

Solutions of novel soliton molecules and their interactions of (2+1)-dimensional potential Boiti–Leon–Manna–Pempinelli equation

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Solutions of novel soliton molecules and their interactions of $(2+1)$-dimensional potential Boiti–Leon–Manna–Pempinelli equation

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The method of variable separation has always been regarded as a crucial method for solving nonlinear evolution equations. In this paper, we use a new form of variable separation to study novel soliton molecules and their interactions in $(2+1)$-dimensional potential Boiti–Leon-Manna–Pempinelli equation. Dromion molecules, ring molecules, lump molecules, multi-instantaneous molecules, and their interactions are obtained. Then we draw corresponding images with maple software to study their dynamic behavior.

Keywords: variable separation method, Hirota bilinear method, dromion solution, $(2+1)$-dimensional potential Boiti–Leon–Manna–Pempinelli equation

PACS: 02.30.Ik, 04.20.Jb, 04.30.Nk, 05.45.Yv **DOI:** 10.1088/1674-1056/ac4cc0

1. Introduction

The exploration of the linear field is not enough to describe the complex natural phenomena, so the exploration of the nonlinear field is born at the right time, and the study of nonlinear system has become one of the important tasks in the field of nonlinear science. Nonlinear equations are widely used in mathematics, physics, biology, astronomy, communication, and many other fields, it is of great significance to solve nonlinear equations. The variables separation method has undoubtedly become one of the powerful tools for obtaining exact solutions of nonlinear equations. In recent years, scholars have spared no effort to perfect this method, and they have proposed many methods. For example, nonlinearization method[1] (formal separation of variables[2]), multiple linear variable separation method,[3] function variable separation method,[4] derivative dependent variable separation method,[5,6] and so on. Up to now, after the unremitting efforts of scholars, the separation of variables has been successfully used to nonlinear equations.[7–16]

By solving the exact solutions of nonlinear system, many interesting local excitation modes have been found.[17–19] This diversity of exact solutions is also shown in the Hirota bilinear method.[20] In addition, Ma *et al.* proposed a generalized algorithm to test Hirota's condition and to prove the existence of N-soliton solutions for nonlinear equations in Refs. [21–23]. This method has been applied to nonlocally integrable equations.[24]

Recently, the soliton molecules obtained by using the mechanism of velocity resonance has aroused a great deal of interest. The velocity resonance mechanism was first proposed by Lou in Ref. [25], and a variety of soliton molecules were obtained by using this method.[26,27] Using velocity resonance mechanism, module resonance, and long wave limit method, researchers obtained new hybrid solutions between soliton molecules, lump waves, and breather waves, and explored their interactions.[28–36] In this article, we will apply the variable separation method to get the dromion molecules, ring molecules, lump molecules, and multi-instantaneous molecules of the $(2+1)$-dimensional potential Boiti–Leon-Manna–Pempinelli (pBLMP) equation, and we will show the interesting interactions between these molecules. The $(2+1)$-dimensional pBLMP equation[37] is

$$u_{yt}+u_{xxxy}-3u_{xy}u_x-3u_{xx}u_y=0. \tag{1}$$

Many studies have been done for Eq. (1) by predecessors. Seadawy *et al.* used an improved extended assisted mapping method to obtain several exact and isolated wave solutions of Eq. (1).[38,39] Kaplan constructed the analytic solution by using transformation rational function method and $\exp(\Phi(\xi))$ method in Ref. [40]. He *et al.* studied the "new" interaction solutions between block solutions and other multi-soliton (kink or banding) solutions by developing a "new" direct method based on Hirota's bilinear form for $(2+1)$-dimensional BLMP equation.[41,42] Kumar *et al.* solved the exact solution by the theory of Lie group transformation and studied its physical properties in Ref. [43]. Hu *et al.* obtained Pfaffian solution by using Pfaffian technique and real auxiliary function $\phi(y)$.[44] Shen *et al.* derived the periodic wave solution by using Riemann function and constructed the respiratory wave solution by using the extended homoclinic test method.[45]

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By giving the hypothesis, we can get the solutions by separating the variables of Eq. (1). Based on multi-linear variable separation method, we can further obtain many local excitation modes of Eq. (1). In Section 2, we use the multi-linear variable method to solve Eq. (1), and under this assumption, we take it a step further to analyze the general quantities valid for a large number of separable systems with multi-linear variables. In Section 3, we draw three cases of dromion solution and explore the corresponding motion rules. The motion of dromion molecules at common velocity is also given. In Section 4, we draw ring molecule with symmetry and give two cases of ring solutions. In Section 5, we plot three cases of lump molecules that can form $N\times M$ lattices and multi-instanton molecules with energy decaying but the shape unchanged. Finally we make some summaries and put forward some ideas.

