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Universal order-parameter and quantum phase transition for two-dimensional q-state quantum Potts model

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We investigate quantum phase transitions for q-state quantum Potts models ($q = 2, 3, 4$) on a square lattice and for the Ising model on a honeycomb lattice by using the infinite projected entangled-pair state algorithm with a simplified updating scheme. We extend the universal order parameter to a two-dimensional lattice system, which allows us to explore quantum phase transitions with symmetry-broken order for any translation-invariant quantum lattice system of the symmetry group G. The universal order parameter is zero in the symmetric phase, and it ranges from zero to unity in the symmetry-broken phase. The ground-state fidelity per lattice site is computed, and a pinch point is identified on the fidelity surface near the critical point. The results offer another example highlighting the connection between (i) critical points for a quantum many-body system undergoing a quantum phase-transition and (ii) pinch points on a fidelity surface. In addition, we discuss three quantum coherence measures: the quantum Jensen–Shannon divergence, the relative entropy of coherence, and the l_1 norm of coherence, which are singular at the critical point, thereby identifying quantum phase transitions.

Keywords: quantum phase transitions, universal order parameter, fidelity

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1. Introduction

Quantum phase transitions (QPTs)[1] occur at absolute zero temperature due to the Heisenberg uncertainty relation and are driven by quantum fluctuations. The core of the QPT consists of the Hamiltonian and its energy spectrum, which have been well studied through the development of various numerical methods. In particular, tensor network algorithm[2–12] is a very powerful tool to study the strongly correlated quantum lattice systems. To date, most QPTs can be described based on the spontaneous-symmetry-breaking order of the Landau–Ginzburg–Wilson paradigm. However, this local order parameter is model dependent and hard to be defined. In addition, not all phases can be described by symmetry-broken order, such phases correspond to QPTs beyond the Landau–Ginzburg–Wilson paradigm.[13–16]

Recently, for one-dimensional systems of infinite size, Liu *et al.*[17] presented a universal order-parameter concept based on the fidelity between a ground state and its symmetry-transformed counterpart. The advantage of the universal order parameter over local order parameters is the former's universality in characterizing QPTs in quantum lattice many-body systems, in the sense that the universal order parameter is not model dependent, in contrast with model-dependent order parameters. Reference [18] extends the universal order parameter from one-dimensional systems of infinite size to one-dimensional finite-size systems. Herein we further extend the use of the universal order parameter to two-dimensional quantum systems. To test the pertinence of the universal order parameter to describe two-dimension lattice symmetry, we also calculate the ground-state fidelity per lattice site and quantum coherence.

Quantum fidelity is a basic notion in quantum information science and is an approach to study QPTs in strongly correlated many-body systems.[19–28] As a measure of distance between two quantum states, quantum fidelity provides a measure of the similarity between two quantum states. When a system undergoes a QPT, the ground-state fidelity dramatically changes upon passing through the critical point in phase space, with the "pinch point" being the signature of the phase transition. In addition, the quantification of quantum coherence[29] has also revealed intriguing connections between correlation and quantum coherence.[30–32] A variety of quantum coherence measures have been introduced to detect QPTs, such as the quantum Jensen–Shannon divergence,[33] the relative entropy of coherence,[29] and the l_1 norm of coherence.[29]

The present work investigates 2-, 3-, and 4-state quantum Potts models on a square lattice and the quantum Ising model on a honeycomb lattice by using the infinite projected entangled-pair state algorithm[3,4,7] in two dimensions. Here, we employ a simplified scheme to update the ground-state wave function. The infinite time-evolving block decimation (iTEBD) method[2] will be used to calculate iPEPS ground-state wave function with χ being the bond dimension. In or-

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der to calculate the observable physical quantity, the corner transfer matrix renormalization method[34] is implemented to contracted vironmental tensor with M being the environment dimension. The remainder of this paper is organized as follows. In Section 2, the two-dimensional q-state quantum Potts mode is introduced. Section 3 discusses the universal order parameter for 2-, 3-, and 4-state quantum Potts models on a square lattice and the quantum Ising model on a honeycomb lattice, and explains how the universal order parameter provides a model-independent way to characterize QPTs in many-body systems. Section 4 discusses the ground-state fidelity per lattice site, and Section 5 shows how quantum coherence measures can serve to detect QPTs. Finally, a summary is given in Section 6.

