

Constructing the three-qudit unextendible product bases with strong nonlocality

Bichen Che(车碧琛), Zhao Dou(窦钊), Xiubo Chen(陈秀波), Yu Yang(杨榆), Jian Li(李剑), and Yixian Yang(杨义先)
Citation: Chin. Phys. B, 2022, 31 (**6**): 060302. DOI: 10.1088/1674-1056/ac4a62
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Coherence migration in high-dimensional bipartite systems

Zhi-Yong Ding(丁智勇), Pan-Feng Zhou(周攀峰), Xiao-Gang Fan(范小刚),Cheng-Cheng Liu(刘程程), Juan He(何娟), and Liu Ye(叶柳)
Chin. Phys. B, 2022, 31 (**6**): 060308. DOI: 10.1088/1674-1056/ac48f8

Short-wave infrared continuous-variable quantum key distribution over satellite-to-submarine channels

Qingquan Peng(彭清泉), Qin Liao(廖骎), Hai Zhong(钟海), Junkai Hu(胡峻凯), and Ying Guo(郭迎)
Chin. Phys. B, 2022, 31 (**6**): 060306. DOI: 10.1088/1674-1056/ac490a

Digraph states and their neural network representations

Ying Yang(杨莹) and Huaixin Cao(曹怀信)
Chin. Phys. B, 2022, 31 (**6**): 060303. DOI: 10.1088/1674-1056/ac401d

Local sum uncertainty relations for angular momentum operators of bipartite permutation symmetric systems

I Reena, H S Karthik, J Prabhu Tej, Sudha, A R Usha Devi, and A K Rajagopal
Chin. Phys. B, 2022, 31 (**6**): 060301. DOI: 10.1088/1674-1056/ac5fa3

Beating standard quantum limit via two-axis magnetic susceptibility measurement

Zheng-An Wang(王正安), Yi Peng(彭益), Dapeng Yu(俞大鹏), and Heng Fan(范桁)
Chin. Phys. B, 2022, 31 (**4**): 040309. DOI: 10.1088/1674-1056/ac4229

Constructing the three-qudit unextendible product bases with strong nonlocality

Bichen Che(车碧琛)[1], Zhao Dou(窦钊)[1,†], Xiubo Chen(陈秀波)[1], Yu Yang(杨榆)[2], Jian Li(李剑)[2], and Yixian Yang(杨义先)[1]

[1] *Information Security Center, State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing* 100876, *China*

[2] *Information Security Center, Beijing University of Posts and Telecommunications, Beijing* 100876, *China*

(Received 10 September 2021; revised manuscript received 20 December 2021; accepted manuscript online 12 January 2022)

Unextendible product bases (UPBs) are interesting members of a family of orthogonal product bases. Here, we investigate the construction of 3-qudit UPBs with strong nonlocality. First, a UPB set in $C^3\otimes C^3\otimes C^3$ of size 19 is presented based on the shift UPBs. By mapping the system to a Rubik's cube, we provide a general method of constructing UPBs in $C^d\otimes C^d\otimes C^d$ of size $(d-1)^3+2d+5$, whose corresponding Rubik's cube is composed of four parts. Second, for the more general case where the dimensions of parties are different, we extend the classical tile structure to the 3-qudit system and propose the tri-tile structure. By means of this structure, a $C^4\otimes C^4\otimes C^5$ system of size 38 is obtained based on a $C^3\otimes C^3\otimes C^4$ system of size 19. Then, we generalize this approach to the $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$ system which also consists of four parts. Our research provides a positive answer to the open question raised in by Halder *et al.* [*Phys. Rev. Lett.* **122** 040403 (2019)], indicating that there do exist UPBs that can exhibit strong quantum nonlocality without entanglement.

Keywords: strong nonlocality, unextendible product bases, tri-tile structure, construction method

PACS: 03.67.–a, 03.65.Aa, 03.65.Ud **DOI:** 10.1088/1674-1056/ac4a62

1. Introduction

With its potential applications, quantum information has brought new changes to the information industry.[1–4] Entanglement between quantum states is a kind of quantum resource, which can be used to achieve tasks that cannot be accomplished by classical resources,[5–8] such as quantum teleportation,[9,10] quantum algorithm,[11–13] and quantum dense coding.[14] For a long time, it has been believed that the nonlocality is closely related to properties of entangled states. However, in 1999, Bennett *et al.*[15] proposed a set of orthogonal product states with nonlocality, which aroused a wide discussion on the relationship between entanglement and nonlocality.

Unextendible product bases (UPBs) are a class of incomplete orthogonal product states, whose complementary space does not contain product states.[16–18] They cannot be distinguished perfectly by local operations and classical communication (LOCC). Researchers[19–21] have shown that it can be used in the production of bound entangled (BE) states and some special bipartite entangled states that remain positive under partial transpose (PPT). Nevertheless, most of the current efforts are devoted to the construction of 2-qubit UPBs,[22–24] while a little progress has been made on multi-qudit UPBs.[25,26] Chen *et al.*[22] investigated the minimum size of UPBs with local dimension of 2, and analyzed the proposed sets using orthogonal graphs. Bej *et al.*[23] proposed that a set of reducible unextendible product states (UPSs) with high dimension can be obtained by adding several orthogonal product states on the bases of a set of UPSs with low dimension in a bipartite system. Recently, a method to construct UPBs of different sizes in $C^m\otimes C^n$ was put forward by Shi *et al.*,[24] which uses U-tile structure. Shi *et al.*[25] also constructed the strongly nonlocal UPB in a tripartite system, which contains d^3-4d+4 states when d is odd, and contains d^3-4d+8 states when d is even. Multi-qudit UPBs are not only valuable in quantum circuit construction or cryptography experiments, but also often used to construct tight Bell inequalities without quantum violations.[27] Therefore, a general construction method for multi-qudit UPBs, especially for the system with any dimension, is needed, which is the first motivation of this paper.

As one of the research hotspots in quantum information theory, the quantum state discrimination problem is the basis of other quantum problems.[28–30] The distinguishable quantum states can be applied to some quantum information processing tasks, such as distributed quantum computing, while the indistinguishable quantum states are very common in design of quantum cryptography protocols. Bennett *et al.*[31] proposed the theorem that any set of orthogonal product states in $2\otimes N$ are distinguishable by LOCC. Zhang *et al.*[32] gave a general method to construct indistinguishable multipartite orthogonal product states in $C^{d_1}\otimes C^{d_2}\otimes\cdots\otimes C^{d_n}$ $\left(C^{d_{1,2,\cdots,n}}\geq 3, n\geq 4\right)$. In addition to the above work, Halder *et al.*[33] recently found that in some nonlocal tripartite

†Corresponding author. E-mail: dou@bupt.edu.cn

© 2022 Chinese Physical Society and IOP Publishing Ltd

systems, when two parties measure together, there is a possibility that the shared state can be distinguished. Hence, the concept of strong nonlocality has been proposed and widely discussed. Based on Halder's work, Zhang *et al.*[34] constructed two sets of quantum states with strong nonlocality, $C^3\otimes C^3\otimes C^3$ and $C^3\otimes C^3\otimes C^3\otimes C^3$. Shi *et al.*[35] provided the process of constructing a set consisting of entangled states in the $C^d\otimes C^d\otimes C^d$ system based on Rubik's cube. However, there is no relevant research on the construction of multi-qudit UPBs with strong nonlocality, which is the second motivation of this study.

Graph theory is an effective representation to help intuitively analyze the abstract structure of state sets.[36,37] It is also widely used in the construction of quantum state sets,[38–40] especially UPBs.[31,41–43] Johnston *et al.*[37] analyzed the structure of UPBs in $\left(C^2\right)^{\otimes p}$ and proposed the minimum size of this quantum system by using the orthogonal graph. Bennett *et al.*[31] first proposed a tripartite UPB and provided two classical construction methods based on the graphs, namely, pyramid structure and tile structure. Hence, our third motivation is to use graph theory to analyze and display the internal structure and relations of the strongly nonlocal states.

Therefore, based on the above three motivations, the main focus in this paper is to construct a general set of 3-qudit UPBs with strong nonlocality. First, based on the shift UPBs, a UPB set in $C^3\otimes C^3\otimes C^3$ is obtained. By observing the orthogonal graph of each qutrit and the corresponding $3\times3\times3$ Rubik's cube, the structure of the states set and the relationship between qudits are analyzed in Figs. 3 and 4. Then, following this construction method, we give a general 3-qudit UPB with strong nonlocality of size $(d-1)^3+2d+5$ for the $C^d\otimes C^d\otimes C^d$ system. The general expression of the states set is provided in detail. Second, after reviewing the connection between UPBs and tile structure, we extend the tile structure from 2-qubit to 3-qudit systems and propose the tritile structure in definition 5. Moreover, by applying proposition 1, we generalize the construction of $C^4\times C^4\times C^5$ and show a general approach to construct a strongly nonlocal UPB set $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$ based on a UPB set with low dimension, $C^{(d_1-1)}\otimes C^{(d_2-1)}\otimes C^{(d_3-1)}$. Our research provides a general form of UPBs with strong nonlocality, which can be further applied in many quantum protocols to fundamentally ensure its security.

The rest of this paper is organized as follows. In Section 2, we briefly introduce the notations and several preliminaries of UPBs. Sections 3 and 4 consist of the main contributions of the present work. Based on graph theory, the general construction method of 3-qudit UPBs with the same dimensions and different dimensions are proposed. Finally, we summarize the results and discuss some open problems in Section 5.

2. Notations and preliminaries

In this section, we briefly introduce the preliminary knowledge and some notations. A multi-qudit pure state $|v\rangle\in C^{d_1}\otimes\cdots\otimes C^{d_p}$ is considered to be separable if and only if it can be written in the form

$$|v\rangle=|v_1\rangle\otimes\cdots\otimes|v_p\rangle.$$

The standard bases of C^d is $\{|0\rangle,\ |1\rangle,\ \ldots,\ |d-1\rangle\}$. Note that throughout this paper, the states and operators are not normalized for simplicity, and only pure states and positive operator valued measurement (POVM) are considered.

Definition 1: Unextendible product bases[44] Consider a p-partite quantum system $\mathcal{H}=\otimes_{i=1}^{p}\mathcal{H}_i$. The orthogonal product bases (PBs) are a set S of pure orthogonal product states spanning a subspace $\mathcal{H}_S$ of $\mathcal{H}$. The uncompletable product bases (UCPBs) are PBs whose complementary subspace $\mathcal{H}_S^{\perp}$ contains fewer mutually orthogonal product states than its dimension. The unextendible product bases (UPBs) are uncompletable product bases that do not contain any product state in complementary subspace $\mathcal{H}_S^{\perp}$.

UPBs are nonlocal and cannot be perfectly distinguished by LOCC. However, when discussing the nonlocality of a multi-qudit system, it is found that there is a certain probability that states can be distinguished when several qudits are joined. Based on this phenomenon, the definition of strong nonlocality is given.

Definition 2: Locally irreducibility[40] A set of orthogonal product states on $\mathcal{H}=\otimes_{i=1}^{n}\mathcal{H}_i$, $n\geq2$, dim $\mathcal{H}_i\geq2$ is locally irreducible if it is impossible to eliminate one or more states from the set by orthogonality-preserving local measurements.

Definition 3: I-irreducible system In a multiparty system $C^{d_1}\otimes C^{d_2}\otimes\cdots\otimes C^{d_n}$ $(d_{1,2,\ \ldots,\ n}\geq3,\ n\geq4)$, if a set of orthogonal product states is arbitrarily divided into i parts and the entire system is still locally irreducible in every new i parts, then the system can be called being i-irreducible, $i=2,3,\ \ldots,\ n-1$.

Definition 4: Strongly nonlocal system In $C^{d_1}\otimes C^{d_2}\otimes\cdots\otimes C^{d_n}$ $(d_{1,2,\ \ldots,\ n}\geq3,\ n\geq4)$, if a set of orthogonal product states is $(n-1)$-irreducible, $(n-2)$-irreducible, $\ldots$, and 2-irreducible simultaneously, we can say that this system is strongly nonlocal.

For the measurement of nonlocal state sets, multiple rounds of measurement are required for multiple participants. Therefore, the nontrivial orthogonality-preserving measurement shall be required to obtain useful information without affecting the characteristics of the state set, which is defined in Definition 4.

