U_1}{\partial X^2}-\eta_1\dfrac{\partial^2U_1}{\partial Y^2} \\ -2\sigma_1U_1A_0^2-2\tau_{12}U_2A_0B_0=0, \\ \dfrac{\partial U_2}{\partial T}+\alpha_1\dfrac{\partial^2P_2}{\partial X^2}+\eta_1\dfrac{\partial^2P_2}{\partial Y^2}=0, \\ \dfrac{\partial P_2}{\partial T}-\alpha_1\dfrac{\partial^2U_2}{\partial X^2}-\eta_1\dfrac{\partial^2U_2}{\partial Y^2} \\ -2\sigma_1U_1A_0B_0-2\tau_{12}U_2B_0^2=0. \end{cases} \tag{54}$$

Substituting U_j and P_j into Eq. (54) yields

$$\begin{cases} -\mu_1 \sin(mX+nY+\Omega T)\Omega - \alpha_1\rho_1 m^2 \sin(mX+nY+\Omega T) - \eta_1\rho_1 n^2 \sin(mX+nY+\Omega T) = 0, \\ \rho_1 \cos(mX+nY+\Omega T)\Omega + \alpha_1\mu_1 m^2 \cos(mX+nY+\Omega T) + \eta_1\mu_1 n^2 \cos(mX+nY+\Omega T) \\ -2\sigma_1\mu_1 A_0^2 \cos(mX+nY+\Omega T) - 2\tau_{12}\mu_2 A_0 B_0 \cos(mX+nY+\Omega T) = 0, \\ -\mu_2 \sin(mX+nY+\Omega T)\Omega - \alpha_1\rho_2 m^2 \sin(mX+nY+\Omega T) - \eta_1\rho_2 n^2 \sin(mX+nY+\Omega T) = 0, \\ \rho_2 \cos(mX+nY+\Omega T)\Omega + \alpha_1\mu_2 m^2 \cos(mX+nY+\Omega T) + \eta_1\mu_2 n^2 \cos(mX+nY+\Omega Tt) \\ -2\sigma_1\mu_1 A_0 B_0 \cos(mX+nY+\Omega T) - 2\tau_{12}\mu_2 B_0^2 \cos(mX+nY+\Omega T) = 0. \end{cases} \tag{55}$$

It can be obtained from the existence condition of homogeneous linear equations

$$\begin{aligned} &\Omega_1 = \alpha_1 m^2 + \eta_1 n^2, \quad \Omega_2 = -(\alpha_1 m^2 + \eta_1 n^2), \\ &\Omega_3 = -\sqrt{(\alpha_1 m^2 + \eta_1 n^2)^2 - 2\tau_{12}B_0^2(\alpha_1 m^2 + \eta_1 n^2) - 2\sigma_1 A_0^2(\alpha_1 m^2 + \eta_1 n^2)}, \\ &\Omega_4 = \sqrt{(\alpha_1 m^2 + \eta_1 n^2)^2 - 2\tau_{12}B_0^2(\alpha_1 m^2 + \eta_1 n^2) - 2\sigma_1 A_0^2(\alpha_1 m^2 + \eta_1 n^2)}. \end{aligned} \tag{56}$$