2. The q-state quantum Potts model in two dimensions

We now consider the q-state quantum Potts model[27,35] in a transverse magnetic field in two dimensions. The q state is described by the Hamiltonian

$$H_q = -\sum_{(\boldsymbol{r},\boldsymbol{r}')}\left(\sum_{p=1}^{q-1} M_{x,\,p}^{[\boldsymbol{r}]} M_{x,\,q-p}^{[\boldsymbol{r}']}\right) - \sum_{\boldsymbol{r}} \lambda M_z^{[\boldsymbol{r}]}. \tag{1}$$

Here, $(\boldsymbol{r},\boldsymbol{r}')$ runs over all possible nearest-neighbor pairs on the square lattice or honeycomb lattice, λ is the transverse magnetic field and $M_{\alpha,p}^{[\boldsymbol{r}]}$ with $p \in [1, q-1]$ ($\alpha = x, z$) are the q-state Potts spins at site $\boldsymbol{r}$. The spin matrices for the q-state quantum Potts model are given by

$$M_{x,1} = \begin{pmatrix} 0 & I_{q-1} \\ 1 & 0 \end{pmatrix},$$

$$M_z = \begin{pmatrix} q-1 & 0 \\ 0 & -I_{q-1} \end{pmatrix},$$

where I_{q-1} is the $(q-1)\times(q-1)$ identity matrix, and $M_{x,p} = (M_{x,1})^p$. Note that $(M_{x,1})^q$ is a $q \times q$ matrix. Here, we consider quantum transverse Ising model ($q = 2$).

The model is invariant under the symmetry operation $M_{x,p}^{\vec{[r]}} \longrightarrow (\omega_q^p)^{m-1} M_{x,p}^{\vec{[r]}}$ and $M_z^{\vec{[r]}} \longrightarrow M_z^{\vec{[r]}}$, where $\omega = \exp(2\pi \mathrm{i}/q)$ and $m \in [1, q]$, which yields Z_q symmetry. The ground state is q-fold degenerate in the Z_q-symmetry-breaking phase for the q-state quantum Potts model. The classical mean-field solutions[36] and many computations by the Monte Carlo method, series expansions, etc.[37–42] have suggested that the 3D classical q-state Potts model undergo a continuous phase transition for $q \leq 2$ and a first-order phase transition for $q > 2$. The numerical results in Ref. [27] show that quantum q-state Potts model undergo a continuous (discontinuous) quantum phase transition for $q = 2$ ($q = 3$, 4) on the square lattice.

3. Universal order parameter

Reference [17] explains the concept of a universal order parameter. For any translation-invariant quantum lattice system of the symmetry group G, if the system undergoes a QPT with symmetry order, the universal order parameter can be used to characterize the QPT. The ground-state fidelity $F(\psi, g\psi)$ between a ground state $|\psi\rangle$ and its symmetry-transformed counterpart $g|\psi\rangle$, where $g \in G$ is any symmetry operation, can be written as $F(\psi, g\psi) = |\langle\psi|g|\psi\rangle|$. The ground-state fidelity asymptotically scales as $[f_g(\lambda)]^L$,[20–26] with L being the system size. As detailed in Refs. [17,18,28], the universal order is

$$I_g(\lambda) = \sqrt{1 - |f_g(\lambda)|^2}. \tag{2}$$

Given a control parameter λ in the symmetry phase, then $I_g(\lambda) = 0$. For a control parameter λ in the symmetry-broken phase, the universal order parameter $I_g(\lambda)$ ranges from zero to unity. Note that $I_g(\lambda)$ satisfies features of the order parameter. In fact, to characterize a quantum lattice many-body system with global symmetry group G spontaneously broken, the universal order parameter is model independent.