2. Multilinear variable separation approach

Supposing the form of solution is[46]

$$u=-2(\ln f)_x+u_0+u_1, \tag{2}$$

where $u_0=u_0(x,y), u_1=u_1(t)$ are seed solutions, by substituting Eq. (2) into Eq. (1) and integrating about x once, we obtain

$$\begin{aligned}&\big(D_yD_t-3(u_{0x}+u_{0y})D_x^2+D_x^3D_y\\&+h_1(x,y)+h_0(t)\big)f\cdot f=0,\end{aligned} \tag{3}$$

where h_1 is a function of x and y, h_0 is a function of t, and the Hirota's bilinear operator is defined by

$$\begin{aligned}D_x^lD_y^nD_t^m(f\cdot f)\ =\ &(\partial_x-\partial_{x'})^l(\partial_y-\partial_{y'})^n(\partial_t-\partial_{t'})^m\\&\times f(x,y,t)\cdot f(x',y',t')|_{x=x',y=y',t=t'}.\end{aligned}$$

Supposing f has the following form

$$f=a_0+a_1p(x,t)+a_2q(y)+a_3p(x,t)q(y), \tag{4}$$

taking Eq. (4) into Eq. (3), then we acquire

$$\begin{aligned}&q_y(p_t+p_{xxx})(a_0a_3-a_1a_2)+3(u_{0x}+u_{0y})(a_1+a_3q)\\&\times\big((a_1+a_3q)(p_x^2-pp_{xxx})-(a_0+a_2q)p_{xx}\big)\\&+(h_1+h_0)f\cdot f=0.\end{aligned} \tag{5}$$

Obviously, when

$$h_0=h_0(t)=0,\ \ h_1=h_1(x,y)=0,$$

equation (1) can be solved by variabls separation. Therefore, we consider Eq. (1) as one of the multilinear variable separation approach (MLVSA) solvable models, and there are a lot of local excitation patterns. In the case of Eq. (4), equation (2) becomes

$$u=-2\frac{a_1p_x+a_3p_xq}{a_0+a_1p+a_2q+a_3pq}+u_0+u_1, \tag{6}$$

the potential field $U\equiv u_y$ from Eq. (6) is written as

$$U=u_y=-2\frac{(a_0a_3-a_1a_2)p_xq_y}{(a_0+a_1p+a_2q+a_3pq)^2}+u_{0y}, \tag{7}$$

where U is a general quantity valid for a large number of separable systems with multiple linear variables. It is the randomicity of the seed solution u_0 that leads to the existence of a large number of different local excitation modes.[17,18,47] In the following sections, we will map four novel soliton molecules and their interactions.

3. Dromions, dromion molecules, and their interactions

In order to gain the dromion molecule of Eq. (1), we take p and q as follows:

$$\begin{aligned}p&=b+\sum_{i=1}^{N}(w_i-\operatorname{sech}(k_ix+v_it+r_i))^i,\\q&=B+\cosh(\beta-(Ky)^2).\end{aligned} \tag{8}$$

The parameter of Eq. (4) in this part is going to take the value of $a_0=2, a_1=1, a_2=1, a_3=1$.

Case 1 For $N=2$, by taking the parameters $b=5$, $w_1=3$, $k_1=1$, $v_1=3$, $r_1=5$, $w_2=1.5$, $k_2=1$, $v_2=-2$, $r_2=-5$, $B=2$, $\beta=2$, $K=1$, we observed the collision phenomenon of dromion molecules with time increasing, then they separate, and there is no energy loss in collision processes, so it is considered as an elastic collision as shown in Fig. 1. If the parameters satisfy $k_1/k_2=v_1/v_2$, two dromion "atom" can be considered to have a common velocity and bound to a one dromion molecule by taking $k_1=3$, $k_2=1$, $v_1=6$, $v_2=2$, and other parameters remain unchanged.

As shown in Fig. 2, we observed that the new dromion molecule moves negatively along x axis with time increasing, but does not move in y axis. We also note that the parameter w_i is the main factor affecting the shape of dromion molecules.

Case 2 For $N=3$, by taking $b=3$, $w_1=3$, $k_1=1$, $v_1=3$, $r_1=5$, $w_2=2$, $k_2=1$, $v_2=-3$, $r_2=-5$, $w_3=1$, $k_3=2$, $v_3=-1$, $r_3=3$, $B=0$, $\beta=0$, $K=1$, we plot a 3D diagram for dromion molecule of Eq. (8) as shown in Fig. 3, it is obvious that the three dromion molecules move from their original position to opposite direction with the increase of time. When $t=-2$, the three dromion molecules collide and fuse together, the collision is also elastic. And then they separate again. We found that the parameter β affects the internal spacing of each molecule. When $\beta>0$, the internal spacing of each molecule increases with the increase of β. When β reaches a certain value, we get six dromions which are situated in two rows and three columns.

Case 3 When $N=3$, we also study two cases of the common velocity. The first kind of circumstance is two molecules keeping pace consistent, by taking $k_1=1$, $k_3=2$, $v_1=2$, and

$v_3 = 4$, other parameters being $b = 3$, $w_1 = 3$, $r_1 = 8$, $w_2 = 2$, $k_2 = 1$, $v_2 = -3$, $r_2 = 7$, $w_3 = 1$, $r_3 = 2$, $K_1 = 1$, $B = 0$, $\beta = 0$. Under these conditions, we draw a 3D plot of three-dromion molecules moving with time increasing as shown in Fig. 4, two dromion "atoms" are bound to a single dromion molecule, which moves forward at a common speed and collides with the third "atom" at $t = -2.7$. After this process, they move forward in the original direction. And the velocity does not change. The second case is to make a total of three molecules, namely under the parameters of the first kind of circumstance, changing by $k_2 = 1$, $v_2 = 2$, then meetimg $k_i/k_j = v_i/v_j$, $1 \le i < j \le 3$, Fig. 5, we observed the three molecules along the x axis at the same rate to the negative direction.

