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Citation: Chin. Phys. B, 2022, 31 (**6**): 060202. DOI: 10.1088/1674-1056/ac48fa
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Ergodic stationary distribution of a stochastic rumor propagation model with general incidence function

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(Received 19 August 2021; revised manuscript received 29 December 2021; accepted manuscript online 7 January 2022)

In daily lives, when emergencies occur, rumors will spread widely on the internet. However, it is quite difficult for the netizens to distinguish the truth of the information. The main reasons are the uncertainty of netizens' behavior and attitude, which make the transmission rates of these information among social network groups be not fixed. In this paper, we propose a stochastic rumor propagation model with general incidence function. The model can be described by a stochastic differential equation. Applying the Khasminskii method via a suitable construction of Lyapunov function, we first prove the existence of a unique solution for the stochastic model with probability one. Then we show the existence of a unique ergodic stationary distribution of the rumor model, which exhibits the ergodicity. We also provide some numerical simulations to support our theoretical results. The numerical results give us some possible methods to control rumor propagation. Firstly, increasing noise intensity can effectively reduce rumor propagation when $\widehat{\mathcal{R}}_0 > 1$. That is, after rumors spread widely on social network platforms, government intervention and authoritative media coverage will interfere with netizens' opinions, thus reducing the degree of rumor propagation. Secondly, speed up the rumor refutation, intensify efforts to refute rumors, and improve the scientific quality of netizen (i.e., increase the value of β and decrease the value of α and γ), which can effectively curb the rumor propagation.

Keywords: rumor propagation model, general incidence function, Itô's formula, ergodic stationary distribution

PACS: 02.50.Ey, 02.50.Fz, 05.40.–a, 87.10.Mn **DOI:** 10.1088/1674-1056/ac48fa

1. Introduction

Recently, due to the pandemic of COVID-19, various rumors related to COVID-19 have been circulated on the internet. The rumors not only easily cause people's emotional panic, but also cause other social problems. For example, "Novel coronavirus can be eliminated by using the blow dryer to blow the face and hands for a long time", "Virus can be killed by drinking bleaching disinfectant" and other pseudoscience or malicious "conspiracy theory". Although some authoritative media and experts have voiced against the rumors, there still are exponential forwarding and diffusion of rumors about the epidemic on social media, just like a virus with great replication ability, which urgently needs to respond with actions matching it. Thus, it is of great significance to study rumor propagation in online social networks for strengthening the supervision of network public opinion and purifying the cyberspace.

To minimize the harmfulness of rumors on online social network platforms as much as possible, it is necessary to understand the rules of rumor spreading on online social network platforms. In the past period of time, some mathematical models were established to investigate rumor spreading in the complex network.[10,13,23] In fact, the research of classical rumor spreading model started from 1960s. In 1964 and 1965, in order to compare the differences between the rumor spreading and virus spreading, Daley and Kendal[5,6] proposed the typical DK model of rumor spreading. In their model, they first divided people into three categories: ignorants (people who do not hear the rumor), spreaders (people who spread the rumor) and stiflers (people who know but do not spread the rumor). Since then, the original DK model had been modified and extended continuously, which made it possible to interpret the rumors in more detail. By changing the propagation mechanism, Maki and Thomson[26] further modified the DK model and established a new rumor propagation model-MT model. At the beginning of the 21st century, Zanette first extended the problems of rumor spreading to complex networks, who mainly investigated the rules of rumor propagation on a small-world network and proposed an epidemic-like model.[34,35] Afterwards, some scholars devoted to studying some rumor propagation models in order to better control the rumors, see Refs. [8,18]. Recently, Tuwairqi *et al.*[1] proposed a rumor propagation model with incubation mechanism. In their work, some important results, such as the basic reproduction number of the model, the local stability and global asymptotic stability of the rumor-free and rumor equilibrium, were obtained under suitable conditions. Additional literature relating to the rumor propagation models in different patterns of groups, see Refs. [22,28,33] and the references therein.

To ensure these models can provide the realistic description of the rumor propagation, the incidence rate plays an important role. In 1927, Kermack and McKendrick[19] in-

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troduced the specific bilinear incidence rate βSI for some disease spreading models. However, there were many problems when it was used in describing the disease spreading. To fit the phenomena of disease propagation, Capasso and Serio[4] proposed the saturated incidence rate $\beta SI/(1+\alpha I)$. After the works, many kinds of incidence rate in various biological models were considered by many scholars, see Refs. [2,9,17,24,25,30–32]. From these literature, one can find that models with more general incidence can provide more complex dynamics than those with the bilinear or standard incidence rate.[3,20] Therefore, on social networks, one may consider the following deterministic rumor propagation model with general incidence function:

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \Lambda - I(t)f(S(t)) - \mu I(t),\\
\frac{\mathrm{d}H}{\mathrm{d}t} &= (1-\theta)I(t)f(S(t)) - (\gamma+\delta+\mu)H(t),\\
\frac{\mathrm{d}S}{\mathrm{d}t} &= \theta I(t)f(S(t)) + \gamma H(t) - (\beta+\mu)S(t),\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \delta H(t) + \beta S(t) - \mu R(t). \qquad (1)
\end{aligned}$$

In sociology, $I(t)$ represents individuals on online social networks who have never heard rumors or been contacted to forward rumors at time t, $H(t)$ indicates those people that have heard rumors but hesitated to spread the rumors at time t, $S(t)$ refers to those people who choose to believe and forward rumors but not actively create rumors at time t, $R(t)$ means the people who have no interest to the rumors or no longer believe in rumors, and hence will not spread the rumors at time t. The function $f(\cdot)$ represents the general incidence function. More sociological meanings about the parameters appearing in system (1) are explained in the sequel:

Λ : the number of entering individuals in the whole group;

μ : leaving rate of the population;

δ : conversion rate from hesitating individuals to stiflers;

γ : conversion rate from hesitating individuals to spreaders;

β : transmission rate from spreaders to stiflers due to lose interest in spreading rumors or forget rumors;

θ : portion of those hearing the rumor who will spread the it, and $\theta \in [0,1]$.

However, in the real world, the individuals' psychological state (e.g., hesitating, forgetting etc.) and individuals' behavior (e.g., forwarding, comment etc.) sometimes are disturbed by the environmental noise through the process of rumor dissemination. Therefore, in the noisy environment, the deterministic models not only have difficulty in fitting the data but also have strong limitations. Thus, it is meaningful to consider the influence of the social randomness in rumor propagation. More precisely, the effect of uncertainty should be considered in the parameters estimations, such as transmission rates between the four groups of people. Recently, some researchers have devoted in studying the rumor propagation models with stochastic perturbations, see e.g., Refs. [7,12,14–16,37]. From the literature,[38] it is known that the deterministic models can be transformed into stochastic models by introducing the time Markov chain, parameter perturbation, and robustness of positive equilibria to the deterministic models or the environmental noise that is proportional to the variables. This gives us the motivation to investigate the stochastic effect to the model (1).