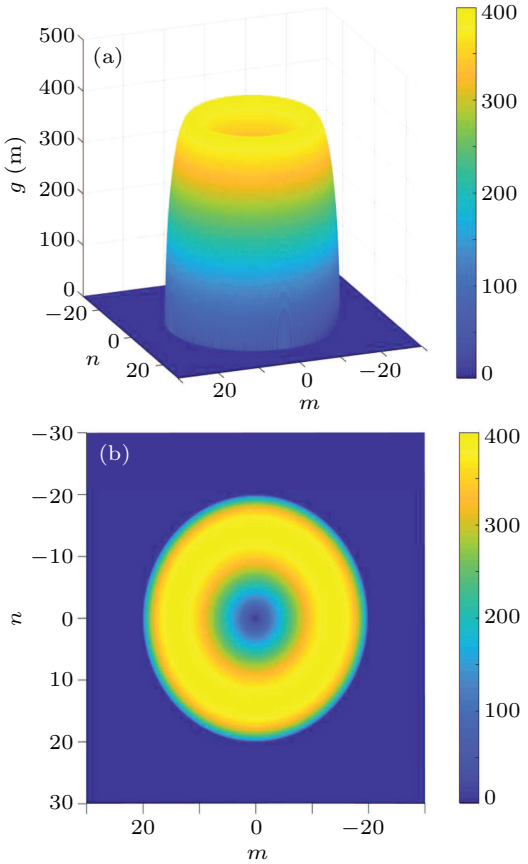


Fig. 5. Modulational instability gain of Eq. (36) with $a = 10$, $\alpha_1 = \eta_1 = \tau_{12} = \sigma_1 = 2$. (a) Modulational instability shown in three dimensions. (b) Top view of modulational instability.

Equation (56) shows that when $(\alpha_1 m^2 + \eta_1 n^2)^2 - 2\tau_{12}B_0^2(\alpha_1 m^2 + \eta_1 n^2) - 2\sigma_1 A_0^2(\alpha_1 m^2 + \eta_1 n^2) > 0$, the frequency Ω is always real and belongs to the stable region. When $(\alpha_1 m^2 + \eta_1 n^2)^2 - 2\tau_{12}B_0^2(\alpha_1 m^2 + \eta_1 n^2) - 2\sigma_1 A_0^2(\alpha_1 m^2 + \eta_1 n^2) < 0$, the frequency Ω is complex when the wave numbers m and n are set to the value of a particular item, that is the modulational instability region. For modulational instability, when$(\alpha_1 m^2 + \eta_1 n^2)^2 - 2\tau_{12}B_0^2(\alpha_1 m^2 + \eta_1 n^2) - 2\sigma_1 A_0^2(\alpha_1 m^2 + \eta_1 n^2) < 0$, it can be known from the definition of gain spectrum that

$$g(m) = \mathrm{Im}|\Omega|. \tag{57}$$

We can know from Eq. (57) that A and B play the same role in the change of gain spectrum. Assuming $A_0 = B_0 = a$, the $g(m)$ changes to

$$g(m) = \sqrt{2(\alpha_1 m^2 + \eta_1 n^2)(\tau_{12} + \sigma_1)a^2 - (\alpha_1 m^2 + \eta_1 n^2)^2}. \tag{58}$$

Figure 5 shows the modulational instability in three dimensions.

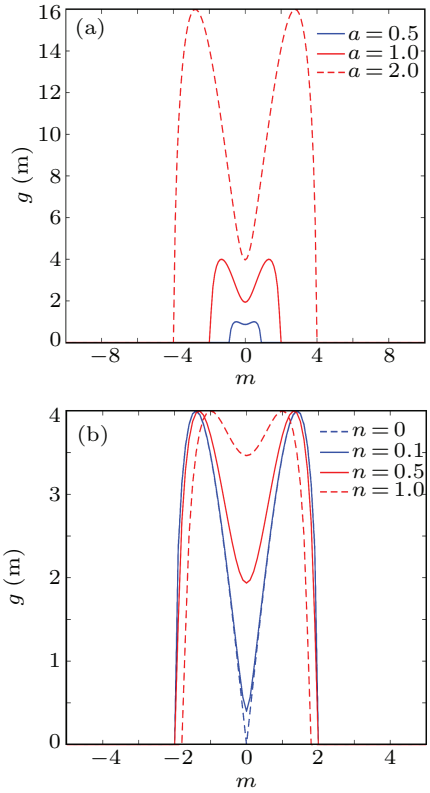


Fig. 6. Comparison chart of modulational instability with different parameters $a = 10$, $\alpha_1 = \eta_1 = \tau_{12} = \sigma_1 = 2$. (a) The values of these parameters are $n = 0.5$ and $a = 0.5, 1, 2$. (b) The values of these parameters are $a = 1$ and $n = 0, 0.1, 0.5, 1$.

In Fig. 5, the area of figures can be split into two parts, one part is a circle with a radius of 19.1 at the origin and this part is the modulational instability region, the Rossby waves are unstable on this part. The other one is the dark blue part, this part and the central point are both in the modulational stability region, so the Rossby waves are stable on this area. Next, the influence of different factors on modulational instability is discussed.