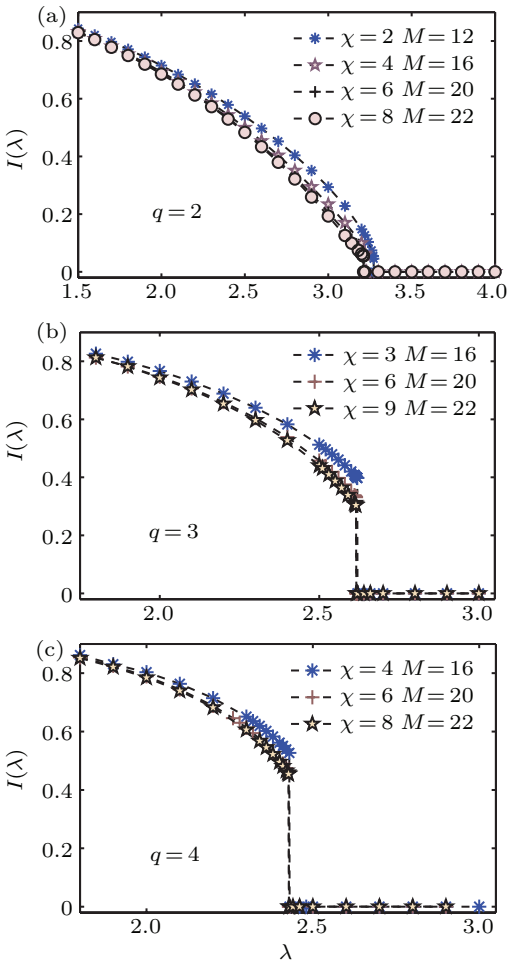


Fig. 1. The universal order parameter $I(\lambda)$ for (a) the quantum Ising model, (b) the quantum three-state Potts model, and (c) the quantum four-state Potts model in a transverse magnetic field λ on the square lattice for different bond dimension χ.

Figures 1(a)–1(c) show the universal order parameter $I(\lambda)$ as a function of transverse magnetic field λ for the quantum Ising model, the three-state quantum Potts model, and the four-state quantum Potts model, respectively, with a transverse magnetic field imposed on a square lattice and for various bond dimensions χ. The results show that the universal order parameter $I(\lambda)$ is nonzero in the Z_q symmetry-broken phase but zero in the symmetry phase. The variations in the universal order parameter $I(\lambda)$ indicate that the many-body system undergoes a QPT when the control parameter λ crosses the phase-transition point. The pseudo critical points are as follows:

(a) $\lambda_c = 3.273, 3.222, 3.213$ and 3.212 with the bond dimension $\chi = 2, 4, 6, 8$ and the environment dimension $M = 12, 16, 20, 22$ for the quantum Ising model, respectively. Indeed, the pseudo critical points obtained by using the simplified updating scheme for the quantum transverse Ising model on the square lattice compares rather poorly with the result $\lambda_c = 3.044330(6)$ from the most accurate quantum Monte Carlo;[43]

(b) $\lambda_c = 2.62, 2.616$ and 2.616 with the bond dimension $\chi = 3, 6, 9$ and the environment dimension $M = 16, 20, 22$ for the three-state quantum Potts model, respectively. The estimate $\lambda_c = 2.616$ is consistent with the known estimate $\lambda_c \sim 2.58$;[44]

(c) $\lambda_c = 2.43, 2.428$ and 2.428 with the bond dimension $\chi = 4, 6, 8$ and the environment dimension $M = 16$, 20, 22 for the four-state quantum Potts model, respectively.

These results are consistent with those of Ref. [27].

Figure 2 plots the universal order parameter $I(\lambda)$ as a function of λ for the quantum Ising model on a honeycomb lattice. The pseudo critical point λ_c occurs at $\lambda_c = 2.216$, 2.206 and 2.20 for the bond dimensions $\chi = 4$, 6, 9 and the environment dimension $M = 16$, 20, 22, respectively. The quantum Ising model on a honeycomb lattice has been much less studied. A high-precision Monte Carlo estimate gives a critical point of $\lambda_c = 2.13250(4)$.[45]

These results indicate that pseudo critical points in two-dimensional systems can be located by using the universal order parameter $I(\lambda)$. In addition, the universal order parameter $I(\lambda)$ is continuous for the Ising model, while the universal order parameter $I(\lambda)$ is abrupt for $q = 3$ and $q = 4$. The continuous (discontinuous) behavior of the universal order parameter $I(\lambda)$ indicates that Ising model on the square (honeycomb) lattice undergoes a continuous phase transition and $q = 3$ and $q = 4$ state Potts undergo a discontinuous phase transition on the square lattice.

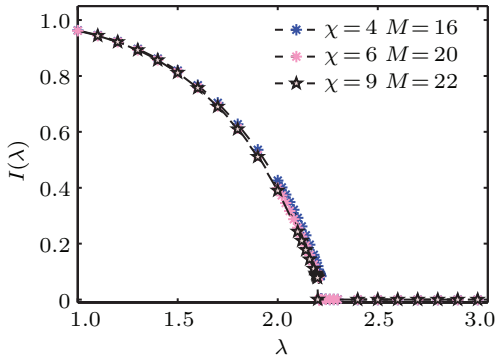


Fig. 2. The universal order parameter $I(\lambda)$ as a function of λ for quantum Ising model on the honeycomb lattice.

