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Coherence migration in high-dimensional bipartite systems

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The conservation law for first-order coherence and mutual correlation of a bipartite qubit state was firstly proposed by Svozilík *et al.*, and their theories laid the foundation for the study of coherence migration under unitary transformations. In this paper, we generalize the framework of first-order coherence and mutual correlation to an arbitrary $(m\otimes n)$-dimensional bipartite composite state by introducing an extended Bloch decomposition form of the state. We also generalize two kinds of unitary operators in high-dimensional systems, which can bring about coherence migration and help to obtain the maximum or minimum first-order coherence. Meanwhile, the coherence migration in open quantum systems is investigated. We take depolarizing channels as examples and establish that the reduced first-order coherence of the principal system over time is completely transformed into mutual correlation of the $(2\otimes 4)$-dimensional system-environment bipartite composite state. It is expected that our results may provide a valuable idea or method for controlling the quantum resource such as coherence and quantum correlations.

Keywords: first-order coherence, mutual correlation, coherence migration, high-dimensional

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1. Introduction

Coherence, being an inevitable consequence of the superposition principle, is one of the most important concepts in quantum optics research, which can be used to describe the interference capability of interacting fields and the characteristics of photon stream.[1–5] It even plays an important role in some interdisciplinary fields, such as quantum thermodynamics,[6–8] quantum metrology,[9–11] and quantum biology.[12–14] To quantify coherence of a quantum state, many measurements have been proposed from different perspectives. Some are based on quantum optical methods,[15,16] and others are based on the viewpoints of resource theory.[17–20] Although coherence is usually considered to be the characteristics of the whole physical system, we need to investigate the internal distribution of coherence between subsystems and their correlations for predicting coherence evolution (migration). For a composite quantum state, coherence between subsystems can be looked as its ability to create entanglement.[21,22] Therefore, it is of great significance to explore the rule of coherence migration for a bipartite composite system.

In 2015, Svozilík *et al*. proposed a conservation law between the first-order coherence and mutual correlation for arbitrary bipartite qubit states under global unitary transformations. They analyzed the characteristics of coherence migration and introduced two unitary operators for obtaining minimum or maximum first-order coherence for a given bipartite qubit state.[15] Meanwhile, Ge *et al.* studied coherence migration and conservation relation of nonclassicality and entanglement for Gaussian states in a beam splitter.[23] Especially, Černoch *et al.* demonstrated the theory of coherence migration through two interesting experimental schemes, among which, one is based on linear optical controlled-phase quantum gate and the other employs nonlinear optical process.[24] Despite the inspiring success in theory and experiment, the conservation law for coherence and correlation are carried out only for a bipartite qubit state. It is then natural to question whether there is a more general conclusion for high-dimensional bipartite composite state. Inspired by the aforementioned research, we consider such a problem in this paper. We firstly construct an extended Bloch decomposition form of an arbitrary $(m\otimes n)$-dimensional bipartite composite state, and then generalize the concepts such as first-order coherence, mean coherence, mutual correlation and the conservation law to an arbitrary high-dimensional system. Meanwhile, coherence migration under global unitary transformations in high-dimensional systems is investigated, and two kinds of unitary operators are generalized. We also show through explicit examples its capacity for obtaining the maximum and minimum first-order coherence.

On the other hand, the evolution of quantum states in open systems is an important research content of quantum information science.[25] It goes without saying that the inescapable interaction between a quantum system and its surrounding environment may lead to decoherence.[26–31] A popular method

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to investigate the dynamics of an open quantum system is to regard the principal system and the interacting environment as a larger composite closed quantum state, whose evolution can be described by a unitary transformation.[25] Here, this composite state is called as system-environment bipartite composite state. In general, the dimensions of the principal system and the environment are not the same. With assuming that the dimension of the system is d, the environment can be modeled in a Hilbert space of no more than n ($n \leq d^2$) dimensions. Therefore, the system-environment state can be regarded as a typical $(d \otimes n)$-dimensional bipartite composite state. In this work, we explore the coherence migration of an arbitrary qubit system in a depolarizing channel in detail. The results show that the reduced first-order coherence of the principal system over time is transformed into mutual correlation of the system-environment bipartite composite state. The study of coherence migration in high-dimensional systems helps us not only understand the characteristics of first-order coherence and mutual correlation from the perspective of resource theory, but also design different schemes to enhance or weaken coherence. It is worth mentioning that our results might present a new mind of manipulating coherence and quantum correlation.

2. Conservation law of coherence and correlation in high-dimensional bipartite systems

The qubit is a fundamental concept of quantum computation and quantum information. Any single qubit state can be graphically represented by the Bloch vector.[25] If we extend the concept of qubit from two dimensions to finite dimensions, we use qudit to represent a quantum state in d-dimensional Hilbert space. In this section, we first review the decomposition form of a qudit state, and propose an extended Bloch representation for any $(m \otimes n)$-dimensional bipartite composite state. Then, we generalize the framework of first-order coherence, mean coherence, mutual correlation, and conservation law for coherence and correlation.