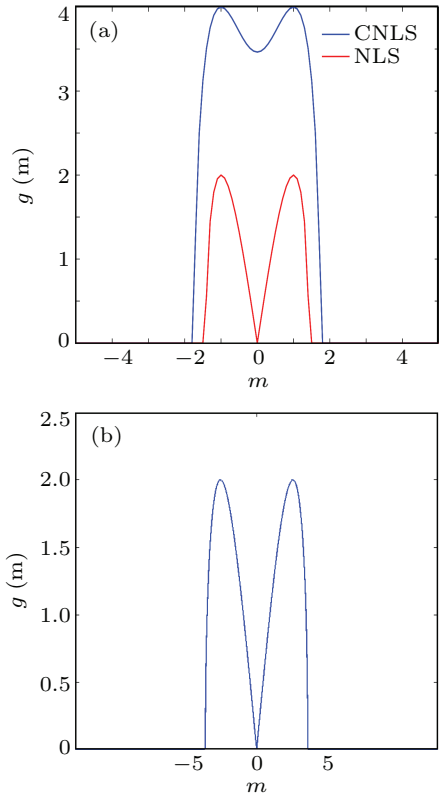


Fig. 7. Modulational instability comparison diagrams of different types of equations with $a = 1$, $\alpha_1 = \eta_1 = \tau_{12} = \sigma_1 = 2$. (a) The modulational instability of standard NLS and CNLS with $n = 1$. (b) The modulational instability of standard NLS.

Because these figures are symmetric, we analyze only half of them. There are two points seen in Fig. 6. The first one is that as the value of amplitude increases, the gain and the width of the unstable region of Rossby waves increases. When the parameter $a = 2$, the width of the unstable region of the uniform Rossby waves can reach 4 and the gain of the Rossby waves can reach 16. However, when $a = 0.5$, the former only reaches 1 and the latter reaches 1. These data suggest that the amplitude determines the gain and width of the unstable region of the Rossby waves. The width of the unstable region and the gain shrinks with the contraction of the amplitude. They are larger when the amplitude is greater. The second one is that dimension does not affect the gain because the gain at $n = 0$ is the same as the gain at other values of n. What the dimension really affects is the stability at the central point and the width of the modulational instability region. When $n = 0$, which means that the wave number in the y direction is zero and there is no y term, the uniform Rossby waves are stable at the central point. However, when the wave number n takes different values, the central point is no longer stable, and the gain at the central point is larger as the value of n increases. The width of the unstable region of Rossby waves decreases with the increase of n, it can be seen from Fig. 6(b) that when $n = 1$, the width of the unstable region of Rossby waves is minimal. On the contrary, when $n = 0.1$, it is maximal. However, the gain of the uniform Rossby waves are fixed no matter what n is, so the dimension has no effect on the gain. In Fig. 7, comparing the modulational instability image of the standard NLS equation with that of the CNLS equations, we can find something that when the dimension of space is changed from one to two and the equations are changed from single to couple, the gain and the width of the modulational instability region change simultaneously. At the same time, the gain of two waves interacting with each other is greater than that of a single wave.

5. Conclusions

In summary, we have obtained a dynamic model describing the propagation and interaction of Rossby waves in stratified fluids using perturbation analysis and scale expansion. When two Rossby waves with slightly different wavenumbers propagate in stratified fluids, their solitons will collide, which will change the trajectory of solitons. The intensity of bright soliton is also related to the dark soliton coefficient. The intensity of bright soliton decreases with the increase of dark soliton coefficient. In addition, in the propagation process of two waves, there is an unstable area. The size of this area is related to the amplitude and wave number in the y direction. When the amplitude is constant, the smaller the wave number is, the larger the width of the unstable area is. However, when the wave number is constant, the larger the amplitude is, the larger the width of the modulational instability area is, meanwhile its gain is also larger. According to the figures, the modulational instability area of a single wave in propagation is smaller than that of two waves.