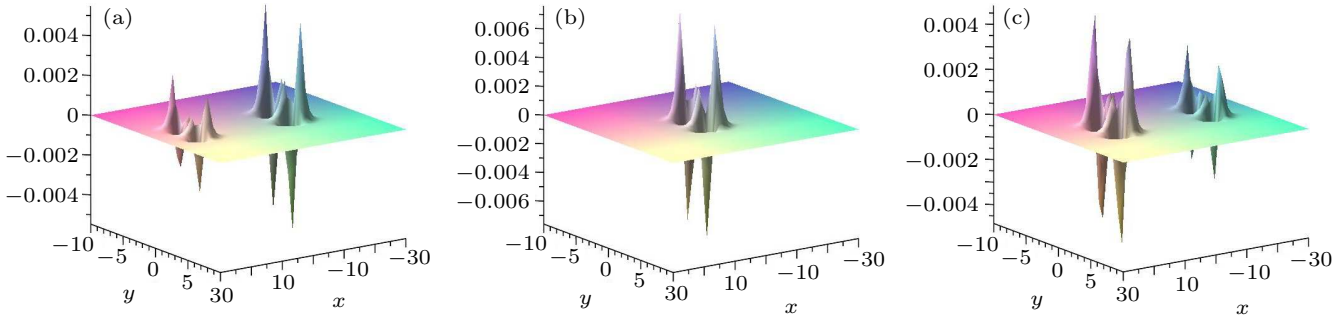


Fig. 1. A three-dimensional (3D) plot of the dromion solution of Eq. (8) for $N = 2$: (a) $t = -8$, (b) $t = -2.2$, and (c) $t = 4$.

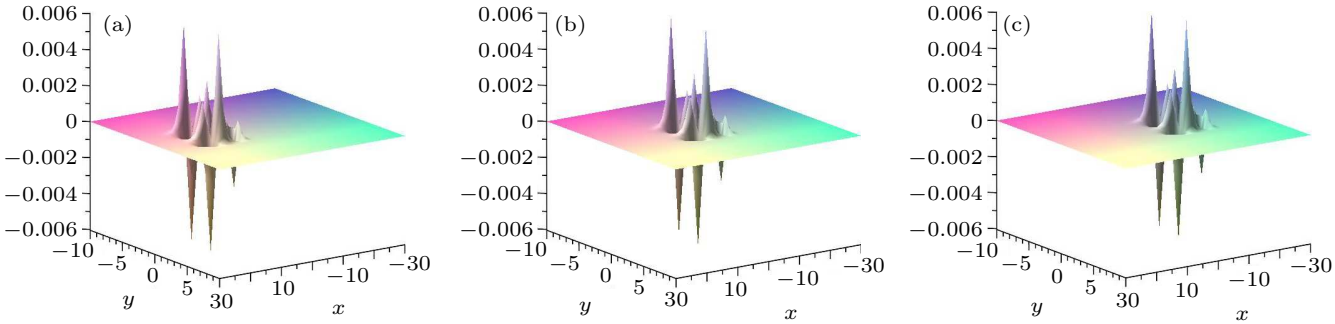


Fig. 2. The 3D plot of dromion molecule of Eq. (8) for $N = 2$ when $k_1/k_2 = v_1/v_2$: (a) $t = -5$, (b) $t = 0$, and (c) $t = 5$.