Definition 5: Nontrivial orthogonality-preserving measurement[45] If a set of mutually orthogonal quantum states remain mutually orthogonal after measurement, the measurement used to distinguish the quantum states is defined as orthonormal preserving measurement (OPM). Furthermore, such a measurement is called being nontrivial if all the measurement matrices constituting the OPM are not proportional to the identity operator, otherwise, it is trivial.

Tile structure is one of the classical structures used to construct UPB. The $C^m \otimes C^n$ can correspond to an $m \times n$ rectangle Γ, which is paved by disjoint tiles $\{t_i\}$, denoted by $\Gamma = \bigcup_i t_i$. A tile t_i should be a rectangle that can be separated. In particular, we show how to construct a 2-qubit UPB in Fig. 1.

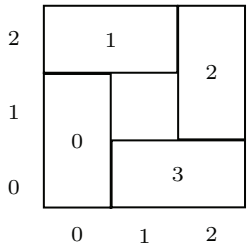


Fig. 1. Tile structure.

Example 1: The set Ψ represents UPBs in the $C^3 \otimes C^3$ system denoting the tile structure.

From Fig. 1, we obtain a set of complete orthogonal product bases as Eq. (1), which are denoted as Φ:

$$\begin{aligned}
&\left|\psi_0^{(0)}\right\rangle = |0\rangle|0+1\rangle, \quad \left|\psi_0^{(1)}\right\rangle = |0\rangle|0-1\rangle,\\
&\left|\psi_1^{(0)}\right\rangle = |0+1\rangle|2\rangle, \quad \left|\psi_1^{(1)}\right\rangle = |0-1\rangle|2\rangle,\\
&\left|\psi_2^{(0)}\right\rangle = |2\rangle|1+2\rangle, \quad \left|\psi_2^{(1)}\right\rangle = |2\rangle|1-2\rangle,\\
&\left|\psi_3^{(0)}\right\rangle = |1+2\rangle|0\rangle, \quad \left|\psi_3^{(1)}\right\rangle = |1-2\rangle|0\rangle. \qquad (1)
\end{aligned}$$

Let

$$|S\rangle = |0+1+2\rangle|0+1+2\rangle$$

be a stopper state to force the unextendibility.

We claim that the set Ψ is a UPB set in the $C^3 \otimes C^3$ system.

$$\Psi = \Phi \cup \{|S\rangle\} \setminus \left\{\left|\psi_i^{(0)}\right\rangle\right\}_{i=0}^{3}.$$

The shift UPBs in $C^2 \otimes C^2 \otimes C^2$ were proposed by Bennett in 1999, which provide one of the oldest examples of nontrivial UPBs and consist of the following four states:

$$\begin{aligned}
|\psi_0\rangle &= |0\rangle|1\rangle|0-1\rangle,\\
|\psi_1\rangle &= |1\rangle|0-1\rangle|0\rangle,\\
|\psi_2\rangle &= |0-1\rangle|0\rangle|1\rangle,\\
|\psi_3\rangle &= |0+1\rangle|0+1\rangle|0+1\rangle. \qquad (2)
\end{aligned}$$

These UPBs can be simply generalized to a UPB set over any number of parties, each with a one-qubit Hilbert space (see Ref. [19]). In this paper, the construction of 3-qudit UPBs is also based on the structure of the shift UPBs.

3. Tripartite system with same dimensions

In Ref. [31], it is proved that members of a UPB are not perfectly distinguishable by LOCC, that is, UPBs is always nonlocal. Then, we set out to investigate what kind of structure of multi-qudit UPBs is strongly nonlocal.

In Ref. [43], it was shown that starting from a two-qubit unextendible entangled basis, it is possible to construct a three-qutrit unextendible entangled basis. Therefore, in this section, we give a construction method of the strongly nonlocal UPBs in $C^3 \otimes C^3 \otimes C^3$ based on shift UPBs in Lemma 1. Similarly, we propose the set of UPBs in $C^4 \otimes C^4 \otimes C^4$ in Lemma 2. Furthermore, we generalize these two constructions to $C^d \otimes C^d \otimes C^d$ for any $d \geq 3$ in Theorem 1.

3.1. Construct a UPB in $C^3 \otimes C^3 \otimes C^3$ system based on shift UPB

Shift UPBs is the most classical 3-qubit UPBs available. When its structure is represented by Rubik's cube, it can be found that any subset of three vectors on either side spans the three-dimensional space of that party,[31] preventing any new vector from being orthogonal to all the existing ones. Following this idea, a $C^3 \otimes C^3 \otimes C^3$ state set is obtained as

$$\begin{aligned}
P_3^1: &\begin{cases} |\psi_0\rangle = |0\rangle_A|1\rangle_B|0-1\rangle_C, \\ |\psi_1\rangle = |1\rangle_A|0-1\rangle_B|0\rangle_C, \\ |\psi_2\rangle = |0-1\rangle_A|0\rangle_B|1\rangle_C, \end{cases}\\
P_3^2: &\begin{cases} |\psi_3\rangle = |2\rangle_A|1\rangle_B|1-2\rangle_C, \\ |\psi_4\rangle = |1\rangle_A|1-2\rangle_B|2\rangle_C, \\ |\psi_5\rangle = |1-2\rangle_A|2\rangle_B|1\rangle_C, \end{cases}\\
P_3^3: &\begin{cases} |\psi_6\rangle = |2\rangle_A|0\rangle_B|1-2\rangle_C, \\ |\psi_7\rangle = |1-2\rangle_A|2\rangle_B|0\rangle_C, \\ |\psi_8\rangle = |0\rangle_A|1-2\rangle_B|2\rangle_C, \\ |\psi_9\rangle = |0\rangle_A|2\rangle_B|0-1\rangle_C, \\ |\psi_{10}\rangle = |0-1\rangle_A|0\rangle_B|2\rangle_C, \\ |\psi_{11}\rangle = |2\rangle_A|0-1\rangle_B|0\rangle_C. \end{cases}\\
&\text{Stopper}: |\psi_{12}\rangle = |0+1+2\rangle_A|0+1+2\rangle_B|0+1+2\rangle_C. \qquad (3)
\end{aligned}$$

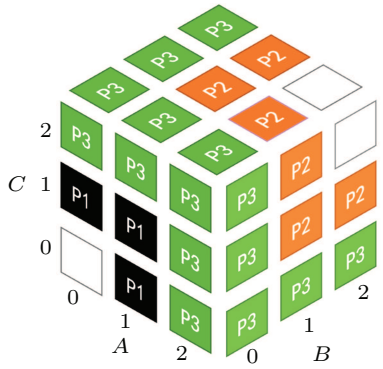


Fig. 2. Rubik's cube corresponding to $C^3 \otimes C^3 \otimes C^3$.

Here, P_3^1 represents the first part of the $C^3\otimes C^3\otimes C^3$ system, and $|\psi_{12}\rangle$ is considered as stopper states since it stops inclusion of any other states from forming complete orthogonal product bases (COPB). Geometrically, by mapping Eq. (3) to Rubik's cube, the structure of UPBs can be displayed more intuitively and clearly, as shown in Fig. 2. The stopper state is not depicted.

The composition of UPBs can be roughly divided into three parts:

P_3^1: The set of UPBs with dimension 2, whose structure is similar to the shift UPBs. It consists of three subcuboids $\{SP^1, SP^2, SP^3\}$ and corresponds to $\{(0)_A,(1)_B,(0,\ 1)_C\}$, $\{(1)_A,(0,\ 1)_B,(0)_C\}$, $\{(0,\ 1)_A,(0)_B,(1)_C\}$, respectively, as shown in Fig. 3(a).

P_3^2: The set of UPBs with dimension 2, whose structure is the same as the shift UPBs. It includes three subcuboids $\{SP^4, SP^5, SP^6\}$, and corresponds to $\{(2)_A,(1)_B,(1,\ 2)_C\}$, $\{(1)_A,(1,\ 2)_B,(2)_C\}$, $\{(1,\ 2)_A,(2)_B,(1)_C\}$, respectively, as shown in Fig. 3(b).

P_3^3: Six edges of the $3\times3\times3$ Rubik's cube, which consists of 6 subcuboids $\{SP^i\}_{i=7}^{12}$, and they correspond to $\{(0,\ 1)_A,\ (0)_B,\ (2)_C\}$, $\{(2)_A,\ (0)_B,\ (1,\ 2)_C\}$, $\{(2)_A,\ (0,\ 1)_B,\ (0)_C\}$, $\{(1,\ 2)_A,\ (2)_B,\ (0)_C\}$, $\{(0)_A,\ (2)_B,\ (0,\ 1)_C\}$, $\{(0)_A,\ (1,\ 2)_B,\ (2)_C\}$, as shown in Fig. 3(c).

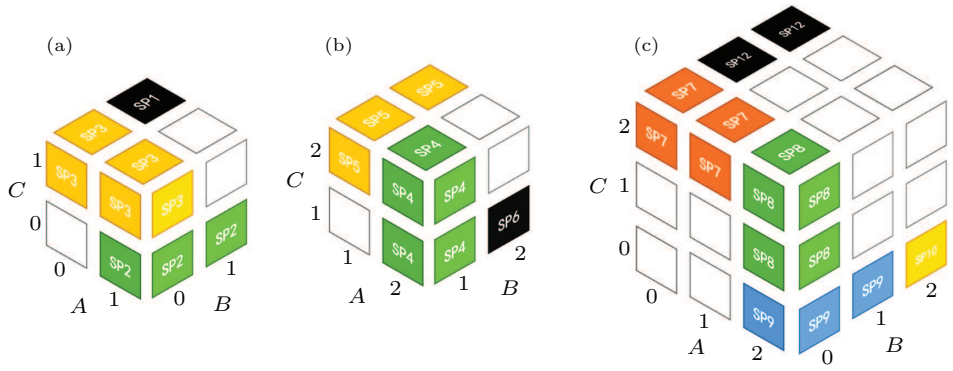


Fig. 3 Composition of UPBs in $C^3\otimes C^3\otimes C^3$.

By observing Rubik's cube, some of the cuboids need to be merged to ensure the principle (see definition 1 in Ref. [42]) that the combination of any two cuboids cannot become a new cuboid. The new merged states corresponding to the cuboids are shown in Eq. (4). In this set, we find that any two of these vectors are not in a two-dimensional plane.

$$
P_3^{1,3}:\begin{cases}
|\psi_{0,1,2}\rangle = |0-w_2^i1\rangle_A|0\rangle_B\left|1-w_2^j2\right\rangle_C,\\
|\psi_{3,4,5}\rangle = |0\rangle_A\left|1-w_2^j2\right\rangle_B|0-w_2^i1\rangle_C,\\
|\psi_{6,7,8}\rangle = \left|1-w_2^j2\right\rangle_A|0-w_2^i1\rangle_B|0\rangle_C,
\end{cases}
$$

$$
P_3^{2,3}:\begin{cases}
|\psi_{9,10,11}\rangle = \left|1-w_2^j2\right\rangle_A|2\rangle_B|0-w_2^i1\rangle_C,\\
|\psi_{12,13,14}\rangle = |2\rangle_A|0-w_2^i1\rangle_B\left|1-w_2^j2\right\rangle_C,\\
|\psi_{15,16,17}\rangle = |0-w_2^i1\rangle_A\left|1-w_2^j2\right\rangle_B|2\rangle_C.
\end{cases}
$$

$$
\text{Stopper}: |\psi_{18}\rangle = |0+1+2\rangle_A|0+1+2\rangle_B|0+1+2\rangle_C. \quad (4)
$$

Here, $w_2=\mathrm{e}^{\frac{2\pi\sqrt{-1}}{2}}$, $i,\ j=0,\ 1,\ ij=0$. Taking into account two qutrits in UPB, if they have the same orthogonal graph after permuting the qutrits and relabeling the vertices, the two qutrits are considered to be equivalent. The original orthogonal graph of each qutrit is drawn in Fig. 4(a) according to Eq. (4). The vertices V_0–V_{17} in the graph correspond to $|\psi_0\rangle$ to $|\psi_{17}\rangle$ in the state set, and the lines between vertices represent the orthogonal relationship between the two states. It can be seen that every edge between any two vertices has appeared in at least one graph, indicating that all states in the set are mutually orthogonal. Figure 4(b) can be obtained after rearranging the vertices and corresponding lines. We find that the three qutrits A, B, and C are equivalent because the orthogonal graph structure of each party is the same. In other words, by permuting each basis of C^3 used in the construction, the original UPBs can be still obtained.