Appendix A: Parameters

When the basic flow $\bar{U}(y)$ is constant, the important coefficients can be obtained through analysis. Now some important coefficients are given below. The solution to Eq. (10) is

$$\phi_n = \sin(m_n y), \tag{A1}$$

according to Eq. (10),

$$\begin{aligned} \phi_n'' &= -m_n^2 \phi_n, \\ C_n &= -\bar{U} - \frac{\beta'}{S_n}, \\ \omega_n &= -\bar{U} k_n - \frac{\beta' k_n}{S_n}, \end{aligned} \tag{A2}$$

where $S_n = k_n^2 + K^2 l_n^2 + m_n^2$. Substituting C_n and ϕ_n into Eq. (19), C_{gn} can be obtained as

$$C_{gn} = -\bar{U} - \frac{\beta'(m_n^2 + K^2 l_n^2 - k_n^2)}{S_n^2}. \tag{A3}$$

Making $e^{R_1}=\mu_{11}$, $e^{\delta_0}=\mu_{13}$, $e^{\delta_0^*}=\mu_{31}$, $e^{R_2}=\mu_{33}$ and the values of some coefficients about soliton solution are given as follows:

$$
\begin{aligned}
e^{Q_{11}} &= -\frac{\alpha_1(k_1+k_1^*)(k_1-il_1)+\eta_1(k_2+k_2^*)(k_2-il_2)}{\alpha_1(k_1+k_1^*)(k_1*+il_1)+\eta_1(k_2+k_2^*)(k_2^*+il_2)}\mu_{11};\\
e^{Q_{12}} &= -\frac{\alpha_1(k_1+k_3^*)(k_1-il_1)+\eta_1(k_2+k_4^*)(k_2-il_2)}{\alpha_1(k_1+k_3^*)(k_3*+il_1)+\eta_1(k_2+k_4^*)(k_4^*+il_2)}\mu_{13};\\
e^{Q_{21}} &= -\frac{\alpha_1(k_3+k_1^*)(k_3-il_1)+\eta_1(k_4+k_2^*)(k_4-il_2)}{\alpha_1(k_3+k_1^*)(k_1*+il_1)+\eta_1(k_4+k_2^*)(k_2^*+il_2)}\mu_{31};\\
e^{Q_{22}} &= -\frac{\alpha_1(k_3+k_3^*)(k_3-il_1)+\eta_1(k_4+k_4^*)(k_4-il_2)}{\alpha_1(k_3+k_3^*)(k_3*+il_1)+\eta_1(k_4+k_4^*)(k_4^*+il_2)}\mu_{33};
\end{aligned}
\tag{A4}
$$

$$
\begin{aligned}
e^{R_1} &= \frac{\sigma_1 a_1 a_1^*}{[\alpha_1(k_1+k_1^*)^2+\eta_1(k_2+k_2^*)^2]}\left[2+\tau_{12}|c_1|^2\frac{\alpha_1(k_1+k_1^*)^2+\eta_1(k_2+k_2^*)^2}{|\alpha_1(k_1+k_1^*)(k_1^*+il_1)+\eta_1(k_2+k_2^*)(k_2^*+il_2)|^2}\right]^{-1};\\
e^{\delta_0} &= \frac{\sigma_1 a_1 a_2^*}{[\alpha_1(k_1+k_3^*)^2+\eta_1(k_2+k_4^*)^2]}\left\{2+\tau_{12}|c_1|^2\frac{\alpha_1(k_1+k_3^*)^2+\eta_1(k_2+k_4^*)^2}{H}\right\}^{-1};\\
e^{\delta_0^*} &= \frac{\sigma_1 a_2 a_1^*}{[\alpha_1(k_3+k_1^*)^2+\eta_1(k_4+k_2^*)^2]}\left\{2+\tau_{12}|c_1|^2\frac{\alpha_1(k_3+k_1^*)^2+\eta_1(k_4+k_2^*)^2}{F}\right\}^{-1};\\
e^{R_2} &= \frac{\sigma_1 a_2 a_2^*}{[\alpha_1(k_3+k_3^*)^2+\eta_1(k_4+k_4^*)^2]}\left[2+\tau_{12}|c_1|^2\frac{\alpha_1(k_3+k_3^*)^2+\eta_1(k_4+k_4^*)^2}{|\alpha_1(k_3+k_3^*)(k_3^*+il_1)+\eta_1(k_4+k_4^*)(k_4^*+il_2)|^2}\right]^{-1};\\
e^{\delta_{11}} &= \frac{p}{\alpha_1(k_1^*+k_3)(k_1+k_1^*)+\eta_1(k_2+k_2^*)(k_2^*+k_4)};\\
e^{\delta_{21}} &= \frac{q}{\alpha_1(k_3^*+k_1)(k_3+k_3^*)+\eta_1(k_4+k_4^*)(k_4^*+k_2)};\\
e^{Q_3} &= \frac{[\alpha_1(k_3+k_3^*)(k_3-il_1)+\eta_1(k_4+k_4^*)(k_4-il_2)][\alpha_1(k_1+k_1^*)(k_1-il_1)+\eta_1(k_2+k_2^*)(k_2-il_2)]}{[\alpha_1(k_3+k_3^*)(k_3^*+il_1)+\eta_1(k_4+k_4^*)(k_4^*+il_2)][\alpha_1(k_1+k_1^*)(k_1^*+il_1)+\eta_1(k_2+k_2^*)(k_2^*+il_2)]}e^{R_3};\\
e^{R_3} &= \frac{1}{2\zeta_1-\tau_{12}|c_1|^2\zeta_2}[\sigma_1(\mu_{11}\zeta_3+\mu_{13}\zeta_4+\mu_{31}\zeta_5+\mu_{33}\zeta_6)+\mu_{11}\mu_{33}(\tau_{12}|c_1|^2\zeta_7-2\zeta_8)+\mu_{13}\mu_{31}(\tau_{12}|c_1|^2\zeta_9-2\zeta_{10})];
\end{aligned}
\tag{A5}
$$