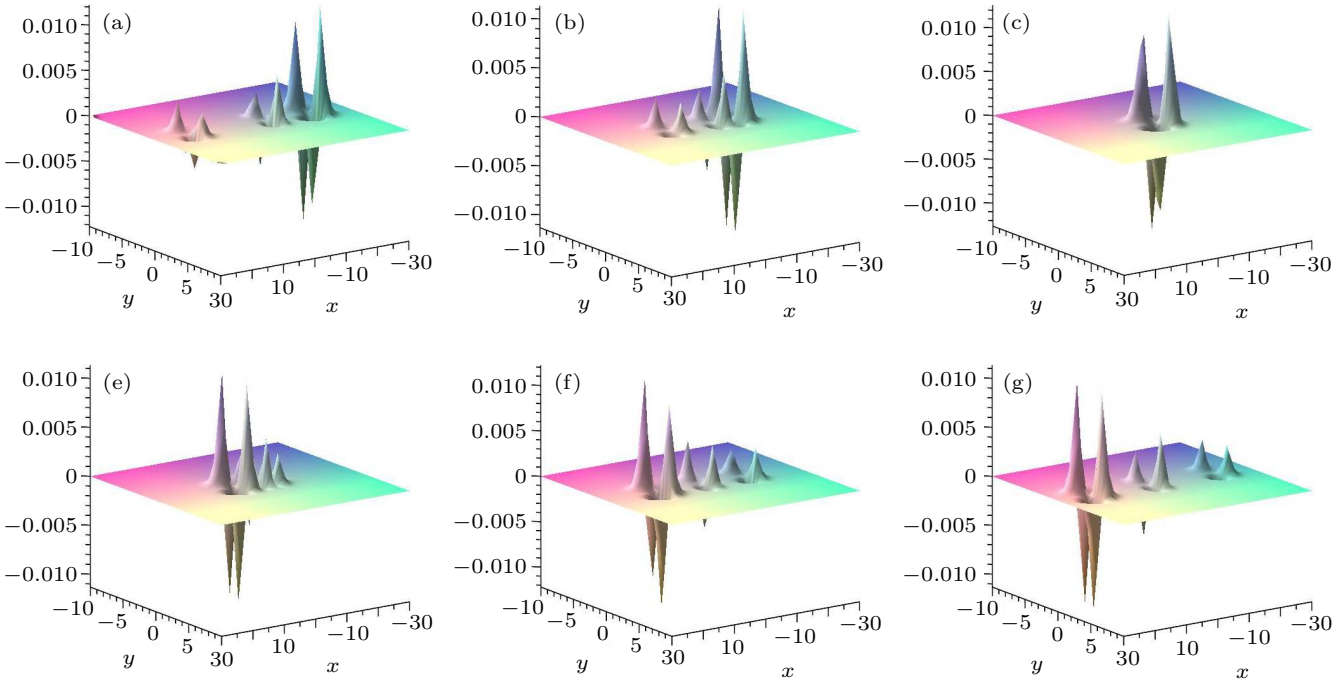


Fig. 3. The 3D plot of three dromion molecules moving with time increasing for $N = 3$: (a) $t = -8$, (b) $t = -5$, (c) $t = -2$, (d) $t = 0$, (e) $t = 3$, and (f) $t = 5$.

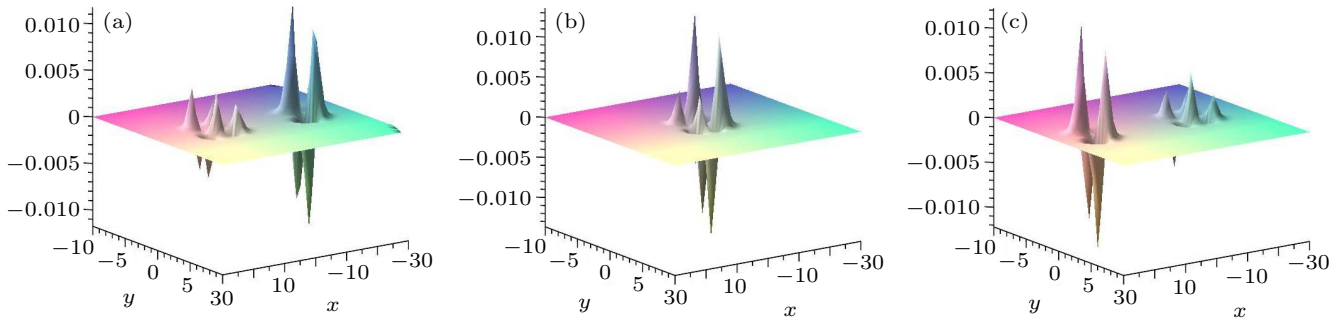


Fig. 4. For $N = 3$, the 3D plot of the elastic collision of dromion molecule and new dromion molecule with time increasing when $k_1/k_3 = v_1/v_3$: (a) $t = -8$, (b) $t = -2.7$, and (c) $t = 4$.

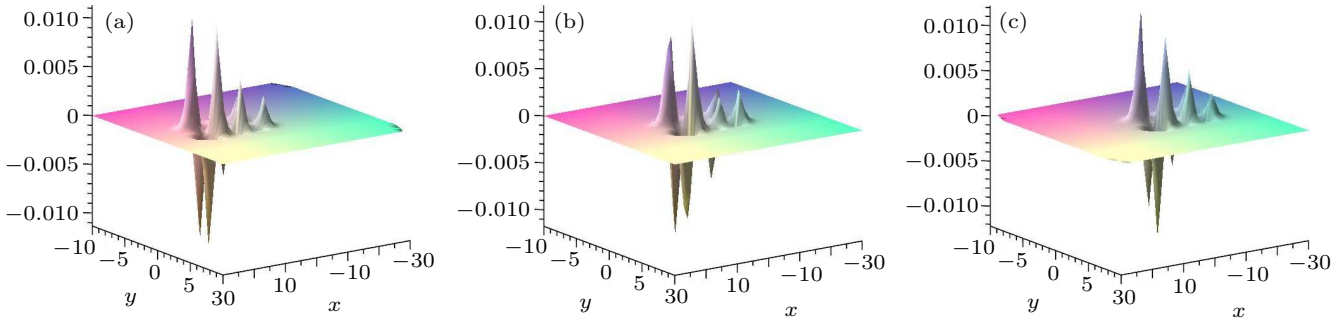


Fig. 5. For $N = 3$, the 3D plot of dromion molecule moving with time increasing when $k_i/k_j = v_i/v_j$: (a) $t = -4$, (b) $t = 0$, and (c) $t = 4$.

4. Rings, ring molecules, and their interactions

In order to obtain the ring molecule of Eq. (1), we take p and q as follows:

$$p = d + \sum_{i=1}^{M} \exp(w_i - (k_i x + v_i t + b_i)^2), \quad q = D + \sum_{i=1}^{N} \frac{2}{\operatorname{csch}(\theta_i + (K_i y + B_i)^2)}. \tag{9}$$

The parameters of Eq. (4) in this part are taken the values of $a_0 = 2$, $a_1 = 1$, $a_2 = 1$, $a_3 = 0$.

Case 1 When $M = N = 1$, by taking parameters $d = 1$, $w_1 = 10$, $k_1 = 1$, $b_1 = 5$, $v_1 = -1$, $D = 1$, $K_1 = -1$, $B_1 = 5$, $\theta_1 = 1$, equation (9) is the ring soliton solution which is the centrosymmetric graph of the central point at $(-5, -5)$ as shown in in Fig. 6.