Adding the environmental noise into the model (1), in this paper, we consider the following stochastic rumor propagation model:

$$\begin{aligned}
\mathrm{d}I &= (\Lambda - I(t)f(S(t)) - \mu I(t))\mathrm{d}t + \sigma_1 I(t)\mathrm{d}B_1(t),\\
\mathrm{d}H &= ((1-\theta)I(t)f(S(t)) - (\gamma+\delta+\mu)H(t))\mathrm{d}t\\
&\quad + \sigma_2 H(t)\mathrm{d}B_2(t),\\
\mathrm{d}S &= (\theta I(t)f(S(t)) + \gamma H(t) - (\beta+\mu)S(t))\mathrm{d}t\\
&\quad + \sigma_3 S(t)\mathrm{d}B_3(t),\\
\mathrm{d}R &= (\delta H(t) + \beta S(t) - \mu R(t))\mathrm{d}t + \sigma_4 R(t)\mathrm{d}B_4(t). \qquad (2)
\end{aligned}$$

The function $B(t) := (B_1(t), B_2(t), B_3(t), B_4(t))$ means the white noise in $\mathbb{R}^4$. σ_i means intensity of the white noises with $i = 1, 2, 3, 4$. $B_i(t)$ denotes independent Brownian motions with $i = 1, 2, 3, 4$. Inspired by the linear function $f(S) = \alpha S$ and the Holling-II type function $f(S) = \alpha S/(1+mS)$, throughout this article, we assume the incidence function $f(\cdot)$ satisfies the assumptions

(A1) $f(u) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ such that $f(0) = 0$, $f'(u) > 0$ and $f'(u) < f(u)/u$ for all $u > 0$;

(A2) $f(u)/u$ is monotonically non-increasing on $(0,\infty)$ and $(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2})[f(u_1) - f(u_2)] \le 0$ $(u_1, u_2 \in [0, \Lambda/\mu])$.

Sociologically, the hypothesis (A1) means that rumors will not spread without rumor-spreading individuals. Mathematically, this assumption can guarantee that there is no rumor equilibrium in the system (2). From (A1), one can see that $f(u)/u \le f'(0)$ for all $u > 0$.

Hypothesis (A2) illustrates the saturation of the incidence: when the number of rumor spreading individuals is high, the response of the incidence rate to the increase of S will be slower than the linear response. Similar assumptions have been used in some rumor propagation models (see Ref. [39]). Obviously, based on the hypotheses (A1) and (A2), the function $f(\cdot)$ contains some special incidence rates. For instance,

- $f(u) = u$, $f(u) = \frac{u}{1+cu}$ $(c > 0)$ and $f(u) = \frac{u}{1+nu^2}$ $(n > 0)$ (Linear incidence rate, nonlinear saturated incidence rate and non-monotonic incidence rate, respectively).
- $\beta f(u) = (\beta - \beta_1 h(u))u$ $(\beta > \beta_1$, and the twice-differentiable function $h(u)$ satisfies $h(0) = 0$, $h'(u) \ge 0$,

$\lim_{t\to\infty} h(u) = 1$ and $f(u) = \mathrm{e}^{-mu}u$ $(m > 0)$) (Nonlinear incidence rates with "media coverage" effect).

- $f(u) = \frac{u}{h_1(u)}$ ($h_1(u)$ satisfies $h(0) = 0$, $h'(u) > 0$ and there is a constant $\varepsilon > 0$ such that $(\frac{u}{h_1(u)})' > 0$ for $0 < u < \varepsilon$ and $(\frac{u}{h_1(u)})' < 0$ for $u > \varepsilon$) (Nonlinear incidence rate with "intervention policy" effect).

The purpose of this paper is to explore the stochastic effect to the stochastic model (2). Under the assumptions (A1) and (A2), we first prove the global existence of solution of system (2) by using the Khasminskii Lyapunov functional method.

Theorem 1 For any initial value $(I(0),H(0),S(0),R(0)) \in \mathbb{R}^4_+$, system (2) admits a unique global solution $(I(t),H(t),S(t),R(t))$ in $\mathbb{R}^4_+$ with probability one.

Then we study the existence of stationary distribution of system (2). To this end, we first define the number

$$\widehat{\mathcal{R}}_0 := \frac{\Lambda f'(0)\gamma(1-\theta)}{\left(\mu+\frac{1}{2}\sigma_1^2\right)\left(\mu+\gamma+\delta+\frac{1}{2}\sigma_2^2\right)\left(\beta+\mu+\frac{1}{2}\sigma_3^2\right)} + \frac{\Lambda f'(0)\theta}{\left(\mu+\frac{1}{2}\sigma_1^2\right)\left(\beta+\mu+\frac{1}{2}\sigma_3^2\right)}. \tag{3}$$

Using the method established by Khasminskii,[21] we provide the sufficient condition to guarantee the existence of an ergodic stationary distribution for system (2). The main result is stated as follows.

Theorem 2 Assume that $\widehat{\mathcal{R}}_0 > 1$, then system (2) admits a unique stationary distribution $\pi(\cdot)$ and the ergodicity holds.

Let us remark that the constant $\widehat{\mathcal{R}}_0$ plays the similar role as the basic reproduction number $\mathcal{R}_0$ of Eq. (1). In fact, when the intensities of environmental noises $\sigma_i = 0$ $(i = 1,2,3,4)$, we have

$$\widehat{\mathcal{R}}_0 = \frac{\Lambda f'(0)\gamma(1-\theta)}{\mu(\mu+\gamma+\delta)(\beta+\mu)} + \frac{\Lambda f'(0)\theta}{\mu(\beta+\mu)} = \mathcal{R}_0. \tag{4}$$

Similar to the arguments of Ref. [22], one can verify that the rumor-free equilibrium of Eq. (1) is globally asymptotically stable when $\mathcal{R}_0 < 1$. If $\mathcal{R}_0 > 1$, then system (1) admits a unique rumor equilibrium which is globally asymptotically stable. Hence, if the noise intensities are small enough, Theorem 2 indicates that the stochastic model (2) still persists the similar dynamics as that of the deterministic model (1).

In addition, we give some numerical simulations to support our theoretical results. According to the numerical results, one can see that the assumption $\widehat{\mathcal{R}}_0 > 1$ and some parameters play an important role in the rumor propagation. Detail discussion can be found in Section 4. Moreover, in the final section, we provide a brief conclusion about the influence of noise intensity and some parameters to the stochastic system (2).