4. Ground-state fidelity per lattice site

Fidelity is a measure of the "similarity" of two quantum states. The ground-state fidelity $F(\lambda_1, \lambda_2) = |\langle\psi(\lambda_2)|\psi(\lambda_1)\rangle|$ asymptotically scales as $F(\lambda_1, \lambda_2) \sim d(\lambda_1, \lambda_2)^L$ with λ_1 and λ_2 being two values of the control parameter λ, where $L = L_x \times L_y$ is the size of the two-dimensional lattice for the two given ground states $|\psi(\lambda_1)\rangle$ and $|\psi(\lambda_2)\rangle$. Here, $d(\lambda_1, \lambda_2)$ is the ground-state fidelity per lattice site, characterizing how fast the fidelity tends to zero in the thermodynamic limit,[20–26] defined as

$$d(|\psi(\lambda_1)\rangle, |\psi(\lambda_2)\rangle) = \lim_{L\to\infty} F(|\psi(\lambda_1)\rangle, |\psi(\lambda_2)\rangle)^{1/L}. \quad (3)$$

The ground-state fidelity per lattice site, $d(\lambda_1, \lambda_2)$, satisfies the inherited properties of (i) range $[0 \le d(\lambda_1, \lambda_2) \le 1]$, (ii) normalization $[d(\lambda, \lambda) = 1]$, and (iii) symmetry $[d(\lambda_1, \lambda_2) = d(\lambda_2, \lambda_1)]$. With the tensor network representation, the ground-state fidelity per lattice site, $d(\lambda_1, \lambda_2)$, is the largest eigenvalue of the transfer matrix.[26]

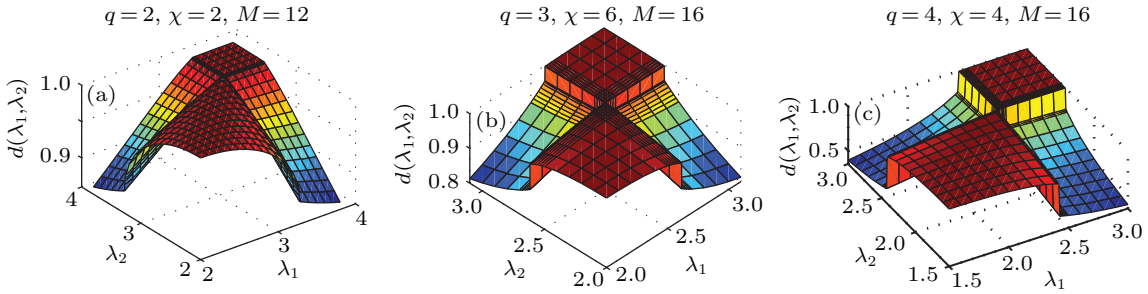


Fig. 3. The ground state fidelity surface defined by the ground state fidelity per site, $d(\lambda_1, \lambda_2)$, as a function of the transverse magnetic field λ_1 and λ_2 for (a) the quantum Ising model, (b) the quantum three-state Potts model, and (c) the quantum four-state Potts model on the square lattice. The pseudo critical point λ_c occurs as a pinch point on the ground state fidelity surface.

Figures 3(a)–3(c) plot the ground-state fidelity per lattice site, $d(\lambda_1,\lambda_2)$, as a function of the control parameters λ_1 and λ_2 for the quantum Ising model, the three-state quantum Potts model, and the four-state quantum Potts model, respectively, with a transverse magnetic field applied to the square lattice. A pinch point corresponding to the QPT point appears on the ground-state fidelity surface. The pinch point is located at $\lambda_c = 3.273$ for the quantum Ising model with the bond dimension $\chi = 2$ and the environment dimension $M = 12$ [Fig. 3(a)], $\lambda_c = 2.616$ for the three-state quantum Potts model with the bond dimension $\chi = 6$ and the environment dimension $M = 16$ [Fig. 3(b)], and $\lambda_c = 2.43$ for the four-state quantum Potts model with the bond dimension $\chi = 4$ and the environment dimension $M = 16$ [Fig. 3(c)].