2.1. High-dimensional bipartite composite state

An arbitrary single qudit state in the finite d-dimensional Hilbert space can be decomposed as[32]

$$\rho = \frac{1}{d}\mathbb{I}_d + \frac{1}{2}\boldsymbol{r}\cdot\boldsymbol{\lambda} = \frac{1}{d}\mathbb{I}_d + \frac{1}{2}\sum_{i=1}^{d^2-1} r_i\lambda_i, \tag{1}$$

where $\mathbb{I}_d$ is the $d \times d$ identity matrix, $\boldsymbol{r}\{r_1, r_2, \ldots, r_{d^2-1}\}$ is the extended Bloch vector of the state with $r_i = \mathrm{Tr}(\rho\lambda_i)$, and $\{\lambda_1, \lambda_2, \ldots, \lambda_{d^2-1}\}$ are generators of the Lie algebra SU(d). For example, $\{\lambda_1, \lambda_2, \lambda_3\}$ (the most commonly used symbols are $\{\sigma_1, \sigma_2, \sigma_3\}$) are Pauli matrices for $d = 2$, and $\{\lambda_1, \lambda_2, \ldots, \lambda_8\}$ are Gell-Mann matrices[33] for $d = 3$.

The systematic construction of generators of SU(d) are given by[32]

$$\{\lambda_i\}_{i=1}^{d^2-1} = \{u_{jk}, v_{jk}, w_l\}, \tag{2}$$

where $u_{jk} = |j\rangle\langle k| + |k\rangle\langle j|$, $v_{jk} = -\mathrm{i}(|j\rangle\langle k| - |k\rangle\langle j|)$, $w_l = \sqrt{\frac{2}{l(l+1)}}(\sum_{j=1}^{l}|j\rangle\langle j| - l|l+1\rangle\langle l+1|)$ with $1 \leq j < k \leq d$ and $1 \leq l \leq d-1$, in which $\{|1\rangle, |2\rangle, \ldots, |j\rangle\}_{j=1}^{d}$ is a complete orthonormal basis of the d-dimensional Hilbert space $\mathcal{H}$.

Consider an arbitrary bipartite state in a composite Hilbert space $\mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$ with $\dim\mathcal{H}_\mathrm{A} = m$ and $\dim\mathcal{H}_\mathrm{B} = n$. Any $(m \otimes n)$-dimensional bipartite composite state ρ_AB can be expanded as

$$\begin{aligned}\rho_\mathrm{AB} &= \frac{1}{mn}(\mathbb{I}_m \otimes \mathbb{I}_n) + \frac{1}{2n}(\boldsymbol{x}\cdot\boldsymbol{\lambda} \otimes \mathbb{I}_n) \\ &\quad + \frac{1}{2m}(\mathbb{I}_m \otimes \boldsymbol{y}\cdot\boldsymbol{\lambda}) + \frac{1}{4}(T\boldsymbol{\lambda} \otimes \boldsymbol{\lambda}) \\ &= \frac{1}{mn}(\mathbb{I}_m \otimes \mathbb{I}_n) + \frac{1}{2n}\sum_{i=1}^{m^2-1}(x_i\lambda_i \otimes \mathbb{I}_n) \\ &\quad + \frac{1}{2m}\sum_{j=1}^{n^2-1}(\mathbb{I}_m \otimes y_j\lambda_j) + \frac{1}{4}\sum_{i=1}^{m^2-1}\sum_{j=1}^{n^2-1}(t_{ij}\lambda_i \otimes \lambda_j),\end{aligned} \tag{3}$$

where $\mathbb{I}_{m(n)}$ is the $m \times m$ ($n \times n$) identity matrix, $\boldsymbol{x}\{x_1, x_2, \ldots, x_{m^2-1}\}$ and $\boldsymbol{y}\{y_1, y_2, \ldots, y_{n^2-1}\}$ are the corresponding extended Bloch vector of subsystems A and B with $x_i = \mathrm{Tr}(\rho_\mathrm{AB}\lambda_i \otimes \mathbb{I}_n)$, $y_j = \mathrm{Tr}(\rho_\mathrm{AB}\mathbb{I}_m \otimes \lambda_j)$, and T is the correlation tensor of composite system with $t_{ij} = \mathrm{Tr}(\rho_\mathrm{AB}\lambda_i \otimes \lambda_j)$.