Therefore, only the properties of one qutrit need to be discussed, and then the properties of three qutrit can be inferred from equivalence. From the composition of the basis, it is obvious that these states have a cyclic property as the cyclic property of the trace. That is to say, the state set has the same properties in the different divisions of $A|B|C$, $B|C|A$ and $C|A|B$.

Lemma 1 In $C^3\otimes C^3\otimes C^3$, the 3-qutrit UPBs of size 19 given by Eq. (4) is strongly nonlocal.

Proof The proof can be summarized into two steps. The first step is to prove that this 3-qutrit system is nonlocal. This step can be omitted in the UPB set since UPBs are incomplete and nonlocal.

The second step is to prove that the whole system is still nonlocal with any partitioning method for three qutrits.

Physically, this method of division means that the subsystems B (Bob) and C (Charlie) are treated together as a 9-dimensional subsystem BC on which joint measurements are now allowed. On account of the original UPB set is nonlocal, so the system will be still nonlocal when Charlie goes first.

Then, we only need to discuss the situation where the *BC* system goes first. In order to make the proof process clearer, we first rewrite the original state, let $|00\rangle \to |0\rangle$, $|01\rangle \to |1\rangle$, …, $|22\rangle \to |8\rangle$. After rewriting the state in Eq. (4), we can get the following states:

$$P_3^{1,3}: \begin{cases} |\psi_{0,1,2}\rangle = |0-w_2^i 1\rangle_{\mathrm{A}} \left|1-w_2^j 2\right\rangle_{\mathrm{BC}}, \\ |\psi_{3,4,5}\rangle = |0\rangle_{\mathrm{A}} \left|3-w_2^i 4-w_2^j 6+w_2^{i+j} 7\right\rangle_{\mathrm{BC}}, \\ |\psi_{6,7,8}\rangle = \left|1-w_2^j 2\right\rangle_{\mathrm{A}} |0-w_2^i 3\rangle_{\mathrm{BC}}, \end{cases}$$

$$P_3^{2,3}: \begin{cases} |\psi_{9,10,11}\rangle = \left|1-w_2^j 2\right\rangle_{\mathrm{A}} |6-w_2^i 7\rangle_{\mathrm{BC}}, \\ |\psi_{12,13,14}\rangle = |2\rangle_{\mathrm{A}} \left|1-w_2^i 2-w_2^j 4+w_2^{i+j} 5\right\rangle_{\mathrm{BC}}, \\ |\psi_{15,16,17}\rangle = |0-w_2^i 1\rangle_{\mathrm{A}} \left|5-w_2^j 8\right\rangle_{\mathrm{BC}}. \end{cases}$$

$$\text{Stopper}: |\psi_{18}\rangle = |0+1+2\rangle_{\mathrm{A}} |0+1+\cdots+8\rangle_{\mathrm{BC}}. \tag{5}$$

The 18 states in $A|BC$ bipartition correspond to 6 blocks of the 3×9 grid in Fig. 5. It can be seen that this whole graph is centrosymmetric.

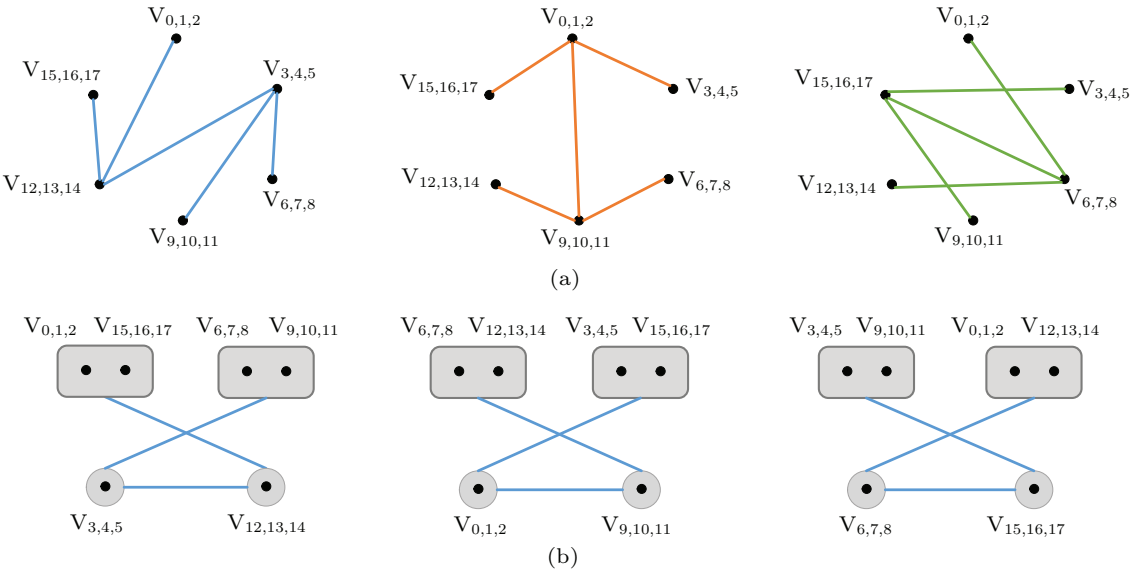


Fig. 4. Orthogonal graph of three qutrits.

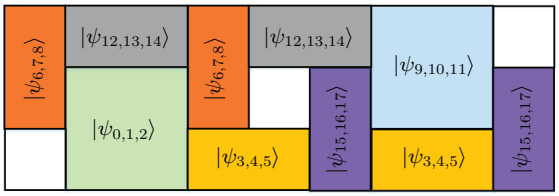


Fig. 5. The corresponding 3×9 grid of Eq. (5).

Suppose that the *BC* system starts with the nontrivial and non-disturbing measurement, represented by a set of POVM elements $M_m^\dagger M_m$ on 9×9 matrix. The POVM measurement in $\{|0\rangle, |1\rangle, \ldots, |8\rangle\}_{\mathrm{BC}}$ basis can be written out, corresponding to the states in Eq. (5).

Therefore, the original matrix can be reduced to

$$M_m^\dagger M_m = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{07} & a_{08} \\ a_{10} & a_{11} & \cdots & a_{17} & a_{18} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{70} & a_{71} & \cdots & a_{77} & a_{78} \\ a_{80} & a_{81} & \cdots & a_{87} & a_{88} \end{bmatrix}.$$

The post-measurement states could be expressed as $(I\otimes M_m)|\psi_i\rangle$, which should be mutually orthogonal. Then $\langle\psi_j|\left(I\otimes M_m^\dagger M_m\right)|\psi_i\rangle = 0$ is obtained. According to this principle, the original matrix could be transformed to

$$M_m^\dagger M_m = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix}.$$

Table 1 shows the detailed derivation process.

Obviously, *BC*'s measurement matrix is proportional to the identity matrix, so it means that the *BC* system starts with a trivial measurement, and cannot get any information about shared state distinction from the measurement result. As for the other two division methods, $B|CA$ and $C|AB$, the proof method is similar to this. In summary, the three qutrits divided into any two parts can keep the strong nonlocality of UPBs.

3.2. Construct a UPB in $C^4\otimes C^4\otimes C^4$ system based on a UPB in $C^3\otimes C^3\otimes C^3$ system

Based on the construction method of state set in Eq. (4), we extend UPBs to the case of dimension equal to 4. Compared with Rubik's cube of $3\times 3\times 3$, the cube of $4\times 4\times 4$ has three new planes, $\{(0,\ldots,3)_{\mathrm{A}}, (0,\ldots,3)_{\mathrm{B}}, 3_{\mathrm{C}}\}$, $\{(0,\ldots,3)_{\mathrm{A}}, 3_{\mathrm{B}}, (0,\ldots,2)_{\mathrm{C}}\}$, $\{3_{\mathrm{A}}, (0,\ldots,2)_{\mathrm{B}}, (0,\ldots,2)_{\mathrm{C}}\}$. According to tile structure, we obtain Fig. 6.

Table 1. POVM elements.

POVM elements	Corresponding states				
$a_{0i}=a_{i0}=0$	$i=1$ $\|\psi_6\rangle,\|\psi_0\rangle$	$i=2$ $\|\psi_6\rangle,\|\psi_1\rangle$	$i=3$ $\|\psi_3\rangle,\|\psi_{18}\rangle$	$i=4$ $\|\psi_6\rangle,\|\psi_{13}\rangle$	$i=5$ $\|\psi_6\rangle,\|\psi_{15}\rangle$
	$i=6$ $\|\psi_6\rangle,\|\psi_9\rangle$	$i=7$ $\|\psi_6\rangle,\|\psi_{10}\rangle$	$i=8$ $\|\psi_6\rangle,\|\psi_{16}\rangle$		
$a_{1i}=a_{i1}=0$	$i=2$ $\|\psi_{12,13}\rangle,\|\psi_{18,14}\rangle$	$i=3$ $\|\psi_1\rangle,\|\psi_7\rangle$	$i=4$ $\|\psi_1\rangle,\|\psi_4\rangle$	$i=5$ $\|\psi_1\rangle,\|\psi_{15}\rangle$	$i=6$ $\|\psi_1\rangle,\|\psi_9\rangle$
	$i=7$ $\|\psi_1\rangle,\|\psi_{10}\rangle$	$i=8$ $\|\psi_1\rangle,\|\psi_{16}\rangle$			
$a_{2i}=a_{i2}=0$	$i=3$ $\|\psi_2\rangle,\|\psi_8\rangle$	$i=4$ $\|\psi_2\rangle,\|\psi_4\rangle$	$i=5$ $\|\psi_2\rangle,\|\psi_{16}\rangle$	$i=6$ $\|\psi_2\rangle,\|\psi_{10}\rangle$	$i=7$ $\|\psi_2\rangle,\|\psi_{11}\rangle$
	$i=8$ $\|\psi_2\rangle,\|\psi_{17}\rangle$				
$a_{3i}=a_{i3}=0$	$i=4$ $\|\psi_7\rangle,\|\psi_{13}\rangle$	$i=5$ $\|\psi_7\rangle,\|\psi_{16}\rangle$	$i=6$ $\|\psi_7\rangle,\|\psi_{10}\rangle$	$i=7$ $\|\psi_7\rangle,\|\psi_{11}\rangle$	$i=8$ $\|\psi_7\rangle,\|\psi_{17}\rangle$
$a_{4i}=a_{i4}=0$	$i=5$ $\|\psi_4\rangle,\|\psi_{15}\rangle$	$i=6$ $\|\psi_{13}\rangle,\|\psi_{19}\rangle$	$i=7$ $\|\psi_{13}\rangle,\|\psi_{10}\rangle$	$i=8$ $\|\psi_4\rangle,\|\psi_{17}\rangle$	
$a_{5i}=a_{i5}=0$	$i=6$ $\|\psi_{15}\rangle,\|\psi_9\rangle$	$i=7$ $\|\psi_{15}\rangle,\|\psi_{10}\rangle$	$i=8$ $\|\psi_{12}\rangle,\|\psi_{18}\rangle$		
$a_{6i}=a_{i6}=0$	$i=7$ $\|\psi_{3,4}\rangle,\|\psi_{18,5}\rangle$	$i=8$ $\|\psi_9\rangle,\|\psi_{16}\rangle$			
$a_{7i}=a_{i7}=0$	$i=8$ $\|\psi_{10}\rangle,\|\psi_{17}\rangle$				
$a_{00}=a_{ii}$	$i=1$ $\|\psi_0\rangle,\|\psi_1\rangle$	$i=2$ $\|\psi_0\rangle,\|\psi_{18}\rangle$	$i=3$ $\|\psi_6\rangle,\|\psi_{18}\rangle$	$i=4$ $\|\psi_{12}\rangle,\|\psi_{18}\rangle$	$i=5$ $\|\psi_{15}\rangle,\|\psi_{16}\rangle$
	$i=6$ $\|\psi_9\rangle,\|\psi_{10}\rangle$	$i=7$ $\|\psi_9\rangle,\|\psi_{18}\rangle$	$i=8$ $\|\psi_{15}\rangle,\|\psi_{18}\rangle$		

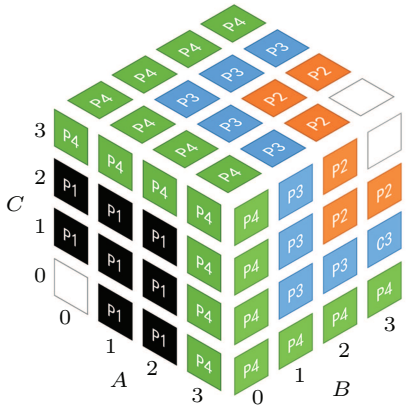


Fig. 6. Composition of UPBs in $C^4\otimes C^4\otimes C^4$.