Case 2 When $M = 2$ and $N = 1$, by taking $d = 1$, $w_1 = 10$, $k_1 = 0.8$, $b_1 = 5$, $v_1 = -5$, $w_2 = 5$, $k_2 = 0.8$, $b_2 = -5$, $v_2 = 2$, $D = 1$, $K_1 = 2$, $B_1 = 5$, $\theta_1 = 1$. We construct two ring molecules that are different in size and exactly the same in shape. In Fig. 7, we plot the images and their corresponding density images at $t = 0.4$, $t = 1.5$, and $t = 2.5$, respectively. When $v_1 \neq v_2 \neq 0$, the two molecules move along x axis at the speed of $t_x = (b_1 v_2 - b_2 v_1)/(k_1 v_2 - k_2 v_1)$.

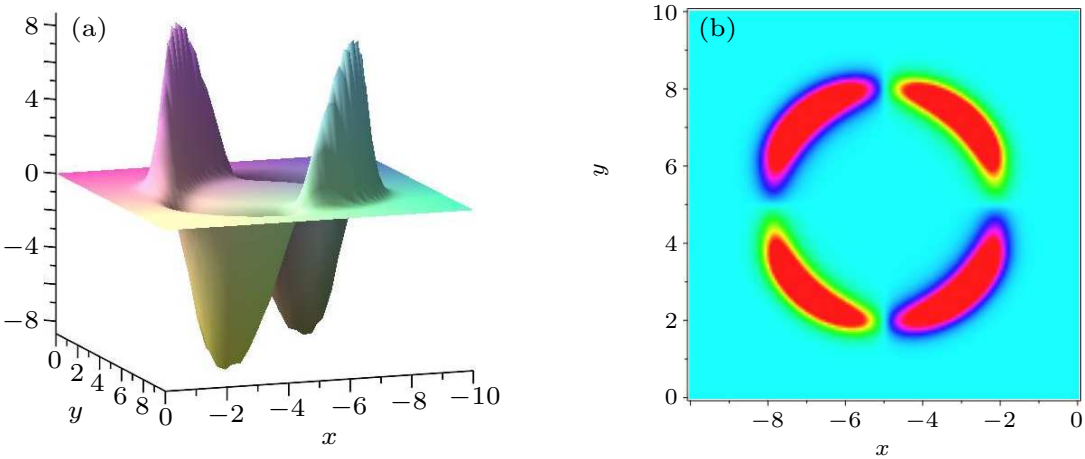


Fig. 6. The 3D plot and density plot of Eq. (9) for $M = N = 1$ at $t = 0$.

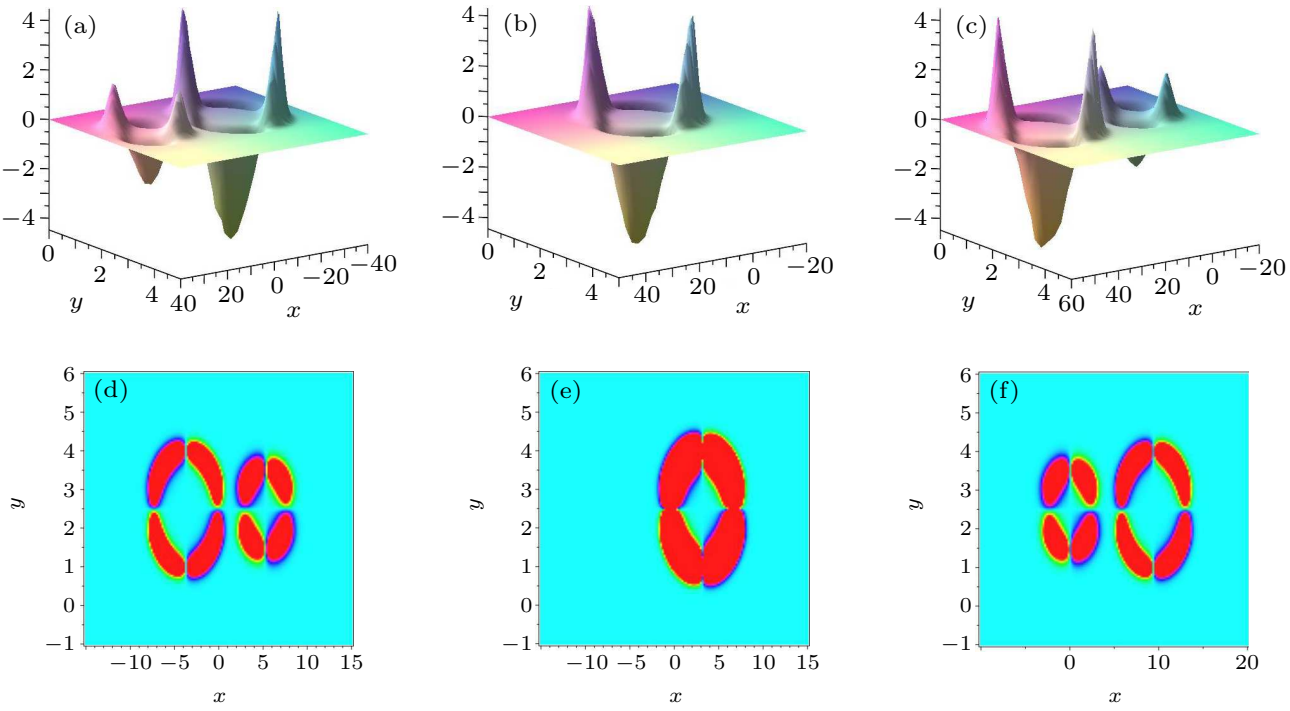


Fig. 7. The 3D plot and density plot of the interaction between two ring molecules when $M = 2$, $N = 1$ for (a) $t = 0.4$, (b) $t = 1.5$, and (c) $t = 2.5$.