$$
\begin{aligned}
p &= a_2[\alpha_1(k_1+k_1^*)(k_3-k_1)+\eta_1(k_4-k_2)(k_2+k_2^*)]\mu_{11}+a_1[\alpha_1(k_1-k_3)(k_3+k_1^*)+\eta_1(k_2-k_4)(k_4+k_2^*)]\mu_{31};\\
q &= a_2[\alpha_1(k_1+k_3^*)(k_3-k_1)+\eta_1(k_4-k_2)(k_2+k_4^*)]\mu_{13}+a_1[\alpha_1(k_1-k_3)(k_3+k_3^*)+\eta_1(k_2-k_4)(k_4+k_4^*)]\mu_{33};\\
H &= [\alpha_1(k_1+k_3^*)(k_3^*+il_1)+\eta_1(k_2+k_4^*)(k_4^*+il_2)][\alpha_1(k_1+k_3^*)(k_1-il_1)+\eta_1(k_2+k_4^*)(k_2-il_2)];\\
F &= [\alpha_1(k_3+k_1^*)(k_1^*+il_1)+\eta_1(k_4+k_2^*)(k_2^*+il_2)][\alpha_1(k_3+k_1^*)(k_3-il_1)+\eta_1(k_4+k_2^*)(k_4-il_2)];\\
\zeta_1 &= \alpha_1(k_1+k_1^*+k_3+k_3^*)^2+\eta_1(k_2+k_2^*+k_4+k_4^*)^2;\\
\zeta_2 &= \frac{J}{|\alpha_1(k_3+k_3^*)(k_3-il_1)+\eta_1(k_4+k_4^*)(k_4-il_2)|^2|\alpha_1(k_1+k_1^*)(k_1-il_1)+\eta_1(k_2+k_2^*)(k_2-il_2)|^2};\\
\zeta_3 &= |a_2|^2\left[\frac{\alpha_1(k_1+k_1^*)(k_3^*-k_1^*)+\eta_1(k_4^*-k_2^*)(k_2+k_2^*)}{\alpha_1(k_1+k_3^*)(k_1+k_1^*)+\eta_1(k_2+k_2^*)(k_2+k_4^*)}+\frac{\alpha_1(k_1+k_1^*)(k_3-k_1)+\eta_1(k_4-k_2)(k_2+k_2^*)}{\alpha_1(k_3+k_1^*)(k_1+k_1^*)+\eta_1(k_2+k_2^*)(k_4+k_2^*)}\right];\\
\zeta_4 &= a_1^*a_2\left[\frac{\alpha_1(k_1+k_3^*)(k_1^*-k_3^*)+\eta_1(k_2^*-k_4^*)(k_2+k_4^*)}{\alpha_1(k_1+k_3^*)(k_1+k_1^*)+\eta_1(k_2+k_2^*)(k_2+k_4^*)}+\frac{\alpha_1(k_1+k_3^*)(k_3-k_1)+\eta_1(k_4-k_2)(k_2+k_4^*)}{\alpha_1(k_1+k_3^*)(k_3+k_3^*)+\eta_1(k_2+k_4^*)(k_4+k_4^*)}\right];\\
\zeta_5 &= a_2^*a_1\left[\frac{\alpha_1(k_1-k_3)(k_1^*+k_3)+\eta_1(k_2-k_4)(k_4+k_2^*)}{\alpha_1(k_3+k_1^*)(k_1+k_1^*)+\eta_1(k_2+k_2^*)(k_4+k_2^*)}+\frac{\alpha_1(k_3+k_1^*)(k_3^*-k_1^*)+\eta_1(k_4^*-k_2^*)(k_4+k_2^*)}{\alpha_1(k_3+k_1^*)(k_3+k_3^*)+\eta_1(k_4+k_4^*)(k_4+k_2^*)}\right];\\