The UPBs with dimension 4 contains the following four parts:

P_4^1: The set of UPBs with dimension 3 correspond to $\{(0,\,1,\,2)_{\rm A},\,(0,\,1,\,2)_{\rm B},\,(0,\,1,\,2)_{\rm C}\}$ of Rubik's cube. It includes 6 subcuboids $\{SP^i\}_{i=1}^{6}$, as shown in Fig. 7(a).

P_4^2: The set of UPBs with dimension 2, whose structure is the same as the shift UPBs. It includes three subcuboids $\{SP^7,\,SP^8,\,SP^9\}$, and corresponds to $\{(3)_{\rm A},(2)_{\rm B},(2,\,3)_{\rm C}\}$, $\{(2)_{\rm A},(2,\,3)_{\rm B},(3)_{\rm C}\}$, $\{(2,\,3)_{\rm A},(3)_{\rm B},(2)_{\rm C}\}$, respectively, as shown in Fig. 7(b).

P_4^3: Six edges of the $4\times4\times4$ Rubik's cube, which consists of 6 subcuboids $\{SP^i\}_{i=10}^{15}$, and they correspond to $\{(0,\,1,\,2)_{\rm A},\,(0)_{\rm B},\,(3)_{\rm C}\}$, $\{(3)_{\rm A},\,(0)_{\rm B},\,(1,\,2,\,3)_{\rm C}\}$, $\{(3)_{\rm A},\,(0,\,1,\,2)_{\rm B}\,,\,(0)_{\rm C}\,\}$, $\{(1,\,2,\,3)_{\rm A},\,(3)_{\rm B},\,(0)_{\rm C}\}$, $\{(0)_{\rm A},\,(3)_{\rm B},\,(0,\,1,\,2)_{\rm C}\}$, $\{(0)_{\rm A},\,(1,\,2,\,3)_{\rm B},\,(3)_{\rm C}\}$, as shown in Fig. 7(c).

P_4^4: The remaining part of the three newly added planes. It can be decomposed into six subcuboids that cannot be further expanded, as shown in Fig. 7(d).

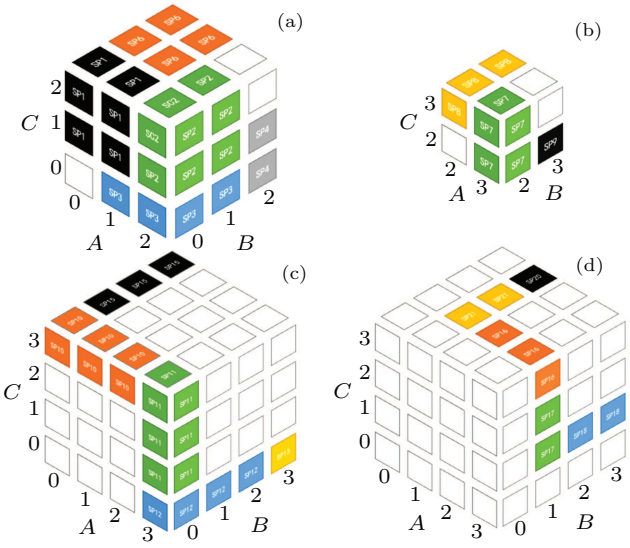


Fig. 7. Composition of UPBs in $C^4\otimes C^4\otimes C^4$.

By matching Rubik's cube with the states set, we get Eq. (6). Compared with $C^3\otimes C^3\otimes C^3$, there is one more part,

P_4^4 in the composition of Eq. (6).

$$
P_4^1:\begin{cases}
|\psi_{0,1,2}\rangle = |0-w_2^i 1\rangle_{\rm A}|0\rangle_{\rm B}|1-w_2^j 2\rangle_{\rm C},\\
|\psi_{3,4,5}\rangle = |0\rangle_{\rm A}|1-w_2^j 2\rangle_{\rm B}|0-w_2^i 1\rangle_{\rm C},\\
|\psi_{6,7,8}\rangle = |1-w_2^j 2\rangle_{\rm A}|0-w_2^i 1\rangle_{\rm B}|0\rangle_{\rm C},\\
|\psi_{9,10,11}\rangle = |1-w_2^j 2\rangle_{\rm A}|2\rangle_{\rm B}|0-w_2^i 1\rangle_{\rm C},\\
|\psi_{12,13,14}\rangle = |2\rangle_{\rm A}|0-w_2^i 1\rangle_{\rm B}|1-w_2^j 2\rangle_{\rm C},\\
|\psi_{15,16,17}\rangle = |0-w_2^i 1\rangle_{\rm A}|1-w_2^j 2\rangle_{\rm B}|2\rangle_{\rm C},
\end{cases}
$$

$$
P_4^2:\begin{cases}
|\psi_{18}\rangle = |3\rangle_{\rm A}|2\rangle_{\rm B}|2-3\rangle_{\rm C},\\
|\psi_{19}\rangle = |2\rangle_{\rm A}|2-3\rangle_{\rm B}|3\rangle_{\rm C},\\
|\psi_{20}\rangle = |2-3\rangle_{\rm A}|3\rangle_{\rm B}|2\rangle_{\rm C},
\end{cases}
$$

$$
P_4^3:\begin{cases}
|\psi_{21,22}\rangle = |0\rangle_{\rm A}|3\rangle_{\rm B}|0+w_3 1+w_3^2 2\rangle_{\rm C},\\
|\psi_{23,24}\rangle = |0+w_3 1+w_3^2 2\rangle_{\rm A}|0\rangle_{\rm B}|3\rangle_{\rm C},\\
|\psi_{25,26}\rangle = |3\rangle_{\rm A}|0+w_3 1+w_3^2 2\rangle_{\rm B}|0\rangle_{\rm C},\\
|\psi_{27,28}\rangle = |3\rangle_{\rm A}|0\rangle_{\rm B}|1+w_3 2+w_3^2 3\rangle_{\rm C},\\
|\psi_{29,30}\rangle = |1+w_3 2+w_3^2 3\rangle_{\rm A}|3\rangle_{\rm B}|0\rangle_{\rm C},\\
|\psi_{31,32}\rangle = |0\rangle_{\rm A}|1+w_3 2+w_3^2 3\rangle_{\rm B}|3\rangle_{\rm C},
\end{cases}
$$

$$
P_4^4:\begin{cases}
|\psi_{33}\rangle = |1-2\rangle_{\rm A}|3\rangle_{\rm B}|1\rangle_{\rm C},\\
|\psi_{34}\rangle = |1\rangle_{\rm A}|1-2\rangle_{\rm B}|3\rangle_{\rm C},\\
|\psi_{35}\rangle = |3\rangle_{\rm A}|1\rangle_{\rm B}|1-2\rangle_{\rm C},\\
|\psi_{36}\rangle = |2-3\rangle_{\rm A}|1\rangle_{\rm B}|3\rangle_{\rm C},\\
|\psi_{37}\rangle = |3\rangle_{\rm A}|2-3\rangle_{\rm B}|1\rangle_{\rm C},\\
|\psi_{38}\rangle = |1\rangle_{\rm A}|3\rangle_{\rm B}|2-3\rangle_{\rm C}.
\end{cases}
$$

$$
\text{Stopper}: |\psi_{39}\rangle = |0+1+2+3\rangle_{\rm A}|0+1+2+3\rangle_{\rm B}|0+1+2+3\rangle_{\rm C}. \tag{6a}
$$

Here $w_3 = \mathrm{e}^{\frac{2\pi\sqrt{-1}}{3}}$, $i,\ j = 0,\ 1,\ ij = 0$.

Lemma 2 In $C^4\otimes C^4\otimes C^4$, the 3-qudit UPBs of size 40 given by Eq. (6) is strongly nonlocal.

The proof of Lemma 2 is given in Appendix A.

3.3. Construct a UPB in $C^d\otimes C^d\otimes C^d$ system based on a UPB in $C^{d-1}\otimes C^{d-1}\otimes C^{d-1}$ system

After analyzing the structure of UPBs proposed in Lemma 1 and Lemma 2, we give a UPB in $C^d\otimes C^d\otimes C^d$ of size $(d-1)^3+2d+5$ based on the set in $C^{d-1}\otimes C^{d-1}\otimes C^{d-1}$, and then obtain Proposition 1.

The general 3-qudit UPB set is always built on the basis of lower dimensional state set, like an onion. First, the three planes, $\{(0,\ldots,d-1)_{\rm A},\ (0,\ldots,d-1)_{\rm B},\ (d-1)_{\rm C}\}$, $\{(0,\ldots,d-1)_{\rm A},\ (d-1)_{\rm B},\ (0,\ldots,d-2)_{\rm C}\}$, $\{(d-1)_{\rm A},\ (0,\ldots,d-2)_{\rm B},\ (0,\ldots,d-2)_{\rm C}\}$ are divided into different nonlocal cuboids. Then, the remaining part is a tripartite system with dimension $d-1$. Again, we divide the three planes, $\{(0,\ldots,d-2)_{\rm A},\ (0,\ldots,d-2)_{\rm B},\ (d-2)_{\rm C}\}$, $\{(0,\ldots,d-2)_{\rm A},\ (d-2)_{\rm B},\ (0,\ldots,d-3)_{\rm C}\}$, $\{(d-2)_{\rm A},\ (0,\ldots,d-3)_{\rm B},\ (0,\ldots,d-3)_{\rm C}\}$ in the same way. Continually, until a 3-qubit UPB with $d=2$ is left, it will be divided according to the shift UPBs.

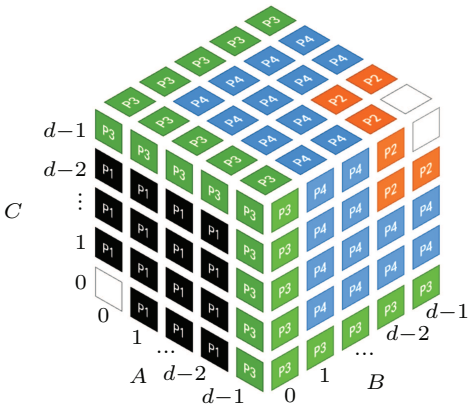


Fig. 8. Composition of UPBs in $C^d\otimes C^d\otimes C^d$.

Proposition 1: In $C^d\otimes C^d\otimes C^d$, the UPBs of size $(d-1)^3+2d+5$ given by Eq. (7) is strongly nonlocal.

$$
\begin{aligned}
&P_d^1:\ P_{d-1}^1,\\
&P_d^2:\ X_d^{\otimes 3}\left(P_{d-1}^2\right),\\
&P_d^3:\begin{cases}
\left|\psi_{(d-2)^3+2d+3+(d-2)}\right\rangle = |0+\cdots+w_{d-1}^{d-2}(d-2)\rangle_{\rm A}|0\rangle_{\rm B}|d-1\rangle_{\rm C},\\
\left|\psi_{(d-2)^3+2d+3+2(d-2)}\right\rangle = |d-1\rangle_{\rm A}|0+\cdots+w_{d-1}^{d-2}(d-2)\rangle_{\rm B}|0\rangle_{\rm C},\\
\left|\psi_{(d-2)^3+2d+3+3(d-2)}\right\rangle = |0\rangle_{\rm A}|d-1\rangle_{\rm B}|0+\cdots+w_{d-1}^{d-2}(d-2)\rangle_{\rm C},\\
\left|\psi_{(d-2)^3+2d+3+4(d-2)}\right\rangle = |1+\cdots+w_{d-1}^{d-2}(d-1)\rangle_{\rm A}|d-1\rangle_{\rm B}|0\rangle_{\rm C},\\
\left|\psi_{(d-2)^3+2d+3+5(d-2)}\right\rangle = |0\rangle_{\rm A}|1+\cdots+w_{d-1}^{d-2}(d-1)\rangle_{\rm B}|d-1\rangle_{\rm C},\\
\left|\psi_{(d-2)^3+2d+3+6(d-2)}\right\rangle = |d-1\rangle_{\rm A}|0\rangle_{\rm B}|1+\cdots+w_{d-1}^{d-2}(d-1)\rangle_{\rm C},
\end{cases}\\
&P_d^4:\ X_{d-1}^{\otimes 3}\left(P_{d-1}^3\right)\cup X_{d-1}^{\otimes 3}\left(P_{d-1}^4\right).\\
&\text{Stopper}:\left|\psi_{(d-1)^3+2d+5}\right\rangle = |0+1+\cdots+(d-1)\rangle_{\rm A}|0+1+\cdots+(d-1)\rangle_{\rm B}|0+1+\cdots+(d-1)\rangle_{\rm C}.
\end{aligned}
\tag{7}
$$

Here $w_{d-1} = \mathrm{e}^{\frac{2\pi\sqrt{-1}}{d-1}}$, P_d^k represents the k-th part of the $C^d \otimes C^d \otimes C^d$ system. $X_d^{\otimes 3}\left(P_{d-1}^2\right)$ means performing $X_d \otimes X_d \otimes X_d$ operations on the quantum states in P_{d-1}^2 part. Taking $|3\rangle_{\mathrm{A}}|2\rangle_{\mathrm{B}}|2-3\rangle_{\mathrm{C}}$ as an example. After performing $X_5^{\otimes 3}$, we will obtain $|4\rangle_{\mathrm{A}}|3\rangle_{\mathrm{B}}|3-4\rangle_{\mathrm{C}}$.