2. Preliminaries

To obtain the ergodic stationary distribution of system (2), we first recall some definitions and known results in the sequel.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and the values of all parameters are positive. Moreover, we denote

$$a\vee b := \max\{a,b\},\quad a\wedge b := \min\{a,b\} \text{ for all } a,b\in\mathbb{R},$$

and $C^{2,1}(\mathbb{R}^d\times(0,\infty);\mathbb{R}_+)$ by the function space consisting of all functions $V(x,t) : \mathbb{R}^d\times(0,\infty)\to\mathbb{R}_+$ which are continuously twice differentiable in x and once in t. Then we consider initial value problem of the d-dimensional stochastic differential equation:

$$\begin{aligned} &\mathrm{d}x(t) = f(x(t),t)\,\mathrm{d}t + g(x(t),t)\,\mathrm{d}B(t),\ \text{for } t\geq 0,\\ &x(0) = x_0\in\mathbb{R}^d. \end{aligned} \tag{5}$$

According to Ref. [27], we also consider the differential operator $L[\cdot]$ defined by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}\left[g^{\mathrm{T}}(x,t)g(x,t)\right]\frac{\partial^2}{\partial x_i\partial x_j}. \tag{6}$$

Then, for $V\in C^{2,1}(\mathbb{R}^d\times(0,\infty);\mathbb{R}_+)$, it is clear that

$$\begin{aligned} LV(x,t) &= V_t(x,t) + V_x(x,t)f(x,t)\\ &\quad + \frac{1}{2}\mathrm{trace}[g^{\mathrm{T}}(x,t)V_{xx}(x,t)g(x,t)], \end{aligned} \tag{7}$$

where $V_t = \partial V/\partial t$, $V_x = (\partial V/\partial x_1,\ldots,\partial V/\partial x_d)$, and $V_{xx} = (\partial^2 V/\partial x_i\partial x_j)_{d\times d}$. Let $x(t)$ be the solution of Eq. (5), by the Itô's formula (cf. Ref. [27]), it follows that

$$\mathrm{d}V(x(t),t) = LV(x(t),t)\,\mathrm{d}t + V_x(x(t),t)g(x(t),t)\,\mathrm{d}B(t). \tag{8}$$

Specifically, let $X(t)$ be a homogeneous Markov process in the d-dimensional Euclidean space $\mathbb{R}^d$, which satisfies the autonomous equation

$$\mathrm{d}X(t) = b(X)\,\mathrm{d}t + \sum_{r=1}^{d} g_{\mathrm{r}}(X)\,\mathrm{d}B_{\mathrm{r}}(t). \tag{9}$$

It is well-known that the diffusion matrix corresponding to Eq. (9) has the form

$$A(X) = (a_{ij}(x)),\ \text{where } a_{ij}(x) := \sum_{r=1}^{d} g_r^i(x)g_r^j(x). \tag{10}$$

To guarantee the existence of a stationary distribution to the Markov process $X(t)$, we recall the results of Khasminskii [Ref. [21], Theorems 4.1 and 4.2] in the following proposition.

Proposition 1 Let $U\in\mathbb{R}^d$ be a bounded open domain with regular boundary Γ, which has the following properties:

(H1)′ In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(H2)′ If $x \in \mathbb{R}^d \backslash U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} \mathbb{R}^x \tau < \infty$ for every compact subset $K \subset \mathbb{R}^d$.

Then the Markov process $X(t)$ admits a unique ergodic stationary distribution $\pi(\cdot)$. In this situation,

$$\mathbb{P}\Big\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t))\,\mathrm{d}t = \int_{\mathbb{R}^d} f(x)\pi(\mathrm{d}x) \Big\} = 1, \quad \text{for all } x \in \mathbb{R}^d, \tag{11}$$

where $f(\cdot)$ is an integrable function with respect to the measure π.

To verify the assumption (H1)′, it suffices to show the operator F, defined by

$$Fu := b(x)\frac{\partial u}{\partial x} + \frac{1}{2}\mathrm{trace}\left(A(x)\frac{\partial^2 u}{\partial x^2}\right), \tag{12}$$

is uniformly elliptic in U. According to Refs. [21,29], we only need to show that

(H1) there exists a constant $M > 0$ such that $\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2$, $\forall x \in U$ and $\xi \in \mathbb{R}^d$.

Moreover, from [Ref. [36], Theorem 3.13], the assumption (H2)′ holds provided that

(H2) there is a C^2-function $V \geq 0$ satisfying $LV < 0$ for any $x \in \mathbb{R}^d - U$.

Hence, to derive the statement of Theorem 2, we only need to verify the conditions (H1) and (H2).

3. Existence of an ergodic stationary distribution

Based on the assumptions (A1) and (A2), the right hand side of system (2) satisfies the local Lipschitz condition. Then, given any initial value $(I(0),H(0),S(0),R(0)) \in \mathbb{R}^4_+$, it is known that (2) admits a unique local solution $(I(t),H(t),S(t),R(t))$ on $t \in [0,\tau_e]$ for some $\tau_e > 0$. Note that τ_e means the explosion time.[27] To derive the result of Theorem 1, we only need to show that $\tau_e = \infty$ a.s.

Proof of Theorem 1 Let n_0 be a large number such that $I(0)$, $H(0)$, $S(0)$, and $R(0)$ all lie in the interval $[1/n_0, n_0]$. For each integer $n \geq n_0$, we define the following sequence of stopping time:

$$\tau_n := \inf\left\{t \in [0,\tau) : (I(t),H(t),S(t),R(t)) \notin [1/n,n]^4\right\}. \tag{13}$$

In addition, we set $\inf\emptyset = \infty$ ($\emptyset$ denotes the empty set, as usual). Clearly, τ_n is increasing as $n \to \infty$. Let $\tau_\infty := \lim_{n\to\infty} \tau_n$, we claim that $\tau_\infty = \infty$.

Suppose the claim is false, i.e., $\tau_\infty < \infty$, there are constants $T > 0$ and $\varepsilon \in (0,1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon. \tag{14}$$

Thus, there exists an integer $n_1 \geq n_0$ such that

$$\mathbb{P}\{\tau_n \leq T\} \geq \varepsilon, \ \ \forall n \geq n_1. \tag{15}$$

Then we consider the function $V \in C^2(\mathbb{R}^4_+;\mathbb{R}_+ \cup \{0\})$ defined by

$$\begin{aligned} V(I,H,S,R) :=\ & \left(I - a - a\ln\frac{I}{a}\right) + (H - 1 - \ln H) \\ & + (S - 1 - \ln S) + (R - 1 - \ln R), \end{aligned} \tag{16}$$

where a is a positive constant to be determined later. Applying the Itô's formula, we have

$$\begin{aligned} \mathrm{d}V =\ & LV\,\mathrm{d}t + \sigma_1(I-a)\,\mathrm{d}B_1(t) + \sigma_2(H-1)\,\mathrm{d}B_2(t) \\ & + \sigma_3(S-1)\,\mathrm{d}B_3(t) + \sigma_4(R-1)\,\mathrm{d}B_4(t), \end{aligned} \tag{17}$$

where

$$\begin{aligned} LV =\ & \Lambda + (a+3)\mu + \gamma + \delta + \beta + \frac{a\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\ & + (af(S) - \mu S) - \mu(I + H + R) - \frac{a\Lambda}{I} \\ & - \frac{(1-\theta)If(S)}{H} - \frac{\theta If(S)}{S} - \frac{\gamma H}{S} - \frac{\delta H}{R} - \frac{\beta S}{R} \\ \leq\ & \Lambda + (a+3)\mu + \gamma + \delta + \beta + \frac{a\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\ & + (af(S) - \mu S), \end{aligned} \tag{18}$$

for any $n \geq n_1$ and $t \in [0,\tau_n]$. By the assumptions of the incidence function $f(\cdot)$, we may choose $a = \mu/f'(0)$. Then it follows from Eq. (18) that

$$\begin{aligned} LV \leq\ & \Lambda + (a+3)\mu + \gamma + \delta + \beta \\ & + \frac{a\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} := K. \end{aligned} \tag{19}$$