5. Other molecules

5.1. Lump molecules

In order to gain the lump molecule of Eq. (1), we take p and q as follows:

$$p = c + \sum_{i=1}^{M} \frac{1}{1 + (k_i x + v_i t + b_i)^2}, \quad q = C + \sum_{i=1}^{N} \frac{1}{1 + (K_i y + B_i)^2}. \tag{10}$$

The parameters of Eq. (4) in this part are taken the values of $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$.

Case 1 When $M = N = 2$, the parameters are taken as $c = 1$, $v_1 = 1$, $k_1 = 1$, $b_1 = 5$, $v_2 = 3$, $k_2 = 1$, $b_2 = -5$, $C = 5$, $K_1 = 1$, $B_1 = -8$, $K_2 = 1$, $B_2 = 8$, we draw the 3D image and density diagram at $t = 0$ as shown in Fig. 8. When $M = N = 2$, equation (10) forms the lump molecule constituted of 2×2 lump "atoms".

Case 2 When $M = 3$ and $N = 1$, the parameters are $c = 1$, $v_1 = -3$, $k_1 = 1$, $b_1 = 5$, $v_2 = 1$, $k_2 = 1$, $b_2 = -5$, $v_3 = 3$, $k_3 = 1$, $b_3 = 15$, $C = -5$, $K_1 = 1$, $B_1 = -2$. Under these conditions, we construct the lump molecule for 1×3 lattice. The 3D image and the corresponding density diagram are shown in Fig. 9 at $t = 0$.

Case 3 When $M = 2$ and $N = 3$, the parameters are $c = 1$, $v_1 = 1$, $k_1 = 1$, $b_1 = 5$, $v_2 = 3$, $k_2 = 1$, $b_2 = -5$, $C = 5$, $K_1 = 1$, $B_1 = -4$, $K_2 = 1$, $B_2 = 4$, $K_3 = 1$, $B_3 = 12$. Under this condition, we construct lump molecule for 3×2 lattice. The 3D image and the corresponding density diagram are shown in Fig. 10 at $t = 0$.

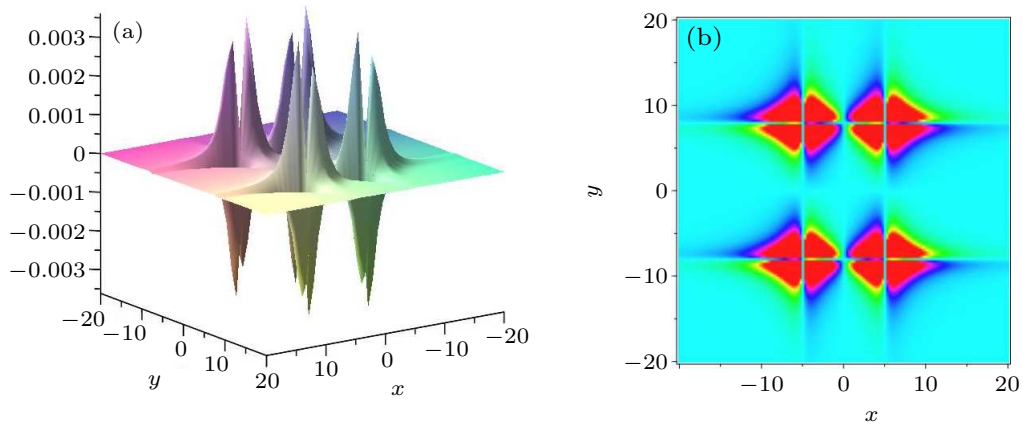


Fig. 8. The 3D plot and density plot of Eq. (10) for $M = N = 2$ at $t = 0$.

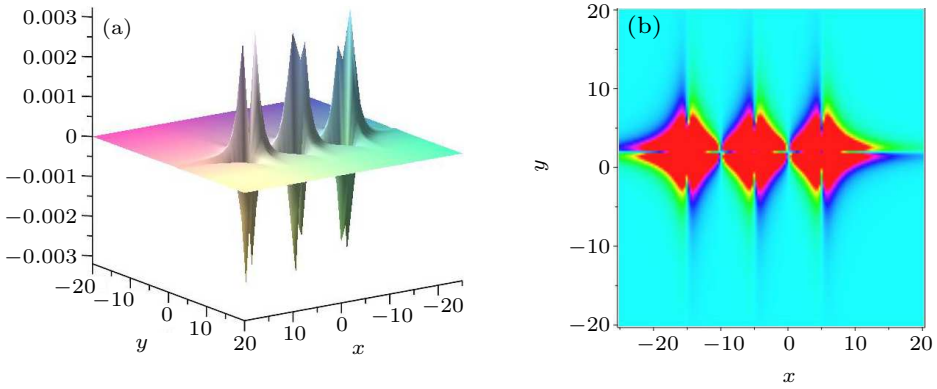


Fig. 9. The 3D plot and density plot of Eq. (10) for $M = 3$, $N = 1$ at $t = 0$.