The UPBs of $C^d \otimes C^d \otimes C^d$ can be roughly divided into the following four parts:

P_d^1: The set of UPBs with dimension $d-1$ corresponding to $\{(0,\ldots,d-2)_{\mathrm{A}}, (0,\ldots,d-2)_{\mathrm{B}}, (0,\ldots,d-2)_{\mathrm{C}}\}$ of Rubik's cube. It includes $3(d-2)^2-6$ subcuboids, as shown in Fig. 9(a).

P_d^2: The set of UPBs with dimension 2, whose structure is the same as the shift UPBs. It contains three subcuboids and corresponds to $\{(d-2,\ d-1)_{\mathrm{A}}, (d-2,\ d-1)_{\mathrm{B}}, (d-2,\ d-1)_{\mathrm{C}}\}$ respectively, as shown in Fig. 9(b).

P_d^3: Six edges of the $d\times d\times d$ Rubik's cube, i.e. $\{(0,\ldots,d-2)_{\mathrm{A}}, (0)_{\mathrm{B}}, (d-1)_{\mathrm{C}}\}$, $\{(d-1)_{\mathrm{A}}, (0)_{\mathrm{B}}, (1,\ldots,d-1)_{\mathrm{C}}\}$, $\{(d-1)_{\mathrm{A}}, (0,\ldots,d-2)_{\mathrm{B}}, (0)_{\mathrm{C}}\}$, $\{(1,\ldots,d-1)_{\mathrm{A}}, (d-1)_{\mathrm{B}}, (0)_{\mathrm{C}}\}$, $\{(0)_{\mathrm{A}}, (d-1)_{\mathrm{B}}, (0,\ldots,d-2)_{\mathrm{C}}\}$, $\{(0)_{\mathrm{A}}, (1,\ldots,d-1)_{\mathrm{B}}, (d-1)_{\mathrm{C}}\}$, as shown in Fig. 9(c).

P_d^4: The remaining part of the three newly added planes. It can be decomposed into $6(d-3)$ subcuboids that cannot be further expanded, which is equivalent to shifting all the previous P3 and P4 part of $C^{d-1}\otimes C^{d-1}\otimes C^{d-1}$, as shown in Fig. 9(d).

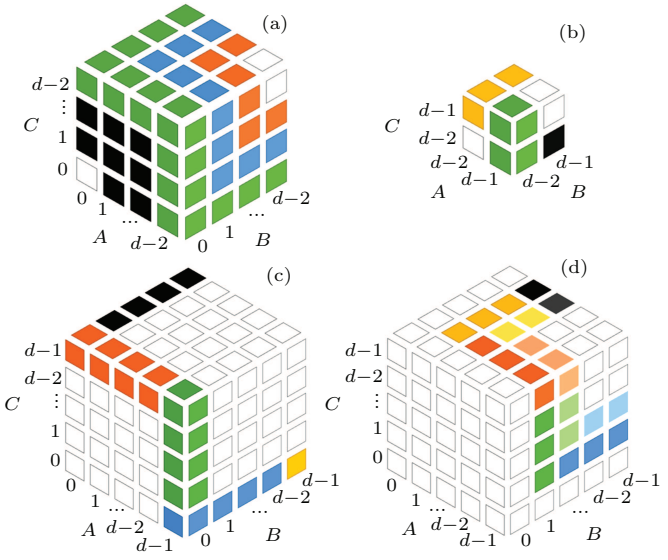


Fig. 9. Composition of UPBs in $C^d \otimes C^d \otimes C^d$.

In addition, there are also some product states in the process of construction that are not orthogonal to the stopper state $|S\rangle$ but orthogonal to all states in $\Psi\backslash\{|S\rangle\}$. We call these states missing states. It is valuable in the discussion of BE states. Any state in the complementary subspace $\mathcal{H}_\Psi^\perp$ is a linear combination of the missing states.

4. Tripartite system with different dimensions

In the above section, UPBs with the same dimensions of three parties are discussed. In this section, we continue to discuss another scenario which is more general in tripartite systems and propose a set of strongly nonlocal UPBs in $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$.

4.1. Construct a UPB in $C^3\otimes C^3\otimes C^4$ system with strong nonlocality

The tile structure is a classical approach in the general construction of 2-qubit UPBs, which was first proposed by Bennet in 1999. Later, many researchers studied the tile structure and some related structures are proposed, such as the Gen tile structure. In tile structures, any two sub-rectangles (i.e. tiles) cannot be combined to form a new rectangle by a simple translation. Following this method, we consider whether tile structures can be extended to the 3-qudit.

The tripartite state set $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$ can always be uniquely mapped to a Rubik's cube whose section is one of the three planes, $C^{d_1}\otimes C^{d_2}$, $C^{d_2}\otimes C^{d_3}$ and $C^{d_1}\otimes C^{d_3}$. Therefore, we extend the classical tile structure to three parties, which is defined in the following.

Definition 5: Tri-tile structure If two sub-cuboids (i.e., tiles) cannot be combined to form a new cuboid (i.e., tiles) by a simple translation, then the structure of the state set is defined as the tri-tile structure.

Moreover, we find that the state set proposed in the above section also satisfies the tri-tile structure. Taking $d=3$ as an example, it is obvious that this $C^3\otimes C^3\otimes C^3$ system satisfies the tri-tile structure.

Proposition 2 If a 3-qudit UPB is strongly nonlocal, then the tripartite state set satisfies the tri-tile structure.

Proof A 3-qudit UPB with strong nonlocality cannot locally eliminate one or more state while performing nontrivial orthogonality-preserving measurement. Hence, all tiles should be irreducible in $\mathcal{H}_{\mathrm{A}}\otimes\mathcal{H}_{\mathrm{B}}\otimes\mathcal{H}_{\mathrm{C}}$ Hilbert space. Therefore, this tripartite state set satisfies the tri-tile structure.

According to the definition of the tri-tile structure, we construct an incomplete $C^3\times C^3\times C^4$ system based on shift UPBs in Eq. (8), which is also strongly nonlocal. This system can lay the foundation for the further construction of 3-qudit UPBs with higher dimension.

$$P_{3,3,4}^1:\begin{cases}|\psi_0\rangle = |1\rangle_{\mathrm{A}}|0\rangle_{\mathrm{B}}|0-1\rangle_{\mathrm{C}},\\ |\psi_1\rangle = |0\rangle_{\mathrm{A}}|0-1\rangle_{\mathrm{B}}|1\rangle_{\mathrm{C}},\\ |\psi_2\rangle = |0-1\rangle_{\mathrm{A}}|1\rangle_{\mathrm{B}}|0\rangle_{\mathrm{C}},\end{cases}$$

$$P_{3,3,4}^2:\begin{cases}|\psi_3\rangle = |1\rangle_{\mathrm{A}}|2\rangle_{\mathrm{B}}|2-3\rangle_{\mathrm{C}},\\ |\psi_4\rangle = |2\rangle_{\mathrm{A}}|1-2\rangle_{\mathrm{B}}|2\rangle_{\mathrm{C}},\\ |\psi_5\rangle = |1-2\rangle_{\mathrm{A}}|1\rangle_{\mathrm{B}}|3\rangle_{\mathrm{C}},\end{cases}$$

$$
P^3_{3,3,4}:\begin{cases}
|\psi_6\rangle = |2\rangle_A|0\rangle_B|1+w_3 2+w_3^2 3\rangle_C,\\
|\psi_7\rangle = |1-2\rangle_A|2\rangle_B|0\rangle_C,\\
|\psi_8\rangle = |0\rangle_A|1-2\rangle_B|3\rangle_C,\\
|\psi_9\rangle = |0\rangle_A|2\rangle_B|0+w_3 1+w_3^2 2\rangle_C,\\
|\psi_{10}\rangle = |0-1\rangle_A|0\rangle_B|3\rangle_C,\\
|\psi_{11}\rangle = |2\rangle_A|0-1\rangle_B|0\rangle_C,
\end{cases}
$$

$$
P^4_{3,3,4}:\begin{cases}
|\psi_{12,13,14}\rangle = \left|0-w_2^i 1\right\rangle_A\left|0-w_2^j 1\right\rangle_B|2\rangle_C,\\
|\psi_{15,16,17}\rangle = \left|1-w_2^i 2\right\rangle_A\left|1-w_2^j 2\right\rangle_B|1\rangle_C.
\end{cases}
$$

$$
\text{Stopper}: |\psi_{18}\rangle = |0+1+2\rangle_A|0+1+2\rangle_B|0+1+2+3\rangle_C. \quad (8)
$$

Here $i, j = 0, 1, ij = 0$.

4.2. Construct a UPB in $C^4 \times C^4 \times C^5$ system based on a UPB in $C^3 \times C^3 \times C^4$ system

When constructing a UPB set of three parties with different dimensions, we adopt the similar practical method as before to construct high-dimensional state sets. Through verifying whether the constructed state set meets the tri-tile structure, we can judge whether the state set is strongly nonlocal more quickly and efficiently. Equation (9) is a set of incomplete UPBs in $C^4 \times C^4 \times C^5$.

$$
P^1_{4,4,5}:\begin{cases}
|\psi_0\rangle = |1\rangle_A|0\rangle_B|0-1\rangle_C,\\
|\psi_1\rangle = |0\rangle_A|0-1\rangle_B|1\rangle_C,\\
|\psi_2\rangle = |0-1\rangle_A|1\rangle_B|0\rangle_C,\\
|\psi_3\rangle = |1\rangle_A|2\rangle_B|2-3\rangle_C,\\
|\psi_4\rangle = |2\rangle_A|1-2\rangle_B|2\rangle_C,\\
|\psi_5\rangle = |1-2\rangle_A|1\rangle_B|3\rangle_C,\\
|\psi_6\rangle = |2\rangle_A|0\rangle_B|1+w_3 2+w_3^2 3\rangle_C,\\
|\psi_7\rangle = |1-2\rangle_A|2\rangle_B|0\rangle_C,\\
|\psi_8\rangle = |0\rangle_A|1-2\rangle_B|3\rangle_C,\\
|\psi_9\rangle = |0\rangle_A|2\rangle_B|0+w_3 1+w_3^2 2\rangle_C,\\
|\psi_{10}\rangle = |0-1\rangle_A|0\rangle_B|3\rangle_C,\\
|\psi_{11}\rangle = |2\rangle_A|0-1\rangle_B|0\rangle_C,\\
|\psi_{12,13,14}\rangle = \left|0-w_2^i 1\right\rangle_A\left|0-w_2^j 1\right\rangle_B|2\rangle_C,\\
|\psi_{15,16,17}\rangle = \left|1-w_2^i 2\right\rangle_A\left|1-w_2^j 2\right\rangle_B|1\rangle_C,
\end{cases}
$$

$$
P^2_{4,4,5}:\begin{cases}
|\psi_{18}\rangle = |3\rangle_A|2\rangle_B|3-4\rangle_C,\\
|\psi_{19}\rangle = |2\rangle_A|2-3\rangle_B|4\rangle_C,\\
|\psi_{20}\rangle = |2-3\rangle_A|3\rangle_B|3\rangle_C,
\end{cases}
$$

$$
P^3_{4,4,5}:\begin{cases}
|\psi_{21}\rangle = |0\rangle_A|3\rangle_B|0+w_4 1+w_4^2 2+w_4^4 3\rangle_C,\\
|\psi_{22}\rangle = |0+w_3 1+w_3^2 2\rangle_A|0\rangle_B|4\rangle_C,\\
|\psi_{23}\rangle = |3\rangle_A|0+w_3 1+w_3^2 2\rangle_B|0\rangle_C,\\
|\psi_{24}\rangle = |3\rangle_A|0\rangle_B|1+w_4 2+w_4^2 3+w_4^4 4\rangle_C,\\
|\psi_{25}\rangle = |1+w_3 2+w_3^2 3\rangle_A|3\rangle_B|0\rangle_C,\\
|\psi_{26}\rangle = |0\rangle_A|1+w_3 2+w_3^2 3\rangle_B|4\rangle_C,
\end{cases}
$$

$$
P^4_{4,4,5}:\begin{cases}
|\psi_{27}\rangle = |2-3\rangle_A|1\rangle_B|4\rangle_C,\\
|\psi_{28}\rangle = |3\rangle_A|1\rangle_B|1+w_3 2+w_3{}^2 3\rangle_C,\\
|\psi_{29,30,31}\rangle = |3\rangle_A\left|2-w_2^i 3\right\rangle_B\left|1-w_2^j 2\right\rangle_C,\\
|\psi_{32,33,34}\rangle = \left|1-w_2^i 2\right\rangle_A|3\rangle_B\left|1-w_2^j 2\right\rangle_C,\\
|\psi_{35}\rangle = |1\rangle_A|3\rangle_B|3-4\rangle_C,\\
|\psi_{36}\rangle = |1\rangle_A|1-2\rangle_B|4\rangle_C.
\end{cases}
$$

$$
\text{Stopper}: |\psi_{37}\rangle = |0+1+2+3\rangle_A|0+1+2+3\rangle_B
|0+1+2+3+4\rangle_C. \quad (9)
$$

Here $w_4 = e^{\frac{2\pi\sqrt{-1}}{4}}$, $i, j = 0, 1, ij = 0$.