Note that K is a constant independent of I, H, S, R, and t. As a consequence of Eqs. (17) and (18), we have

$$\begin{aligned} & \mathbb{E}V\left(I(\tau_n \wedge T), H(\tau_n \wedge T), S(\tau_n \wedge T), R(\tau_n \wedge T)\right) \\ \leq\ & V\left(I(0),H(0),S(0),R(0)\right) + KT. \end{aligned} \tag{20}$$

Moreover, we set $\Omega_n := \{\tau_n \leq T\}$ for any $n \geq n_1$. Then Eq. (15) leads to $\mathbb{P}(\Omega_n) \geq \varepsilon$. Hence, for every $\omega \in \Omega_n$, at least one of $I(\tau_n,\omega)$, $H(\tau_n,\omega)$, $S(\tau_n,\omega)$ or $R(\tau_n,\omega)$ equals to n or $1/n$. Thus, none of $I(\tau_n,\omega)$, $H(\tau_n,\omega)$, $S(\tau_n,\omega)$, and $R(\tau_n,\omega)$ is less than either

$$\begin{aligned} & \left(n - a - a\ln\frac{n}{a}\right) \wedge (n - 1 - \ln n) \\ \text{or} \quad & \left(\frac{1}{n} - a + a\ln(na)\right) \wedge \left(\frac{1}{n} - 1 + \ln n\right). \end{aligned} \tag{21}$$

It follows that

$$V(I(\tau_n,\omega),H(\tau_n,\omega),S(\tau_n,\omega),R(\tau_n,\omega)) \geq \left(n-a-a\ln\frac{n}{a}\right)\wedge(n-1-\ln n)\wedge\left(\frac{1}{n}-a+a\ln(na)\right)\wedge\left(\frac{1}{n}-1+\ln n\right). \quad (22)$$

Combining Eqs. (20) and (22), we can obtain

$$\begin{aligned} V(I(0),H(0),S(0),R(0))+KT &\geq \mathbb{E}[1_{\Omega_n}V(I(\tau_n,\omega),H(\tau_n,\omega),S(\tau_n,\omega),R(\tau_n,\omega))] \\ &\geq \varepsilon\left[\left(n-a-a\ln\frac{n}{a}\right)\wedge(n-1-\ln n)\wedge\left(\frac{1}{n}-a+a\ln(na)\right)\wedge\left(\frac{1}{n}-1+\ln n\right)\right], \end{aligned} \quad (23)$$

where 1_{Ω_n} means the indicator function of Ω_n. Letting $n\to\infty$, we have

$$\infty > V(I(0),H(0),S(0),R(0))+KT = \infty, \quad (24)$$

which leads to a contradiction. Therefore, the claim $\tau_\infty=\infty$ a.s. holds.

Since $\tau_\infty\leq\tau_n$ a.s., we conclude that $\tau_e=\infty$ a.s. Hence, $(I(t),H(t),S(t),R(t))\in\mathbb{R}^4_+$ a.s. for all $t\geq 0$. The proof of Theorem 1 is complete.

Based on Theorem 1, we are ready to prove the result of Theorem 2. For convenience, we denote the constants

$$\begin{aligned} &\sigma := \sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2, \\ &N := I+H+S+R, \\ &N_\rho := I^{\rho+1}+H^{\rho+1}+S^{\rho+1}+R^{\rho+1}, \quad (25) \\ &M := \min_{(I,H,S,R)\in\overline{U}_n}\left\{\sigma_1^2I^2,\sigma_2^2H^2,\sigma_3^2S^2,\sigma_4^2R^2\right\}, \\ &\lambda := \left(\beta+\mu+\frac{1}{2}\sigma_3^2\right)(\widehat{\mathcal{R}}_0-1), \quad (26) \\ &m_1 := \frac{\Lambda f'(0)\theta}{\left(\mu+\frac{1}{2}\sigma_1^2\right)^2}, \\ &m_2 := \frac{\Lambda f'(0)\gamma(1-\theta)}{\left(\mu+\frac{1}{2}\sigma_1^2\right)^2\left(\gamma+\delta+\mu+\frac{1}{2}\sigma_2^2\right)}, \\ &m_3 := \frac{\Lambda f'(0)\gamma(1-\theta)}{\left(\mu+\frac{1}{2}\sigma_1^2\right)\left(\gamma+\delta+\mu+\frac{1}{2}\sigma_2^2\right)^2}. \quad (27) \end{aligned}$$

Proof of Theorem 2 From Eq. (10), we know the diffusion matrix of system (2) has the form

$$A(X) := \operatorname{diag}\left(\sigma_1^2I^2,\sigma_2^2H^2,\sigma_3^2S^2,\sigma_4^2R^2\right). \quad (28)$$

According to Proposition 1, we only need to verify the conditions (H1) and (H2). By elementary computations, we have

$$\begin{aligned} \sum_{i,j=1}^4 a_{ij}(x)\xi_i\xi_j &= \sigma_1^2I^2\xi_1^2+\sigma_2^2H^2\xi_2^2+\sigma_3^2S^2\xi_3^2+\sigma_4^2R^2\xi_4^2 \\ &\geq M|\xi|^2, \end{aligned} \quad (29)$$

for all $(I,H,S,R)\in\overline{U}_n:=[1/n,n]^4$ provided that $n\in\mathbb{Z}_+$ is sufficiently large. Note that $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^4$. Hence the condition (H1) holds.