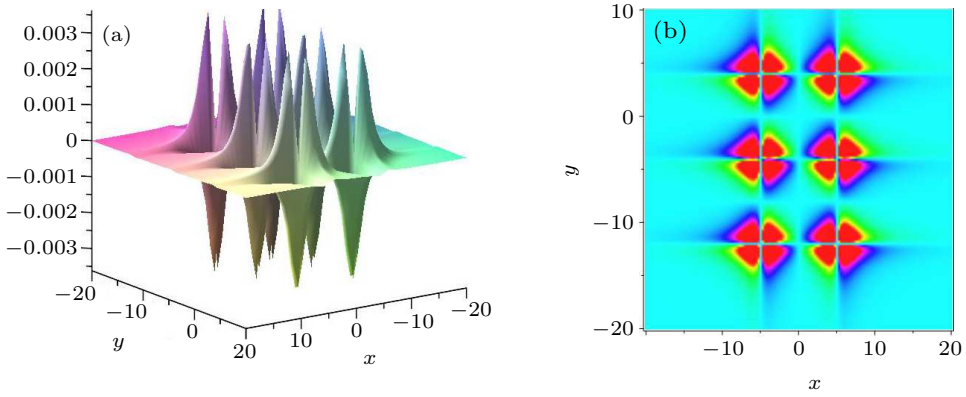


Fig. 10. The 3D plot and density plot of Eq. (10) for $M = 2$, $N = 3$ at $t = 0$.

Through the analysis of the above three cases, it is concluded that the lump solutions constructed by Eq. (10) can form lump molecules with N rows and M columns by choosing appropriate parameters, *i.e.*, the formation of $N \times M$ lattice.

5.2. Multi-instantaneous molecule

In order to gain the multi-instantaneous molecule of Eq. (1), we take p and q as follows:

$$p = \sum_{i=1}^{M} \frac{1}{1 + (k_i x + v_i(\mathrm{sech}(t)) + b_i)^2}, \quad q = \sum_{i=1}^{N} \frac{2(\mathrm{sech}(K_i y))^2}{1 + (K_i y + B_i)^2}. \tag{11}$$

The parameters of Eq. (4) in this part are taken the values of $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$.

When $M = N = 2$, by choosing $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $k_1 = 1$, $v_1 = 1$, $b_1 = 5$, $K_1 = 1$, $B_1 = -8$, $k_2 = 1$, $v_2 = 3$, $b_2 = -5$, $K_2 = 1$, and $B_2 = 8$, we draw a 3D image of Eq. (11) and its density diagram at $t = 5$ in Figs. 11(a) and 11(b). By observing the image, we find the graph of density plot is centrosymmetric and (0,0) is the symmetric point. The image shape and position of the multi-instantaneous molecule does not change with time, but its energy decays with time until it decays to 0. In order to more intuitively observe the changing trend of the energy, we plot the energy curve at $t = 0$, $t = 1$, $t = 2$ in Fig. 11.

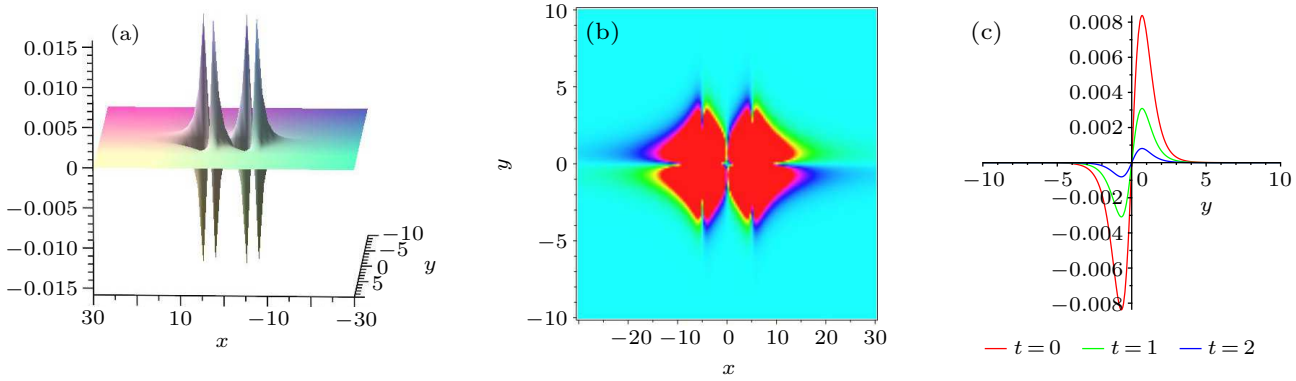


Fig. 11. (a) and (b) The 3D plot and density plot of Eq. (11) for $M = N = 2$ at $t = 5$, (c) the energy curve of Eq. (11) at $t = 0$, $t = 1$, $t = 2$.

6. Conclusion

In this paper, we first use the variable separation method to solve the (2+ 1)-dimensional potential BLMP equation. By taking appropriate assumptions, we conclude that equation (1) can be separated by variables under the given conditions. And the solutions under these conditions satisfy the general formula U applicable to most systems in which variable separation can be carried out. Then we know that Eq. (1) has a large number of local excitation modes in the form of Eq. (7). Therefore, we give four intuitive excitation modes, and draw the corresponding soliton molecules.

We plot dromion molecules when $N = 2$ and $N = 3$, and add the covelocity conditions while studying the motion, so that we could study the motion behavior when two or three dromion "atoms" are bound into a dromion molecule. Through the above studies, we come to the conclusion that no matter how to change their motion forms, the interactions between molecules are always elastic and they move along the x axis without changing their motion direction and speed. Then we plot the ring molecules and find that the interactions between the molecules are still elastic. We also plot two very interesting types of excitation patterns. Under the condition of Eq. (10) , the 3D plot of the lump molecule of $N \times M$ lattice will always be formed by taking appropriate parameters. In addition, we also draw a multi-instantaneous molecule whose shape does not change with time, but the energy decays to zero with time.