Figure 4 plots the ground-state fidelity per lattice site, $d(\lambda_1,\lambda_2)$, as a function of the control parameters λ_1 and λ_2 for the quantum Ising model on a honeycomb lattice. With the square lattice, a pinch point appears on the ground-state fidelity surface. Note that the pinch point is at the intersection of two singular lines that characterize the phase-transition points. The pseudo critical point is located at $\lambda_c = 2.216$ for the quantum Ising model with the bond dimension $\chi = 4$ and the environment dimension $M = 16$ on the honeycomb lattice. Thus, the phase-transition points λ_c obtained from the fidelity per lattice site are consistent with the results from the universal order parameter. In addition, the results constitute another example of the connection between (i) pseudo critical points for a quantum many-body system undergoing a QPT[19–21] and (ii) pinch points on a fidelity surface. The fidelity per lattice site can distinguish continuous (discontinuous) quantum phase transition according to the continuous (discontinuous) behavior of fidelity near the critical point. From the surface of the fidelity per lattice site, we can also see that $q \geq 3$ Potts model undergoes a discontinuous phase transition on the square lattice. Ising model undergoes a continuous phase transition on the square lattice and honeycomb lattice, respectively.

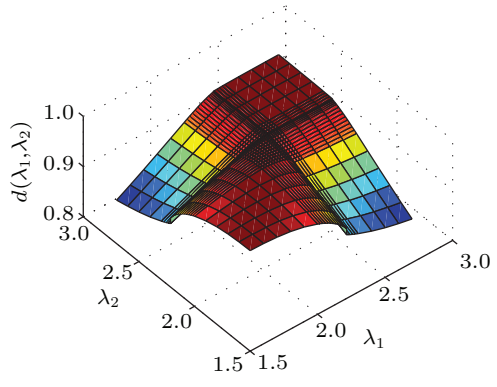


Fig. 4. The ground state fidelity surface defined by the ground state fidelity per site, $d(\lambda_1,\lambda_2)$, as a function of the transverse magnetic field λ_1 and λ_2 for the quantum Ising model on the honeycomb lattice.

5. Quantum coherence measures

This section discusses the connection between various quantum coherence measures and a quantum many-body system undergoing a QPT. We consider three coherence measures: the quantum Jensen–Shannon divergence, the relative entropy of coherence, and the l_1 norm of coherence. The quantum Jensen–Shannon divergence[33] is

$$C_{\mathrm{JS}}(\rho) = \sqrt{S\left(\frac{\rho_{\mathrm{diag}}+\rho}{2}\right) - \frac{S(\rho_{\mathrm{diag}})+S(\rho)}{2}}, \tag{4}$$

where ρ is the density matrix, and ρ_{diag} contains the diagonal elements of the density matrix ρ.

The relative entropy of coherence[29] is

$$C_{\mathrm{re}}(\rho) = S(\rho_{\mathrm{diag}}) - S(\rho). \tag{5}$$

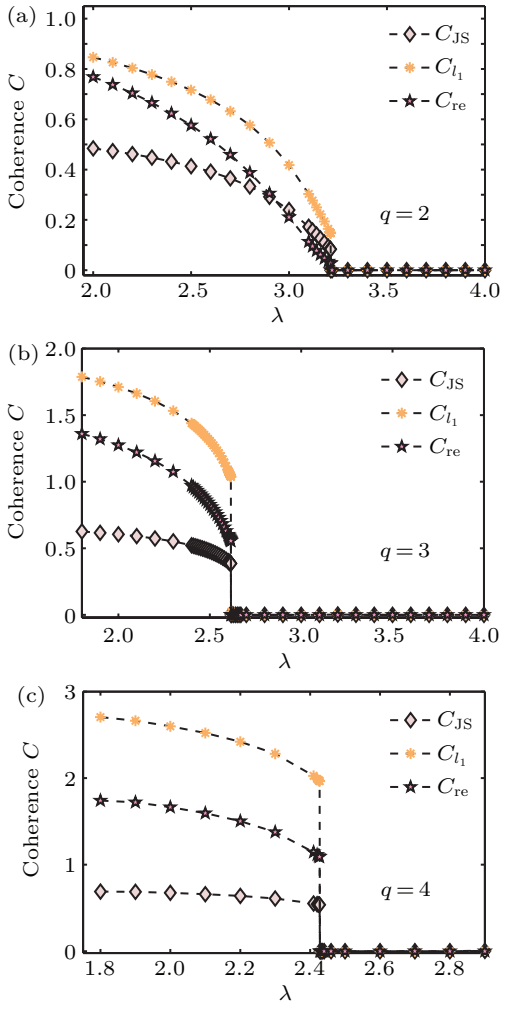


Fig. 5. Quantum coherence measures C as a function of the transverse magnetic field λ for (a) the quantum Ising model with $\chi = 6$ and $M = 20$, (b) the quantum three-state Potts model with $\chi = 9$ and $M = 22$, and (c) the quantum four-state Potts model with $\chi = 8$ and $M = 22$ on the square lattice.