Similarly, the UPBs of $C^4 \times C^4 \times C^5$ can be divided into the following four parts:

$P^1_{4,4,5}$: The system of $C^3 \times C^3 \times C^4$ corresponds to $\{(0, 1, 2)_A, (0, 1, 2)_B, (0, 1, 2, 3)_C\}$ of Rubik's cube. It includes 14 subcuboids $\{SP^i\}_{i=1}^{14}$, as shown in Fig. 10(a).

$P^2_{4,4,5}$: The set of UPBs with dimension 2, whose structure is the same as the shift UPBs. It includes three subcubes $\{SP^{15}, SP^{16}, SP^{17}\}$, and corresponds to $\{(3)_A, (2)_B, (3, 4)_C\}$, $\{(2)_A, (2, 3)_B, (4)_C\}$, $\{(2, 3)_A, (3)_B, (3)_C\}$, respectively, as shown in Fig. 10(b).

$P^3_{4,4,5}$: Six edges of the $4 \times 4 \times 5$ Rubik's cube, which consists of 6 subcuboids $\{SP^i\}_{i=18}^{23}$, and they correspond to $\{(0, 1, 2)_A, (0)_B, (4)_C\}$, $\{(3)_A, (0)_B, (1, 2, 3, 4)_C\}$, $\{(3)_A, (0, 1, 2)_B, (0)_C\}$, $\{(1, 2, 3)_A, (3)_B, (0)_C\}$, $\{(0)_A, (3)_B, (0, 1, 2, 3)_C\}$, $\{(0)_A, (1, 2, 3)_B, (4)_C\}$, as shown in Fig. 10(c).

$P^4_{4,4,5}$: The remaining part of the three newly added planes. It is decomposed into 6 subcuboids that cannot be further expanded, as shown in Fig. 10(d).

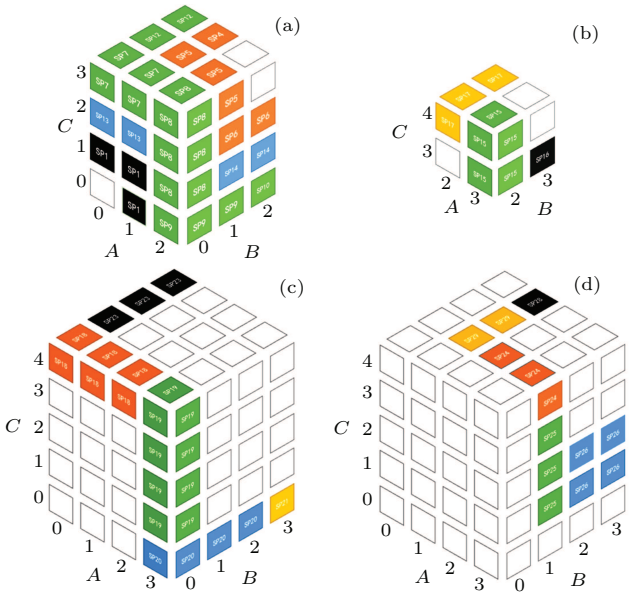


Fig. 10. Composition of UPBs in $C^4 \otimes C^4 \otimes C^5$.

4.3. Construct a UPB in $C^{d_1} \otimes C^{d_2} \otimes C^{d_3}$ system based on a UPB in $C^{(d_1-1)} \otimes C^{(d_2-1)} \otimes C^{(d_3-1)}$ system

In this subsection, we propose a general construction method of UPBs in $C^{d_1} \otimes C^{d_2} \otimes C^{d_3}$ based on UPBs in $C^{(d_1-1)} \otimes C^{(d_2-1)} \otimes C^{(d_3-1)}$ according to the proposed tri-tile structure.

Continuing with the contruction method in the previous subsections, it is obvious that parts $P^1_{d_1,d_2,d_3}$, $P^2_{d_1,d_2,d_3}$, and $P^2_{d_1,d_2,d_3}$ can be easily obtained. As for the remaining part $P^4_{d_1,d_2,d_3}$, it needs to be divided to ensure that the whole cuboid meets the requirements of tri-tile structure.

Proposition 3: Let $\mathcal{H}_{\rm A},\mathcal{H}_{\rm B},\mathcal{H}_{\rm C}$ be Hilbert spaces of dimension d_1, d_2, d_3 respectively. Suppose that $A=\{|1\rangle,\ldots,|d_1\rangle\}, B=\{|1\rangle,\ldots,|d_2\rangle\}, C=\{|1\rangle,\ldots,|d_3\rangle\}$ are ordered orthonormal bases with respect to $\mathcal{H}_{\rm A},\mathcal{H}_{\rm B},\mathcal{H}_{\rm C}$. In $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$, the set of UPBs given by Eq. (10) is pairwise orthogonal and strongly nonlocal.

$$
\begin{aligned}
&P^1_{d_1,d_2,d_3}:\ P^1_{d_1-1,d_2-1,d_3-1},\\
&P^2_{d_1,d_2,d_3}:\ X_d^{\otimes 3}\left(P^2_{d_1-1,d_2-1,d_3-1}\right),\\
&P^3_{d_1,d_2,d_3}:\ \begin{cases}
|\psi_k\rangle=\left|0+\cdots+w_{d_1-1}^{d_1-2}(d_1-2)\right\rangle_{\rm A}|0\rangle_{\rm B}|d_3-1\rangle_{\rm C},\\
|\psi_{k+1}\rangle=|d_1-1\rangle_{\rm A}\left|0+\cdots+w_{d_2-1}^{d_2-2}(d_2-2)\right\rangle_{\rm B}|0\rangle_{\rm C},\\
|\psi_{k+2}\rangle=|0\rangle_{\rm A}|d_2-1\rangle_{\rm B}\left|0+\cdots+w_{d_3-1}^{d_3-2}(d_3-2)\right\rangle_{\rm C},\\
|\psi_{k+3}\rangle=\left|1+\cdots+w_{d_1-1}^{d_1-2}(d_1-1)\right\rangle_{\rm A}|d_2-1\rangle_{\rm B}|0\rangle_{\rm C},\\
|\psi_{k+4}\rangle=|0\rangle_{\rm A}\left|1+\cdots+w_{d_2-1}^{d_2-2}(d_2-1)\right\rangle_{\rm B}|d_3-1\rangle_{\rm C},\\
|\psi_{k+5}\rangle=|d_1-1\rangle_{\rm A}|0\rangle_{\rm B}\left|1+\cdots+w_{d_3-1}^{d_3-2}(d_3-1)\right\rangle_{\rm C},
\end{cases}\\
&P^4_{d_1,d_2,d_3}:\ \begin{cases}
|\psi_{k+6}\rangle=\left|1+\cdots+w_{d_1-3}^{d_1-4}(d_1-3)\right\rangle_{\rm A}\left|1+\cdots+w_{d_2-2}^{d_2-3}(d_2-2)\right\rangle_{\rm B}|d_3-1\rangle_{\rm C},\\
|\psi_{k+7}\rangle=|(d_1-2)-(d_1-1)\rangle_{\rm A}\left|1+\cdots+w_{d_2-3}^{d_2-2}(d_2-3)\right\rangle_{\rm B}|d_3-1\rangle_{\rm C},\\
|\psi_{k+8}\rangle=|d_1-1\rangle_{\rm A}\left|1+\cdots+w_{d_2-3}^{d_2-4}(d_2-3)\right\rangle_{\rm B}\left|1+\cdots+w_{d_3-2}^{d_3-3}(d_3-2)\right\rangle_{\rm C},\\
|\psi_{k+9}\rangle=\left|1+\cdots+w_{d_1-2}^{d_1-3}(d_1-2)\right\rangle_{\rm A}|d_2-1\rangle_{\rm B}\left|1+\cdots+w_{d_3-3}^{d_3-4}(d_3-3)\right\rangle_{\rm C},\\
|\psi_{k+10}\rangle=\left|1+\cdots+w_{d_1-3}^{d_1-4}(d_1-3)\right\rangle_{\rm A}|d_2-1\rangle_{\rm B}|(d_3-2)-(d_3-1)\rangle_{\rm C},\\
|\psi_{k+11}\rangle=|d_1-1\rangle_{\rm A}|(d_2-2)-(d_2-1)\rangle_{\rm B}\left|1+\cdots+w_{d_3-3}^{d_3-4}(d_3-3)\right\rangle_{\rm C}.
\end{cases}\\
&\text{Stopper}:\ |S\rangle=|0+\cdots+(d_1-1)\rangle_{\rm A}|0+\cdots+(d_2-1)\rangle_{\rm B}|0+\cdots+(d_3-1)\rangle_{\rm C}.
\end{aligned}
\tag{10}
$$

Here $w_k=\mathrm{e}^{\frac{2\pi\sqrt{-1}}{k}}$.

By mapping the set in Eq. (10) to Rubik's cube, the structure of UPBs is shown in Fig. 11.

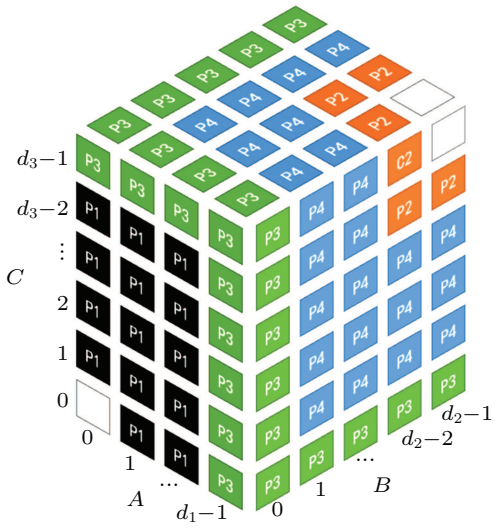


Fig. 11. Composition of UPBs in $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$.

The UPBs of $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$ can be divided into the following four parts:

$P^1_{d_1,d_2,d_3}$: The system of $C^{(d_1-1)}\otimes C^{(d_2-1)}\otimes C^{(d_3-1)}$ correspond to $\{(0,\ldots,d_1-2)_{\rm A},\ (0,\ldots,d_2-2)_{\rm B},\ (0,\ldots,d_3-2)_{\rm C}\}$ of Rubik's cube.

$P^2_{d_1,d_2,d_3}$: The set of UPBs with dimension 2, whose structure is the same as the shift UPBs. It includes three subcubes and corresponds to $\{(d_1-2,\ d_1-1)_{\rm A},\ (d_2-2,\ d_2-1)_{\rm B},\ (d_3-2,\ d_3-1)_{\rm C}\}$, respectively.