To verify (H2), let us define the C^2-nonnegative function $Q(\cdot):\mathbb{R}^4_+\to\mathbb{R}$ by

$$Q(I,H,S,R) := pV_1+V_2-\ln I-\ln H-\ln R, \quad (30)$$

with

$$\begin{aligned} V_1 &:= -(m_1+m_2)\ln I-m_3\ln H-\ln S, \\ V_2 &:= \frac{1}{\rho+1}N^{\rho+1}. \end{aligned} \quad (31)$$

We assume that p and ρ are positive constants satisfying the conditions

$$-p\lambda+D\leq -2 \text{ and } 0<\rho<2\mu/\sigma, \quad (32)$$

where

$$\begin{aligned} D := \sup_{(I,H,S,R)\in\mathbb{R}^4_+}\Bigg\{&-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)N_\rho+A+3\mu \\ &+\gamma+\delta+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_4^2)\Bigg\}, \end{aligned} \quad (33)$$

and A is a positive constant determined later in Eq. (37). In addition, we denote the regions $\mathbb{U}_k:=(1/k,k)^4$ for $k\in\mathbb{Z}_+$. It is clear that

$$\liminf_{k\to\infty,(I,H,S,R)\in\mathbb{R}^4_+\setminus\mathbb{U}_k} Q(I,H,S,R) = +\infty. \quad (34)$$

It follows that $Q(I,H,S,R)$ is a continuous function which has a minimum at the point $(\bar{I}_0,\ \bar{H}_0,\ \bar{S}_0,\ \bar{R}_0)$ lying in the interior of $\mathbb{R}^4_+$. Thus we define the nonnegative function $V(\cdot)\in C^2(\mathbb{R}^4_+,\mathbb{R})$ by

$$V(I,H,S,R) := Q(I,H,S,R)-Q(\bar{I}_0,\bar{H}_0,\bar{S}_0,\bar{R}_0). \quad (35)$$

Since $f(u)/u$ is monotonically non-increasing on $(0,\infty)$, for any $\varepsilon>0$, the Itô's formula implies that there exists a constant φ_ε such that $f(u)/u>f'(0)-\varepsilon$ for all $0<u\leq\varphi_\varepsilon$. Then,

$$\begin{aligned} LV_1 = &-\frac{\Lambda}{I}(m_1+m_2)+(m_1+m_2)f(S)+m_1\left(\mu+\frac{1}{2}\sigma_1^2\right) \\ &+m_2\left(\mu+\frac{1}{2}\sigma_1^2\right)-\frac{m_3(1-\theta)If(S)}{H} \\ &+m_3\left(\gamma+\delta+\mu+\frac{1}{2}\sigma_2^2\right)-\frac{\theta If(S)}{S} \end{aligned}$$

$$
\begin{aligned}
&-\frac{\gamma H}{S}+\left(\beta+\eta+\mu+\frac{1}{2}\sigma_3^2\right)\\
\leq&\left(f'(0)-\frac{2f(S)}{S}\right)\frac{\Lambda\theta}{\mu+\frac{1}{2}\sigma_1^2}\\
&+\left(2f'(0)-\frac{3f(S)}{S}\right)\frac{\Lambda\gamma(1-\theta)}{\left(\mu+\frac{1}{2}\sigma_1^2\right)\left(\gamma+\delta+\mu+\frac{1}{2}\sigma_2^2\right)}\\
&+\left(\beta+\mu+\frac{1}{2}\sigma_3^2\right)+(m_1+m_2)f(S)\\
\leq&-\left(\beta+\mu+\frac{1}{2}\sigma_3^2\right)\left(\widehat{\mathcal{R}}_0-1\right)\\
&+\left(\frac{2\varepsilon\Lambda\theta}{\mu+\frac{1}{2}\sigma_1^2}+\frac{3\varepsilon\Lambda\gamma(1-\theta)}{\left(\mu+\frac{1}{2}\sigma_1^2\right)\left(\gamma+\delta+\mu+\frac{1}{2}\sigma_2^2\right)}\right.\\
&\left.+(m_1+m_2)f'(0)\right)S. \qquad (36)
\end{aligned}
$$

Moreover, by taking

$$
A:=\sup_{N\in(0,\infty)}\left\{\Lambda N^{\rho}-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)N^{\rho+1}\right\}<\infty, \qquad (37)
$$

it is easy to see

$$
\begin{aligned}
&LV_2=\Lambda N^{\rho}-\left(\mu-\frac{1}{2}\rho\sigma\right)N^{\rho+1}\leq A-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)N_{\rho},\\
&-L\ln I\leq-\frac{b}{I}+f'(0)S+\mu+\frac{1}{2}\sigma_1^2, \qquad (38)\\
&-L\ln H\leq-\frac{(1-\theta)(f'(0)-\varepsilon)SI}{H}+(\gamma+\delta+\mu)+\frac{1}{2}\sigma_2^2,\\
&-L\ln R=-\frac{\delta H}{R}-\frac{\beta S}{R}+\mu+\frac{1}{2}\sigma_4^2. \qquad (39)
\end{aligned}
$$

Summing up the above estimations, we can derive

$$
\begin{aligned}
LV\leq&-p\left(\beta+\mu+\frac{1}{2}\sigma_3^2\right)(\widehat{\mathcal{R}}_0-1)+p\left(\frac{2\varepsilon\Lambda\theta}{\mu+\frac{1}{2}\sigma_1^2}+\frac{3\varepsilon\Lambda\gamma(1-\theta)}{(\mu+\frac{1}{2}\sigma_1^2)(\gamma+\delta+\mu+\frac{1}{2}\sigma_2^2)}+(m_1+m_2)f'(0)\right)S\\
&+f'(0)S-\frac{1}{2}(\mu-\frac{1}{2}\rho\sigma)N_{\rho}-\frac{\Lambda}{I}-\frac{(1-\theta)(f'(0)-\varepsilon)SI}{H}-\frac{\delta H}{R}-\frac{\beta S}{R}+A+\gamma+\delta+3\mu+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_4^2). \qquad (40)
\end{aligned}
$$

Next, we consider the compact subset

$$
D_{\varepsilon_1}:=\left\{\varepsilon_1\leq I\leq 1/\varepsilon_1,\varepsilon_1\leq S\leq 1/\varepsilon_1,\varepsilon_1^3\leq H\leq 1/\varepsilon_1^3,\varepsilon_1^3\leq R\leq 1/\varepsilon_1^3\right\}, \qquad (41)
$$

where ε_1 is sufficiently small positive constant such that

$$
-p\lambda+(pB+f'(0))\varepsilon_1+D\leq-1,\quad E\leq-1+\min\left\{\frac{\Lambda}{\varepsilon_1},\frac{(1-\theta)(f'(0)-\varepsilon)}{\varepsilon_1},\frac{\beta}{\varepsilon_1^2}\right\}, \qquad (42)
$$

$$
F,G\leq-1+\frac{1}{4}\left(\mu-\frac{1}{2}\rho\sigma\right)\frac{1}{\varepsilon_1^{\rho+1}},\quad J,K\leq-1+\frac{1}{4}\left(\mu-\frac{1}{2}\rho\sigma\right)\frac{1}{\varepsilon_1^{3\rho+3}}, \qquad (43)
$$

in the region $\mathbb{R}^4_+\backslash D_{\varepsilon_1}$, while B E, F, G, J, and K are all positive constants defined by

$$
B:=\frac{2\varepsilon\Lambda\theta}{\mu+\frac{1}{2}\sigma_1^2}+\frac{3\varepsilon\Lambda\gamma(1-\theta)}{\left(\mu+\frac{1}{2}\sigma_1^2\right)\left(\gamma+\delta+\mu+\frac{1}{2}\sigma_2^2\right)}+(m_1+m_2)f'(0), \qquad (44)
$$