In this article, we adopted a relatively simple assumption for the $(2+1)$-dimensional potential BLMP equation and carried out variable separations. In fact, there could be some other forms of solutions that could be used to separate variables. Besides the four moleculars mentioned in this paper, there are still a lot of interesting excitation modes waiting for readers to explore. We hope that our results can further explain and enrich the dynamic behavior of nonlinear evolution equations, and also hope that our research can make some contributions to the research of mathematics and physics.

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References

[1] Cao C W 1990 *Phys. Lett. A* **175** 22
[2] Lou S Y and Chen L L 1999 *J. Math. Phys.* **40** 6491
[3] Shen S F 2005 *Commun. Theor. Phys.* **44** 779
[4] Zhang S L and Lou S Y 2007 *Commun. Theor. Phys.* **48** 385
[5] Lou S Y, Zhang S L and Qu C Z 2005 *Chin. Phys. Lett.* **22**
[6] Zhang S L and Lou S Y 2004 *Commun. Theor. Phys.* **41** 161
[7] Tang X Y and Lou S Y 2002 *Commun. Theor. Phys.* **38** 1
[8] Tang X Y and Lou S Y 2003 *Chin. Phys. Lett.* **20** 335
[9] Ying J P and Lou S Y 2003 *Chin. Phys. Lett.* **20** 1448
[10] Lou S Y, Qian X M and Hu X B 2004 *J. Phys. A: Math. Gen.* **37** 2401
[11] Qian X M, Lou S Y, and Hu X B 2004 *Zeitschrift für Naturforschung A* **59** 645
[12] Lou S Y, Hu H C and Liu Q P 2003 *Chin. Phys. Lett.* **20** 1413
[13] Hu H C, Lou S Y, Tang X Y and Liu Q P 2004 *Chaos, Solitons and Fractals* **22** 327
[14] Tang X Y and Lou S Y 2009 *Front. Phys. China* **4** 235
[15] Tang X Y and Liang Z F 2006 *Phys. Lett. A* **351** 398
[16] Lou S Y, Zhang S L and Qu C Z 2005 *Chin. Phys. Lett.* **22** 1029
[17] Tang X Y and Lou S Y 2003 *J. Math. Phys.* **44** 4000
[18] Lou S Y, Tang X Y and Zhang Y 2002 *Phys. Rev. E* **66** 046601
[19] Jin X W and Lin J 2020 *J. Magn. Magn. Mater.* **502** 166590
[20] Ma W X 2003 *Phys. Lett. A* **319** 325
[21] Ma W X 2021 *Math. Comput. Simul.* **190** 270
[22] Ma W X 2021 *J. Geom. Phys.* **165** 104191
[23] Ma W X, Yong X and Lü X 2021 *Wave Motion* **103** 102719
[24] Ma W X 2022 *Physica D* **430** 133078
[25] Lou S Y 2020 *J. Phys. Commun.* **4** 041002
[26] Yan Z W and Lou S Y 2020 *Appl. Math. Lett.* **104** 106271
[27] Zhao Q L, Lou S Y and Jia M 2020 *Commun. Theor. Phys.* **72** 085005
[28] Zhang Z, Yang S X and Li B 2019 *Chin. Phys. Lett.* **36** 120501
[29] Dong J J, Li B and Yuen M W 2020 *Commun. Theor. Phys.* **72** 025002
[30] Yang X Y, Fan R and Li B 2020 *Phys. Scripta* **95** 045213
[31] Xu D H and Lou S Y 2020 *Acta Phys. Sin.* **69** 014208 (in Chinese)
[32] Yang X Y, Zhang Z and Li B 2020 *Chin. Phys. B* **29** 100501
[33] Ma H C, Huang H Y and Deng A P 2021 *J. Geom. Phys.* **170** 104362
[34] Ma H C, Huang H Y and Deng A P 2021 *Mod. Phys. Lett. B* **35** 8
[35] Ma H C, Cheng Q X and Deng A P 2020 *Commun. Theor. Phys.* **72** 095001
[36] Ma H C, Gao Y D and Deng A P 2021 *Mod. Phys. Lett. B* **35** 2150482
[37] Nimmo J J C, Gilson C R and Willox R 1993 *Phys. Lett. A* **180** 337
[38] Seadawy A R, Ali A, and Helal M A 2021 *Math. Meth. Appl. Sci.* **44** 14292
[39] Ashraf F, Younis M, Iqbal H, Rizvi S T R, Seadawy A R and Baleanu D 2020 *Results Phys.* **19** 103661
[40] Kaplan M 2018 *Chin. J. Phys.* **56** 2523
[41] Ma W X, He C H, Tang X Y and Ma J L 2019 *Nonlinear Dyn.* **95** 29
[42] Luo L 2011 *Phys. Lett. A* **375** 1059
[43] Kumar M and Tiwari A K 2018 *Comput. Math. Appl.* **75** 1434
[44] Jia S L, Su J J, Hu L, Gao Y T and Deng G F 2019 *Mod. Phys. Lett. B.* **33** 1950376
[45] Zhang C R, Tian H Y, Shen Y, Tian B and Liu S H 2021 *Mod. Phys. Lett. B.* **35** 2150261
[46] Tang X Y, Cui C J and Cui Y J 2020 *Appl. Math. Lett.* **102** 106109
[47] Li H M, Lin J, Ren B and Li Y S 2008 *Phys. Rev. E* **77** 036605