$P^3_{d_1,d_2,d_3}$: Six edges of the $d_1\otimes d_2\otimes d_3$ Rubik's cube, which consists of 6 subcuboids and they correspond to $\{(0,\ldots,d_1-2)_{\rm A},(0)_{\rm B},(d_3-1)_{\rm C}\}$, $\{(d_1-1)_{\rm A},(0)_{\rm B},(1,\ldots,d_3-1)_{\rm C}\}$, $\{(d_1-1)_{\rm A},\ (0,\ldots,d_2-2)_{\rm B},(0)_{\rm C}\}$, $\{(1,\ldots,d_1-1)_{\rm A},\ (d_2-1)_{\rm B},(0)_{\rm C}\}$, $\{(0)_{\rm A},\ (d_2-1)_{\rm B},(0,\ldots,d_3-2)_{\rm C}\}$, $\{(0)_{\rm A},(1,\ldots,d_2-1)_{\rm B},(d_3-1)_{\rm C}\}$.

$P^4_{d_1,d_2,d_3}$: The remaining part of the three newly added planes. It is divided into six subcuboids. When constructing this part, the partitioning method is similar to that in part $P^3_{d_1,d_2,d_3}$, but the limitation of the tri-tile structure needs to be considered.

5. Conclusion

In summary, we have concentrated on the general construction method of 3-qudit UPBs exhibiting strong nonlocality. Firstly, a strongly nonlocal set $C^3\otimes C^3\otimes C^3$ of size 19 is constructed based on the shift UPBs. After a simple observation of its orthogonal graphs and corresponding $3\times3\times3$ Rubik's cube, the relationship among three qutrits is discussed and the structure of UPBs is analyzed. According to the above results, following the idea of deducing the UPBs with high dimension from the UPBs with low dimension, we try to construct UPBs with higher dimension and obtain a general set in $C^d\otimes C^d\otimes C^d$ of size $(d-1)^3+2d+5$. Second, we extend the 2-qubit tile structure to 3-qudit and define it as a tri-tile structure, which is conducive for us to judge whether the state set is strongly nonlocal more quickly and efficiently. Based on this method, we give the construction process of 3-qudit UPBs with different dimensions. The construction of UPBs in $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$ is similar to that of $C^d\otimes C^d\otimes C^d$ system, which also consists of four parts.

It is noted that the state set constructed using this general construction method includes a considerable number of states. Considering that reducing the number of quantum states is of great significance for exploring which states affect the nonlocality of the system, it is a valuable research direction to discuss the minimum number of states in 3-qudit UPBs. In addition, there is another kind of interesting state that is closely related to UPBs, BE states. It is also a worthy research direction to produce BE states of small rank by constructing UPBs.

Appendix A: Proof of Lemma 2

Lemma 2 In $C^4\otimes C^4\otimes C^4$, the 3-qudit UPBs of size 40 given by Eq. (6) is strongly nonlocal.

Proof In order to make the proof process clearer, we first rewrite the original state, let $|00\rangle\to|0\rangle$, $|01\rangle\to|1\rangle$, …, $|33\rangle\to|15\rangle$. After rewriting the state in Eq. (6), we can obtain the following states:

$$P_4^1:\begin{cases}|\psi_{0,1,2}\rangle=\left|0-w_2^i1\right\rangle_{\rm A}\left|1-w_2^j2\right\rangle_{\rm BC},\\ \left|\psi_{3,4,5}\right\rangle=|0\rangle_{\rm A}\left|4-w_2^j5-w_2^i8+w_2^{i+j}9\right\rangle_{\rm BC},\\ \left|\psi_{6,7,8}\right\rangle=\left|1-w_2^j2\right\rangle_{\rm A}\left|0-w_2^i4\right\rangle_{\rm BC},\\ |\psi_{9,10,11}\rangle=\left|1-w_2^j2\right\rangle_{\rm A}\left|8-w_2^i9\right\rangle_{\rm BC},\\ |\psi_{12,13,14}\rangle=|2\rangle_{\rm A}\left|1-w_2^j2-w_2^i5+w_2^{i+j}6\right\rangle_{\rm BC},\\ \left|\psi_{15,16,17}\right\rangle=\left|0-w_2^i1\right\rangle_{\rm A}\left|6-w_2^j10\right\rangle_{\rm BC},\end{cases}$$

$$P_4^2:\begin{cases}|\psi_{18}\rangle=|3\rangle_{\rm A}|10-11\rangle_{\rm BC},\\ |\psi_{19}\rangle=|2\rangle_{\rm A}|11-15\rangle_{\rm BC},\\ |\psi_{20}\rangle=|2-3\rangle_{\rm A}|14\rangle_{\rm BC},\end{cases}$$

$$P_4^3:\begin{cases}|\psi_{21,22}\rangle=|0\rangle_{\rm A}\left|12+w_3 13+w_3^2 14\right\rangle_{\rm BC},\\ |\psi_{23,24}\rangle=\left|0+w_3 1+w_3^2 2\right\rangle_{\rm A}|3\rangle_{\rm BC},\\ \left|\psi_{25,26}\right\rangle=|3\rangle_{\rm A}\left|0+w_3 4+w_3^2 8\right\rangle_{\rm BC},\\ |\psi_{27.28}\rangle=|3\rangle_{\rm A}\left|1+w_3 2+w_3^2 3\right\rangle_{\rm BC},\\ |\psi_{29.30}\rangle=\left|1+w_3 2+w_3^2 3\right\rangle_{\rm A}|12\rangle_{\rm BC},\\ |\psi_{31,32}\rangle=|0\rangle_{\rm A}\left|7+w_3 11+w_3^2 15\right\rangle_{\rm BC},\end{cases}$$

$$P_4^4:\begin{cases}|\psi_{33}\rangle=|1-2\rangle_{\rm A}|13\rangle_{\rm BC},\\ |\psi_{34}\rangle=|1\rangle_{\rm A}|7-11\rangle_{\rm BC},\\ |\psi_{35}\rangle=|3\rangle_{\rm A}|5-6\rangle_{\rm BC},\\ |\psi_{36}\rangle=|2-3\rangle_{\rm A}|7\rangle_{\rm BC},\\ |\psi_{37}\rangle=|3\rangle_{\rm A}|9-13\rangle_{\rm BC},\\ |\psi_{38}\rangle=|1\rangle_{\rm A}|14-15\rangle_{\rm BC}.\end{cases}$$

$$\text{Stopper}:\ |\psi_{39}\rangle=|0+1+2+3\rangle_{\rm A}|0+1+\cdots+15\rangle_{\rm BC}. \tag{11a}$$

Suppose that the BC system starts with the nontrivial and non-disturbing measurement, represented by a set of POVM elements $M_m^\dagger M_m$ on 16×16 matrix. The POVM measurement in $\{|0\rangle,|1\rangle,\cdots,|15\rangle\}_{\rm BC}$ basis corresponding to the states in Eq. (11), can be written as

$$M_m^\dagger M_m=\begin{bmatrix}a_{00}&a_{01}&\cdots&a_{0(14)}&a_{0(15)}\\ a_{10}&a_{11}&\cdots&a_{1(14)}&a_{1(15)}\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ a_{(14)0}&a_{(14)1}&\cdots&a_{(14)(14)}&a_{(14)(15)}\\ a_{(15)0}&a_{(15)1}&\cdots&a_{(15)(14)}&a_{(15)(15)}\end{bmatrix}.$$

The post-measurement states could be expressed as $(I\otimes M_m)|\psi_i\rangle$, which should be mutually orthogonal. Then $\langle\psi_j|\left(I\otimes M_m^\dagger M_m\right)|\psi_i\rangle=0$ is obtained. According to this principle, the original matrix could be transformed to

$$M_m^\dagger M_m=\begin{bmatrix}a&0&\cdots&0\\ 0&a&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 0&0&\cdots&a\end{bmatrix}.$$

Table 2 shows the detailed derivation process.

Obviously, the BC's measurement matrix is proportional to the identity matrix, so it means that the BC system starts with a trivial measurement, and cannot get any information about shared state distinction from the measurement result. As for the other two division methods, i.e., $B|CA$ and $C|AB$, the proof method is similar to this. In summary, the multi-party divided into any two parts can keep the strong nonlocality of the quantum system.

Table 2. POVM elements.