$$
E:=\sup_{(I,H,S,R)\in\mathbb{R}^4_+}\left\{-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)N_{\rho}+\left(pB+f'(0)\right)S+A+3\mu+\gamma+\delta+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_4^2\right)\right\}, \qquad (45)
$$

$$
\begin{aligned}
F:=&\sup_{(I,H,S,R)\in\mathbb{R}^4_+}\left\{-\frac{1}{4}\left(\mu-\frac{1}{2}\rho\sigma\right)I^{\rho+1}-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)\left(H^{\rho+1}+S^{\rho+1}+R^{\rho+1}\right)\right.\\
&\left.+\left(pB+f'(0)\right)S+A+3\mu+\gamma+\delta+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_4^2\right)\right\}, \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
G:=&\sup_{(I,H,S,R)\in\mathbb{R}^4_+}\left\{-\frac{1}{4}\left(\mu-\frac{1}{2}\rho\sigma\right)S^{\rho+1}-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)\left(I^{\rho+1}+H^{\rho+1}+R^{\rho+1}\right)\right.\\
&\left.+\left(pB+f'(0)\right)S+A+3\mu+\gamma+\delta+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_4^2\right)\right\}, \qquad (47)
\end{aligned}
$$

$$
\begin{aligned}
J:=&\sup_{(I,H,S,R)\in\mathbb{R}^4_+}\left\{-\frac{1}{4}\left(\mu-\frac{1}{2}\rho\sigma\right)H^{\rho+1}-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)\left(I^{\rho+1}+S^{\rho+1}+R^{\rho+1}\right)\right.\\
&\left.+\left(pB+f'(0)\right)S+A+3\mu+\gamma+\delta+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_4^2\right)\right\}, \qquad (48)
\end{aligned}
$$

$$K := \sup_{(I,H,S,R)\in\mathbb{R}^4_+}\left\{-\frac{1}{4}\left(\left[\mu-\frac{1}{2}\rho\sigma\right)R^{\rho+1}-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)\left(I^{\rho+1}+S^{\rho+1}+H^{\rho+1}\right)\right.$$
$$\left.+\left(pB+f'(0)\right)S+A+3\mu+\gamma+\delta+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_4^2\right)\right\}. \tag{49}$$

Let us claim that

$$LV\le -1,\ \text{for all}\ (I,H,S,R)\in\mathbb{R}^4_+\backslash D_{\varepsilon_1}. \tag{50}$$

To prove the claim (50), we divide the region $\mathbb{R}^4_+\backslash D_{\varepsilon_1}$ into the following eight subregions:

$$D_1:=\left\{(I,H,S,R)\in\mathbb{R}^4_+:0<I<\varepsilon_1\right\},\ D_2:=\left\{(I,H,S,R)\in\mathbb{R}^4_+:0<S<\varepsilon_1\right\}, \tag{51}$$
$$D_3:=\left\{(I,H,S,R)\in\mathbb{R}^4_+:I\ge\varepsilon_1,S\ge\varepsilon_1,0<H<\varepsilon_1^3\right\}, \tag{52}$$
$$D_4:=\left\{(I,H,S,R)\in\mathbb{R}^4_+:S\ge\varepsilon_1,0<R<\varepsilon_1^3\right\}, \tag{53}$$
$$D_5:=\left\{(I,H,S,R)\in\mathbb{R}^4_+:I>1/\varepsilon_1\right\},\ D_6:=\left\{(I,H,S,R)\in\mathbb{R}^4_+:S>1/\varepsilon_1\right\}, \tag{54}$$
$$D_7:=\left\{(I,H,S,R)\in\mathbb{R}^4_+:H>1/\varepsilon_1^3\right\},\ D_8:=\left\{(I,H,S,R)\in\mathbb{R}^4_+:R>1/\varepsilon_1^3\right\}. \tag{55}$$

It is sufficient to show the claim (50) holds in the regions D_1 and D_5, since the proof is similar in other regions.

Case 1 Assume $(I,H,S,R)\in D_1$.

Since ε is an arbitrary positive constant, by the inequalities of (42, we can obtain

$$LV\le -\frac{\Lambda}{I}-p\lambda+(pB+f'(0))S-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)N_\rho+A+\gamma+\delta+3\mu+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_4^2)\le -\frac{\Lambda}{I}+E\le -\frac{\Lambda}{\varepsilon_1}+E. \tag{56}$$

Then it follows from the second inequality of Eq. (42) that $LV\le -1$ for all $(I,H,S,R)\in D_1$.

Case 2 Assume $(I,H,S,R)\in D_5$.

By elementary computations, we have

$$LV\le -\frac{1}{4}\left(\mu-\frac{1}{2}\rho\sigma\right)I^{\rho+1}-\frac{1}{4}\left(\mu-\frac{1}{2}\rho\sigma\right)I^{\rho+1}-\frac{1}{2}\left(\mu-\frac{1}{2}\rho\sigma\right)\left(H^{\rho+1}+S^{\rho+1}+R^{\rho+1}\right)$$
$$+\left(pB+f'(0)\right)S+A+\gamma+\delta+3\mu+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_4^2\right)\le -\frac{1}{4}\left(\mu-\frac{1}{2}\rho\sigma\right)\frac{1}{\varepsilon^{\rho+1}}+F. \tag{57}$$

Then it follows from the F-inequality of Eq. (43) that $LV\le -1$ for all $(I,H,S,R)\in D_5$.

By the first inequality of Eq. (42) and similar argument for Case 1, we can also prove that $LV\le -1$ for all $(I,H,S,R)\in D_2\cup\in D_3\cup\in D_4$. Similarly, by the G,J,K-inequalities of Eq. (43) and similar argument for Case 2, we have $LV\le -1$ for all $(I,H,S,R)\in D_6\cup\in D_7\cup\in D_8$. Therefore, the claim (50) holds, i.e., the condition (H2) holds.

By Proposition 1, we conclude that system (2) admits a unique stationary distribution which exhibits the ergodicity. The proof of Theorem 2 is complete.