\zeta_6 &= |a_1|^2\left[\frac{\alpha_1(k_3+k_3^*)(k_1^*-k_3^*)+\eta_1(k_2^*-k_4^*)(k_4+k_4^*)}{\alpha_1(k_3+k_1^*)(k_3+k_3^*)+\eta_1(k_4+k_4^*)(k_4+k_2^*)}+\frac{\alpha_1(k_3+k_3^*)(k_1-k_3)+\eta_1(k_2-k_4)(k_4+k_4^*)}{\alpha_1(k_1+k_3^*)(k_3+k_3^*)+\eta_1(k_2+k_4^*)(k_4+k_4^*)}\right];\\
\zeta_7 &= \frac{M}{|\alpha_1(k_3+k_3^*)(k_3-il_1)+\eta_1(k_4+k_4^*)(k_4-il_2)|^2|\alpha_1(k_1+k_1^*)(k_1-il_1)+\eta_1(k_2+k_2^*)(k_2-il_2)|^2};\\
\zeta_8 &= \alpha_1(k_3+k_3^*-k_1-k_1^*)^2+\eta_1(k_2+k_2^*-k_4-k_4^*)^2;\\
\zeta_9 &= \frac{N}{|\alpha_1(k_1+k_3^*)(k_1-il_1)+\eta_1(k_2+k_4^*)(k_2-il_2)|^2|\alpha_1(k_3+k_1^*)(k_3-il_1)+\eta_1(k_4+k_2^*)(k_4-il_2)|^2};\\
\zeta_{10} &= \alpha_1(k_1-k_1^*-k_3+k_3^*)^2+\eta_1(k_2-k_2^*-k_4+k_4^*)^2;\\
J &= \{[\alpha_1(k_3+k_3^*)(k_3-il_1)+\eta_1(k_4+k_4^*)(k_4-il_2)][\alpha_1(k_1+k_1^*)(k_1-il_1)+\eta_1(k_2+k_2^*)(k_2-il_2)]\\
&\quad -[\alpha_1(k_3+k_3^*)(k_3^*+il_1)+\eta_1(k_4+k_4^*)(k_4^*+il_2)][\alpha_1(k_1+k_1^*)(k_1^*+il_1)+\eta_1(k_2+k_2^*)(k_2^*+il_2)]\}^2;\\
M &= \{[\alpha_1(k_3+k_3^*)(k_3^*+il_1)+\eta_1(k_4+k_4^*)(k_4^*+il_2)][\alpha_1(k_1+k_1^*)(k_1-il_1)+\eta_1(k_2+k_2^*)(k_2-il_2)]
\end{aligned}
\tag{A7}
$$

$$-[\alpha_1(k_3+k_3^*)(k_3-\mathrm{i}l_1)+\eta_1(k_4+k_4^*)(k_4-\mathrm{i}l_2)][\alpha_1(k_1+k_1^*)(k_1^*+\mathrm{i}l_1)+\eta_1(k_2+k_2^*)(k_2^*+\mathrm{i}l_2)]\}^2;$$

$$N=[|\alpha_1(k_1+k_3^*)(k_1-\mathrm{i}l_1)+\eta_1(k_2+k_4^*)(k_2-\mathrm{i}l_2)|^2-|\alpha_1(k_1+k_3^*)(k_3^*+\mathrm{i}l_1)+\eta_1(k_2+k_4^*)(k_4^*+\mathrm{i}l_2)|^2]^2. \tag{A8}$$

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