POVM elements	Corresponding states							
$a_{0i}=a_{i0}=0$	$i=1$ $\|\psi_6\rangle,\|\psi_0\rangle$	$i=2$ $\|\psi_7\rangle,\|\psi_1\rangle$	$i=3$ $\|\psi_6\rangle,\|\psi_{34}\rangle$	$i=4$ $\|\psi_3\rangle,\|\psi_{39}\rangle$	$i=5$ $\|\psi_6\rangle,\|\psi_{12}\rangle$	$i=6$ $\|\psi_6\rangle,\|\psi_{15}\rangle$	$i=7$ $\|\psi_6\rangle,\|\psi_{25}\rangle$	$i=8$ $\|\psi_6\rangle,\|\psi_9\rangle$
	$i=9$ $\|\psi_6\rangle,\|\psi_{10}\rangle$	$i=10$ $\|\psi_6\rangle,\|\psi_{16}\rangle$	$i=11$ $\|\psi_6\rangle,\|\psi_{19}\rangle$	$i=12$ $\|\psi_6\rangle,\|\psi_{37}\rangle$	$i=13$ $\|\psi_6\rangle,\|\psi_{32}\rangle$	$i=14$ $\|\psi_6\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_6\rangle,\|\psi_{21}\rangle$	
$a_{1i}=a_{i1}=0$	$i=2$ $\|\psi_{12}\rangle,\|\psi_{39}\rangle$	$i=3$ $\|\psi_0\rangle,\|\psi_{34}\rangle$	$i=4$ $\|\psi_1\rangle,\|\psi_8\rangle$	$i=5$ $\|\psi_0\rangle,\|\psi_3\rangle$	$i=6$ $\|\psi_1\rangle,\|\psi_{16}\rangle$	$i=7$ $\|\psi_0\rangle,\|\psi_{30}\rangle$	$i=8$ $\|\psi_0\rangle,\|\psi_9\rangle$	$i=9$ $\|\psi_0\rangle,\|\psi_{10}\rangle$
	$i=10$ $\|\psi_1\rangle,\|\psi_{17}\rangle$	$i=11$ $\|\psi_1\rangle,\|\psi_{30}\rangle$	$i=12$ $\|\psi_0\rangle,\|\psi_{37}\rangle$	$i=13$ $\|\psi_0\rangle,\|\psi_{32}\rangle$	$i=14$ $\|\psi_0\rangle,\|\psi_{33}\rangle$	$i=15$ $\|\psi_0\rangle,\|\psi_{21}\rangle$		
$a_{2i}=a_{i2}=0$	$i=3$ $\|\psi_1\rangle,\|\psi_{34}\rangle$	$i=4$ $\|\psi_2\rangle,\|\psi_8\rangle$	$i=5$ $\|\psi_1\rangle,\|\psi_4\rangle$	$i=6$ $\|\psi_0\rangle,\|\psi_{15}\rangle$	$i=7$ $\|\psi_{14}\rangle,\|\psi_{25}\rangle$	$i=8$ $\|\psi_1\rangle,\|\psi_{10}\rangle$	$i=9$ $\|\psi_1\rangle,\|\psi_{11}\rangle$	$i=10$ $\|\psi_0\rangle,\|\psi_{16}\rangle$
	$i=11$ $\|\psi_2\rangle,\|\psi_{29}\rangle$	$i=12$ $\|\psi_1\rangle,\|\psi_{37}\rangle$	$i=13$ $\|\psi_1\rangle,\|\psi_{32}\rangle$	$i=14$ $\|\psi_{14}\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_1\rangle,\|\psi_{21}\rangle$			
$a_{3i}=a_{i3}=0$	$i=4$ $\|\psi_{34}\rangle,\|\psi_7\rangle$	$i=5$ $\|\psi_{34}\rangle,\|\psi_4\rangle$	$i=6$ $\|\psi_{34}\rangle,\|\psi_{15}\rangle$	$i=7$ $\|\psi_{34}\rangle,\|\psi_{26}\rangle$	$i=8$ $\|\psi_{34}\rangle,\|\psi_9\rangle$	$i=9$ $\|\psi_{34}\rangle,\|\psi_{10}\rangle$	$i=10$ $\|\psi_{34}\rangle,\|\psi_{16}\rangle$	$i=11$ $\|\psi_{34}\rangle,\|\psi_{29}\rangle$
	$i=12$ $\|\psi_{34}\rangle,\|\psi_{39}\rangle$	$i=13$ $\|\psi_{34}\rangle,\|\psi_{32}\rangle$	$i=14$ $\|\psi_{34}\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_{34}\rangle,\|\psi_{19}\rangle$				
$a_{4i}=a_{i4}=0$	$i=5$ $\|\psi_7\rangle,\|\psi_{13}\rangle$	$i=6$ $\|\psi_7\rangle,\|\psi_{16}\rangle$	$i=7$ $\|\psi_7\rangle,\|\psi_{26}\rangle$	$i=8$ $\|\psi_7\rangle,\|\psi_{10}\rangle$	$i=9$ $\|\psi_7\rangle,\|\psi_{11}\rangle$	$i=10$ $\|\psi_7\rangle,\|\psi_{17}\rangle$	$i=11$ $\|\psi_6\rangle,\|\psi_{29}\rangle$	$i=12$ $\|\psi_7\rangle,\|\psi_{37}\rangle$
	$i=13$ $\|\psi_7\rangle,\|\psi_{31}\rangle$	$i=14$ $\|\psi_7\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_7\rangle,\|\psi_{19}\rangle$					
$a_{5i}=a_{i5}=0$	$i=6$ $\|\psi_3\rangle,\|\psi_{15}\rangle$	$i=7$ $\|\psi_{12}\rangle,\|\psi_{26}\rangle$	$i=8$ $\|\psi_{12}\rangle,\|\psi_{10}\rangle$	$i=9$ $\|\psi_{13}\rangle,\|\psi_{11}\rangle$	$i=10$ $\|\psi_4\rangle,\|\psi_{16}\rangle$	$i=11$ $\|\psi_3\rangle,\|\psi_{38}\rangle$	$i=12$ $\|\psi_{12}\rangle,\|\psi_{37}\rangle$	$i=13$ $\|\psi_{12}\rangle,\|\psi_{31}\rangle$
	$i=14$ $\|\psi_{12}\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_4\rangle,\|\psi_{38}\rangle$						
$a_{6i}=a_{i6}=0$	$i=7$ $\|\psi_{13}\rangle,\|\psi_{25}\rangle$	$i=8$ $\|\psi_{15}\rangle,\|\psi_9\rangle$	$i=9$ $\|\psi_{15}\rangle,\|\psi_{10}\rangle$	$i=10$ $\|\psi_{28}\rangle,\|\psi_{18}\rangle$	$i=11$ $\|\psi_{15}\rangle,\|\psi_{30}\rangle$	$i=12$ $\|\psi_{15}\rangle,\|\psi_{37}\rangle$	$i=13$ $\|\psi_{15}\rangle,\|\psi_{31}\rangle$	$i=14$ $\|\psi_{13}\rangle,\|\psi_{20}\rangle$
	$i=15$ $\|\psi_{15}\rangle,\|\psi_{38}\rangle$							
$a_{7i}=a_{i7}=0$	$i=8$ $\|\psi_{25}\rangle,\|\psi_{35}\rangle$	$i=9$ $\|\psi_{25}\rangle,\|\psi_9\rangle$	$i=10$ $\|\psi_{16}\rangle,\|\psi_{30}\rangle$	$i=11$ $\|\psi_{25}\rangle,\|\psi_{18}\rangle$	$i=12$ $\|\psi_{25}\rangle,\|\psi_{37}\rangle$	$i=13$ $\|\psi_{25}\rangle,\|\psi_{31}\rangle$	$i=14$ $\|\psi_{25}\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_{25}\rangle,\|\psi_{19}\rangle$
$a_{8i}=a_{i8}=0$	$i=9$ $\|\psi_{35}\rangle,\|\psi_{23}\rangle$	$i=10$ $\|\psi_{10}\rangle,\|\psi_{16}\rangle$	$i=11$ $\|\psi_9\rangle,\|\psi_{29}\rangle$	$i=12$ $\|\psi_9\rangle,\|\psi_{37}\rangle$	$i=13$ $\|\psi_9\rangle,\|\psi_{32}\rangle$	$i=14$ $\|\psi_{10}\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_3\rangle,\|\psi_{38}\rangle$	
$a_{9i}=a_{i9}=0$	$i=10$ $\|\psi_{10}\rangle,\|\psi_{17}\rangle$	$i=11$ $\|\psi_{24}\rangle,\|\psi_{18}\rangle$	$i=12$ $\|\psi_{10}\rangle,\|\psi_{39}\rangle$	$i=13$ $\|\psi_{10}\rangle,\|\psi_{31}\rangle$	$i=14$ $\|\psi_9\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_9\rangle,\|\psi_{19}\rangle$		
$a_{10i}=a_{i10}=0$	$i=11$ $\|\psi_{16}\rangle,\|\psi_{29}\rangle$	$i=12$ $\|\psi_{16}\rangle,\|\psi_{37}\rangle$	$i=13$ $\|\psi_{16}\rangle,\|\psi_{31}\rangle$	$i=14$ $\|\psi_{15}\rangle,\|\psi_{22}\rangle$	$i=15$ $\|\psi_{17}\rangle,\|\psi_{38}\rangle$			
$a_{11i}=a_{i11}=0$	$i=12$ $\|\psi_{18}\rangle,\|\psi_{37}\rangle$	$i=13$ $\|\psi_{19}\rangle,\|\psi_{31}\rangle$	$i=14$ $\|\psi_{19}\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_{30}\rangle,\|\psi_{22}\rangle$				
$a_{12i}=a_{i12}=0$	$i=13$ $\|\psi_{37}\rangle,\|\psi_{32}\rangle$	$i=14$ $\|\psi_{37}\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_{37}\rangle,\|\psi_{19}\rangle$					
$a_{13i}=a_{i13}=0$	$i=14$ $\|\psi_{32}\rangle,\|\psi_{20}\rangle$	$i=15$ $\|\psi_{32}\rangle,\|\psi_{22}\rangle$						
$a_{14i}=a_{i14}=0$	$i=15$ $\|\psi_{33}\rangle,\|\psi_{38}\rangle$							
$a_{00}=a_{ii}$	$i=1$ $\|\psi_0\rangle,\|\psi_{39}\rangle$	$i=2$ $\|\psi_0\rangle,\|\psi_1\rangle$	$i=3$ $\|\psi_{36}\rangle,\|\psi_{39}\rangle$	$i=4$ $\|\psi_6\rangle,\|\psi_7\rangle$	$i=5$ $\|\psi_{27}\rangle,\|\psi_{39}\rangle$	$i=6$ $\|\psi_{15}\rangle,\|\psi_{39}\rangle$	$i=7$ $\|\psi_{29}\rangle,\|\psi_{39}\rangle$	$i=8$ $\|\psi_9\rangle,\|\psi_{39}\rangle$
	$i=9$ $\|\psi_9\rangle,\|\psi_{10}\rangle$	$i=10$ $\|\psi_{15}\rangle,\|\psi_{16}\rangle$	$i=11$ $\|\psi_{18}\rangle,\|\psi_{39}\rangle$	$i=12$ $\|\psi_{33}\rangle,\|\psi_{39}\rangle$	$i=13$ $\|\psi_{24}\rangle,\|\psi_{39}\rangle$	$i=14$ $\|\psi_{22}\rangle,\|\psi_{39}\rangle$	$i=15$ $\|\psi_{19}\rangle,\|\psi_{39}\rangle$	

Acknowledgements

This work was supported by the National Key R&D Program of China (Grant No. 2020YFB1805405), the 111 Project (Grant No. B21049), the Foundation of Guizhou Provincial Key Laboratory of Public Big Data (Grant No. 2019BDKFJJ014), and the Fundamental Research Funds for the Central Universities (Grant Nos. 2019XD-A02 and 2020RC38).

References

[1] Walgate J, Short A J, Hardy L and Vedral V 2000 *Phys. Rev. Lett.* **85** 4972
[2] Zhang C R, Hu M J and Xiang G Y 2020 *Chin. Phys. Lett.* **37** 012103
[3] Rahaman R and Parker M G 2015 *Phys. Lett. A* **91** 022330
[4] DiVincenzo D P, Leung D W and Terhal B M 1988 *IEEE Trans. Inf. Theory* **48** 580
[5] Eggeling T and Werner R F 2002 *Phys. Rev. Lett.* **89** 097905
[6] Cao H J and Song H S 2006 *Chin. Phys. Lett.* **23** 290
[7] Li C Y, Li X H, Deng F G, Zhou P, Liang Y J and Zhou H Y 2006 *Chin. Phys. Lett.* **23** 2896

[8] Xue C, Chen Z Y, Wu Y C and Guo G P 2021 *Chin. Phys. Lett.* **38** 030302
[9] Furusawa A, Srensen J L, Braunstein S, Fuchs C A, Kimble H J and Polzik E S 1998 *Science* **282** 706
[10] Karlsson A and Bourennane M 1998 *Phys. Rev. A* **58** 4394
[11] Hu Y L, Zhang Z and Wu B 2021 *Chin. Phys. B* **30** 020308
[12] Harrow A W, Hassidim A and Lloyd S 2009 *Phys. Rev. Lett.* **103** 150502
[13] Wang G 2017 *Phys. Rev. A* **96** 012335
[14] Bru D, D'Ariano G. M, Lewenstein M, Macchiavello C, Sen A and Sen U 2000 *Phys. Rev. Lett.* **93** 210501
[15] Bennett C H, DiVincenzo D P, Fuchs C A, Mor T, Rains E, Shor P W, Smolin J A and Wootters W K 1999 *Phys. Lett. A* **59** 1070
[16] Jiao F and Hui Z 2001 *Chin. Phys. B* **19** 100309
[17] Johnston N 2014 *J. Phys. A: Math. Theor.* **47** 424034
[18] De Rinaldis S 2004 *Phys. Rev. A* **70** 022309
[19] DiVincenzo D P, Mor T, Shor P W, Smolin J A and Terhal B M 2003 *Commun. Math. Phys.* **238** 379
[20] Agrawal S, Halder S and Banik M 2020 *Phys. Rev. A* **99** 032335
[21] Wieniak M, Pandya P, Sakarya O and Woloncewicz B 2020 *Quantum Rep.* **2** 49
[22] Chen J and Johnston N 2015 *Commun. Math. Phys.* **333** 351
[23] Bej P and Halder S 2021 *Phys. Lett. A* **386** 126992
[24] Shi F and Zhang X and Chen L 2020 *Phys. Rev. A* **101** 062329
[25] Shi F, Li M S, Hu M, Chen L, Yung M H, Wang Y L and Zhang X 2021 arXiv:2101.0073
[26] Sun Y and Chen L 2021 arXiv:2102.11553
[27] Augusiak R, Fritz T, Kotowski M, Kotowski M, Pawowski M, Lewenstein M and Acn A 2012 *Phys. Rev. A* **85** 042113
[28] Fan H 2004 *Phys. Rev. Lett.* **92** 177905
[29] Wang Y M, Li J G, Zou J and Xu B M 2016 *Chin. Phys. B* **25** 120302
[30] Walgate J, Short A J, Hardy L and Vedral V 2000 *Phys. Rev. Lett.* **85** 4972
[31] Bennett C H, DiVincenzo D P, Mor T, Shor P W, Smolin J A and Terhal B M 1999 *Phys. Rev. Lett.* **82** 5385
[32] Zhang Z C, Zhang K J and Gao F 2017 *Phys. Rev. A* **95** 052344
[33] Halder S, Banik M, Agrawal S and Bandyopadhyay S 2019 *Phys. Rev. Lett.* **122** 040403
[34] Zhang Z C and Zhang X 2017 *Phys. Rev. A* **99** 062108
[35] Shi F, Hu M, Chen L and Zhang X 2020 *Phys. Rev. A* **102** 042202
[36] Gross J L and Yellen J 2003 *Handbook of Graph Theory* (Boca Raton: CRC Press) p. 1192
[37] Johnston N 2014 *J. Phys. A: Math. Theor.* **47** 424034
[38] Belhaj A, Sedra M B and Segui A 2014 *J. Phys. A: Math. Theor.* **48** 045401
[39] Markham D and Sanders B C 2008 *Phys. Rev. A* **78** 042309
[40] Halder S, Banik M and Ghosh S 2019 *Phys. Rev. A* **99** 062329
[41] Wang K and Chen L 2020 *Quantum Inf. Process.* **19** 1
[42] Zhang Z C, Wu X and Zhang X 2020 *Phys. Rev. A* **101** 022306
[43] Guo Y, Jia Y and Li X 2015 *Quantum Inf. Process.* **14** 3553
[44] Halder S and Sengupta R 2019 *Phys. Lett. A* **383** 2004
[45] Halder S and Sengupta R 2020 *Phys. Rev. A* **101** 012311