4. Numerical simulations

To better understand the theoretical results, we carry out some numerical simulations to illustrate the dynamic results of the stochastic rumor model under some parameters. Let us choose

$$f(S)=\frac{\alpha S}{1+mS}\ \text{with}\ \alpha=0.4,\ m=1.5.$$

Then, by using the Milstein scheme,[11] the following discretization system is obtained:

$$I_{k+1}=I_k+\left(b-\frac{\alpha S_kI_k}{1+mS_k}-\mu I_k\right)\Delta t$$
$$+\sigma_1I_k\sqrt{\Delta t}\tau_{1,k}+\frac{\sigma_1^2}{2}I_k\Delta t\left(\tau_{1,k}^2-1\right),$$
$$H_{k+1}=H_k+\left(\frac{(1-\theta)\alpha S_kI_k}{1+mS_k}-(\gamma+\delta+\mu)H_k\right)\Delta t$$
$$+\sigma_2H_k\sqrt{\Delta t}\tau_{2,k}+\frac{\sigma_2^2}{2}H_k\Delta t\left(\tau_{2,k}^2-1\right),$$
$$S_{k+1}=S_k+\left(\frac{\theta\alpha S_kI_k}{1+mS_k}+\gamma H_k-(\beta+\mu)S_k\right)\Delta t$$
$$+\sigma_3S_k\sqrt{\Delta t}\tau_{3,k}+\frac{\sigma_3^2}{2}S_k\Delta t\left(\tau_{3,k}^2-1\right),$$
$$R_{k+1}=R_k+(\delta H_k+\beta S_k-\mu R_k)\Delta t$$
$$+\sigma_4R_k\sqrt{\Delta t}\tau_{4,k}+\frac{\sigma_4^2}{2}R_k\Delta t\left(\tau_{4,k}^2-1\right), \tag{58}$$

where $\tau_{i,k}$ $(i=1,2,3,4)$ are four independent the Gaussian random variables with $N(0,1)$ and time increment $\Delta t>0$. In

addition, we assume that the values of parameters are as follows:

$$b=1,\ \ \gamma=0.5,\ \ \theta=0.35,\ \ \alpha=0.4,$$
$$\mu=0.3,\ \ \delta=0.015,\ \ \beta=0.2.$$

Next, through the numerical simulations of discretization system (58), we discuss its influence on rumor propagation from the following two aspects: the intensities of noise and key parameters.

4.1. Impact of the intensities of noise

By Eq. (3), we have

$$\widehat{\mathcal{R}}_0=\begin{cases}1.9230, & \text{when } \sigma_1=\sigma_2=\sigma_3=\sigma_4=0.1,\\ 1.9785, & \text{when } \sigma_1=\sigma_2=\sigma_3=\sigma_4=0.02.\end{cases} \tag{59}$$

Taking the same initial value $(I(0),H(0),S(0),R(0))=(0.8,0.2,0.5,0.1)$, the numerical results for the distribution of solution and density function of system (2) with different intensity of the white noise are given in Figs. 1 and 2, respectively.

According to the above numerical simulations, we can conclude that

- the intensity of white noise decreases, so does the area of the oscillating region;
- for the case with higher noise intensity (cf. Fig. 1), the distribution of the solution inclines to the left of the normal distribution, while the fluctuation amplitude is remarkable;
- for the case with lower noise intensity (cf. Fig. 2), the distribution of the solution is close to the normal distribution, while the fluctuation amplitude is slight.

From their density functions, one can also see that the solution of system (2) has a stationary distribution. In addition, the numerical results reveal the following interesting fact.

When $\widehat{\mathcal{R}}_0>1$, if the intensity of the white noise is sufficiently small, then system (2) will maintain a certain weak stability, which means that the rumor persists. In other words, the extent to which the government intervenes in rumor propagation determines whether rumors spread further in social media.

4.2. Impact of different parameters

In addition to the stationary distribution of model (2), we are also interested in what parameters will affect rumor propagation. Firstly, we analyze the contact transmission rate α, which is of great significance in communication because it is considered to be a key indicator to reflect whether rumors can be further spread. Here, under the same values of other parameters, we select different transmission rate $\alpha=0.1,\ 0.4,$ and 0.8 to study their influence on rumor propagation.

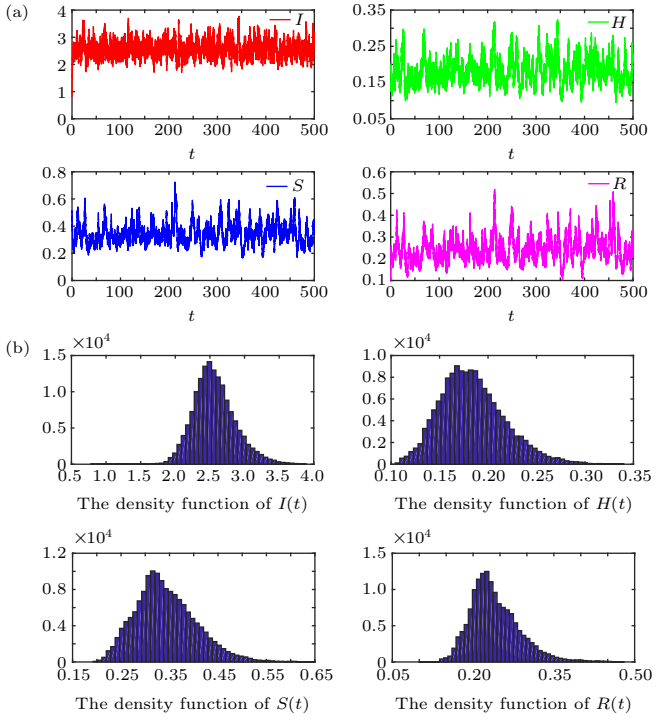


Fig. 1. Stationary distribution of system (2) with $\sigma_1=\sigma_2=\sigma_3=\sigma_4=0.1$. Panels (a) and (b) represent the solution and the density function respectively.

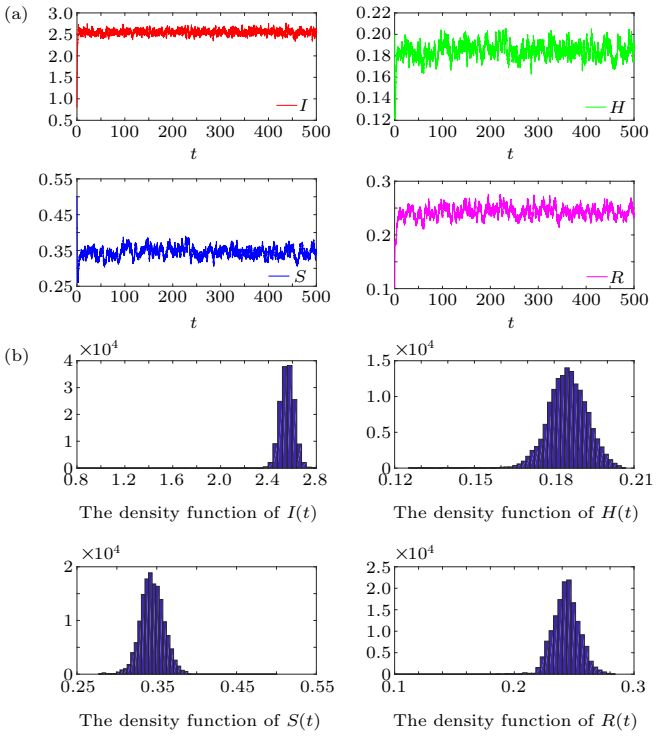


Fig. 2. Stationary distribution of system (2) with $\sigma_1=\sigma_2=\sigma_3=\sigma_4=0.02$. Panels (a) and (b) represent the solution and the density function respectively.