The l_1 norm of coherence[29] is

$$C_{l_1}(\rho) = \sum_{n\neq m} |\rho_{nm}|, \tag{6}$$

where ρ_{nm} denotes the off-diagonal elements of the density matrix ρ. Finally, S is the entanglement entropy and is defined as $S = -\mathrm{Tr}\rho\log_2\rho$.

Figures 5(a)–5(c) plot the three coherence measures as a function of λ for the quantum Ising model, the three-state quantum Potts model, and the four-state quantum Potts model, respectively, on a square lattice. The pseudo critical point is located at $\lambda_c = 3.213$ with $\chi = 6$ and $M = 20$ [Fig. 5(a)], $\lambda_c = 2.616$ with $\chi = 9$ and $M = 22$ [Fig. 5(b)], and $\lambda_c = 2.428$ with $\chi = 8$ and $M = 22$ [Fig. 5(c)]. The result is consistent with the results for the universal order parameter and for the ground-state fidelity per lattice site. Figure 6 also plots the three coherence measures as a function of λ for the quantum Ising model on a honeycomb lattice. The pseudo critical point is at $\lambda_c = 2.20$ with the bond dimension $\chi = 9$ and the environment dimension $M = 22$, which is consistent with the results from the universal order parameter and the ground-state fidelity per lattice site. These results show that quantum coherence measure detects the QPT in a two-dimensional lattice system. In addition, quantum coherence can also distinguish continuous and discontinuous phase transitions.

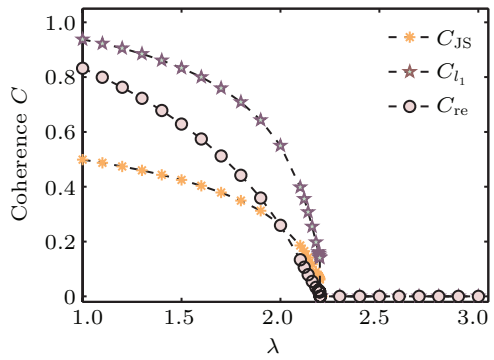


Fig. 6. Three quantum coherence measures C as a function of the transverse magnetic field λ for the quantum Ising model on the honeycomb lattice, with $\chi = 9$ and $M = 22$.

6. Summary

We investigate herein the QPTs for 2-, 3-, and 4-state quantum Potts models on a square lattice and for the Ising model on a honeycomb lattice by using the infinite projected entangled-pair state algorithm with the simplified updating scheme. We extend the universal order parameter to a two-dimensional system. The universal order parameter, ground-state fidelity per lattice site, and a variety of quantum coherence measures are used to characterize QPTs. The universal order parameter can be used to explore QPTs with symmetry-broken order for any translation-invariant quantum lattice system of the symmetry group G. Although the universal order parameter is zero in a symmetric phase, it ranges from zero to unity in symmetry-broken phase. When the control parameter crosses the critical point, the universal order parameter changes, which implies that the system undergoes a QPT at the phase-transition point. We also discuss the ground-state fidelity per lattice site and identify the pinch point near the critical point on the fidelity surface, which corresponds to the QPT point. Finally, we discuss three quantum coherence measures: the quantum Jensen–Shannon divergence, the relative entropy of coherence, and the l_1 norm of coherence. These measures have singularities at the critical point, which identifies the QPTs. By using the universal order parameter, the ground-state fidelity per lattice site and the quantum coherence measures provide consistent positions for the phase-transition point. From the universal order parameter, the surface of the fidelity per lattice site and the quantum coherence measures, Our results show that Ising model undergoes a continuous phase transition on the square lattice and honeycomb lattice, respectively. And 3, 4-state Potts model undergo a discontinuous phase transition on the square lattice. We expect the universal order parameter, ground-state fidelity per lattice site, and quantum coherence measures to provide further insights into critical phenomena in strongly correlated many-body quantum lattice systems of any dimensionality.

Acknowledgements

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