From Fig. 3, we observe that when the value of the transmission rate α is small, the number of rumor spreaders $S(t)$ will decrease rapidly and is close to zero; when the value of the transmission rate α is big, the number of rumor spreaders $S(t)$ will increase in the fluctuation, and the range of the oscillation will also keep increasing. This shows that reasonably cutting off rumor propagation channels is an effective method

to curb rumor propagation on social networks. Especially in the case of emergencies, it is necessary to limit the flow, screen the relevant accounts and other methods to avoid serious negative impact.

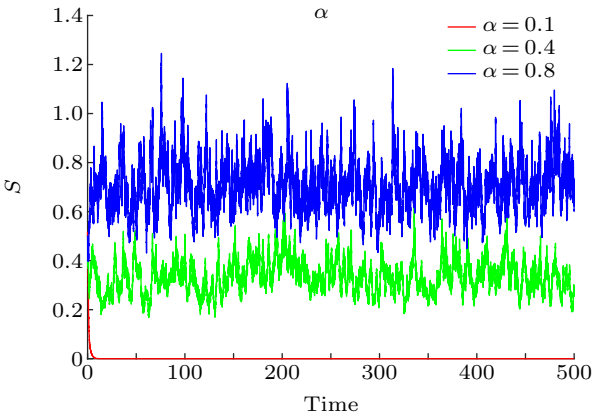


Fig. 3. Trajectories simulations of rumor spreaders $S(t)$ of stochastic rumor propagation model (2) is graphed for $\alpha = 0.1$, 0.4, and 0.8.

Moreover, we discuss the influence of the conversion rate from rumor-hesitating individuals to spreaders on rumor propagation. Thus, we change the value of γ while keeping all other parameter values unchanged. As shown in Fig. 4, one can also see that as the value of γ increases, the number of rumor spreaders increases, and the amplitude of oscillation also increases significantly under the interference of social environment noise. This shows that netizens express their opinions on rumors, which lead to rumor-hesitating individuals can not correctly judge the truth or falseness of information, so that the number of people fluctuates at every moment. In the meanwhile, some rumor-hesitating individuals believe and forward false information only based on the comments of some netizens, resulting in the continuous spread of rumors on the internet. Therefore, government departments should publicize the truth of emergencies to the public in a timely manner, eliminate the doubts of rumor-hesitant individuals, reduce the conversion rate from rumor-hesitant individuals to rumor spreaders, and effectively curb the spread of rumors.

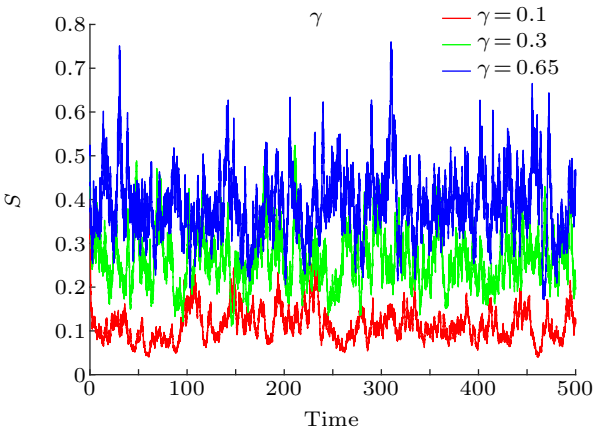


Fig. 4. Trajectories simulations of rumor spreaders $S(t)$ of stochastic rumor propagation model (2) is graphed for $\gamma = 0.1$, 0.3, and 0.65.

Besides the above mentioned, we believe that the scale of rumor spreaders is closely related to individual factors (such as the degree of interest in rumor propagation, and the degree of information memory and forgetting). Therefore, we investigate the influence of β on the rumor spreaders under the premise of other parameter values unchanged. From Fig. 5, we can see that with the increase of the value of β, the number of rumor spreaders shows a downward trend, and the amplitude of the oscillation also decreased significantly. This illustrates that when β is small, rumor spreaders are vulnerable to the comments of netizens and constantly forward rumors to their friends around them; With the increase of β, the attraction of rumor propagation decreases, which makes the rumor spreaders lose interest in rumor propagation or forget rumors and have no intention to spread, and reduce the influence of rumor propagation. Therefore, government needs to strengthen science education for netizens, and timely release effective information to clarify the facts when an emergency occurs,then the value of β will increase correspondingly, so that the rumor will have less influence.

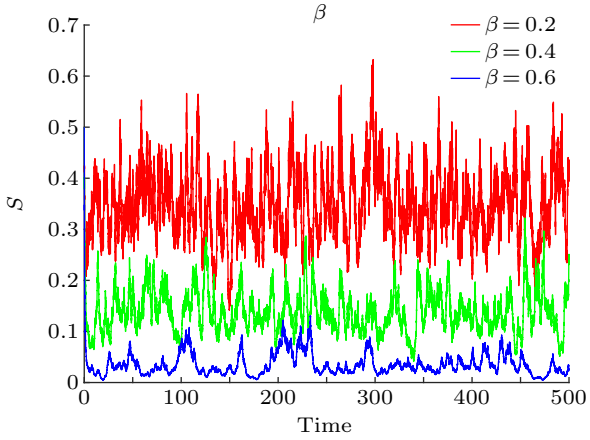


Fig. 5. Trajectories simulations of rumor spreaders $S(t)$ of stochastic rumor propagation model (2) is graphed for $\beta = 0.2$, 0.4, and 0.6.

5. A brief conclusion

In this paper, we propose and consider the dynamics of stochastic rumor propagation model (2) with general incidence function. In fact, we extend the classical rumor propagation model by introducing the random interference, and discuss the influence of random factors on the rumor propagation. More precisely, we show the existence of global positive solution of system (2) and provide some sufficient conditions to guarantee the existence of a unique ergodic stationary distribution. Furthermore, the numerical simulation results show that when $\widehat{\mathcal{R}}_0 > 1$, the noise intensity has a great influence on rumor propagation. This means that when rumors diffuse in social media, the intensity of government control will play a key role in the rumor propagation. In addition, the change of parameter value will significantly affect the number of rumor spreaders, so as to provide decision-making departments with effective

methods to control rumor propagation. Therefore, this work is helpful to analyze the rumor spreading model in theory and put forward corresponding measures for controlling rumor propagation. In practice, we can change the behavior of netizens by strengthening government intervention, authoritative media reports, publicity and education to wake up the awareness of netizens and other environmental interference factors, so as to reduce the interaction among ignorants, rumor-hesitant individuals and spreaders.

In addition to Theorems 1 and 2, we further illustrate the following interesting topics and leave them for our future consideration.

- Examining the existence of positive periodic solutions of system (2).
- Investigating the strategy to effectively control the dynamics of system (2).
- Extending our results to the case with Lévy noise, or other non-autonomous systems.

Acknowledgments

Project supported by the Funding for Outstanding Doctoral Dissertation in NUAA (Grant No. BCXJ18-09), the National Natural Science Foundation of China (Grant No. 72071106), and Postgraduate Research & Practice Innovation Program of Jiangsu Province, China (Grant No. KYCX18_0234).

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