2.2. Conservation law for coherence and correlation

For a $(m \otimes n)$-dimensional bipartite composite state ρ_AB, as shown in Eq. (3), composed of subsystems A and B, we can calculate the reduced matrices of each subsystem as a partial trace, i.e., $\rho_\mathrm{A} = \mathrm{Tr}_\mathrm{B}(\rho_\mathrm{AB})$ and $\rho_\mathrm{B} = \mathrm{Tr}_\mathrm{A}(\rho_\mathrm{AB})$. By promoting the definition of Ref. [15], the degree of first-order coherence of each subsystem can be written as

$$D_\mathrm{A}^2 = m\mathrm{Tr}(\rho_\mathrm{A}^2) - 1 = \frac{m}{2}|\boldsymbol{x}|^2, \tag{4a}$$

$$D_\mathrm{B}^2 = n\mathrm{Tr}(\rho_\mathrm{B}^2) - 1 = \frac{n}{2}|\boldsymbol{y}|^2, \tag{4b}$$

where $|\boldsymbol{x}|^2 = \sum_{i=1}^{m^2-1} x_i^2$ and $|\boldsymbol{y}|^2 = \sum_{j=1}^{n^2-1} y_j^2$ are the square of the vector modules of subsystems A and B, respectively. It should be pointed out that the physical meaning of the first-order coherence for each subsystem represents the length of the extended Bloch vector. When $m = n = 2$, the definition of the first-order coherence in Eqs. (4a) and (4b) return to the original definition in Ref. [15]. Therefore, the mean coherence of first-order coherence for the bipartite composite state is defined as

$$\mathcal{D} = \frac{D_\mathrm{A}^2 + D_\mathrm{B}^2}{mn} = \frac{m|\boldsymbol{x}|^2 + n|\boldsymbol{y}|^2}{2mn}. \tag{5}$$

On the other hand, we also generalize the definition of the mutual correlation between the two high-dimensional subsystems in Ref. [24] as

$$\mathcal{T} = \frac{1}{mn} + \frac{1}{4}\sum_{i=1}^{m^2-1}\sum_{j=1}^{n^2-1} t_{ij}^2. \tag{6}$$

Obviously, the mean coherence is only determined by the parameters of subsystems and independent of the mutual correlation between two subsystems. For the bipartite composite state $\rho_{\rm AB}$, there is a concise conservation law between mean coherence and mutual correlation

$$\mathcal{D} + \mathcal{T} = \mathcal{P}, \tag{7}$$

where $\mathcal{P} = {\rm Tr}(\rho_{\rm AB}^2)$ is the purity of the high-dimensional bipartite composite state. We know that if the evolution of the composite state is unitary transformation, the purity will remain unchanged. That indicates when the mean coherence decreases (increases), the mutual correlation increases (decreases). We refer to the change of coherence between subsystems in this process as coherence migration.

3. Coherence migration under global unitary transformations

For simplicity, we assume that the dimensions of the two subsystems of the bipartite composite system are consistent. Now the research object $\rho_{\rm AB}$ is limited to an arbitrary bipartite state in a composite Hilbert space $\mathcal{H}_{\rm A}\otimes\mathcal{H}_{\rm B}$ with $\dim\mathcal{H}_{\rm A} = \dim\mathcal{H}_{\rm B} = d$. We know that the dynamics of a closed quantum system are described by a unitary transformation, which does not change the purity of the system. Therefore, if the global unitary operator acting on the quantum state $\rho_{\rm AB}$ is represented by U, the evolution state can be expressed as $\rho'_{\rm AB} = U\rho_{\rm AB}U^\dagger$.

3.1. Maximum first-order coherence

Any $(d\otimes d)$-dimensional density matrix $\rho_{\rm AB}$ can always be diagonalized in its eigenvectors $\{|\psi_k\rangle\}_{k=1}^{d^2}$ spanned by its corresponding eigenvalues $\{\chi_k\}_{k=1}^{d^2}$, which is called spectral decomposition,

$$\rho_{\rm AB} = \sum_{k=1}^{d^2}\chi_k|\psi_k\rangle\langle\psi_k| = VEV^\dagger, \tag{8}$$

where the diagonal matrix $E = {\rm diag}\{\chi_1,\chi_2,\ldots,\chi_{d^2}\}$ contains the eigenvalues with $\sum_{i=1}^{d^2}\chi_k = 1$, and the matrix $V = \{|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_{d^2}\rangle\}$ consists of the corresponding eigenvectors, which constitute a complete set of the orthogonal basis.

For any $(d\otimes d)$-dimensional bipartite composite state $\rho_{\rm AB}$, there exists a global unitary operator $U_1 = V^\dagger$. When it acts on $\rho_{\rm AB}$ and transforms it into the evolution state $\rho'_{\rm AB} = U_1\rho_{\rm AB}U_1^\dagger = E$, $\rho'_{\rm AB}$ will obtain the maximum first-order coherence. The certification process is given in Appendix A. The reduced matrix of the subsystem of the evolution state is represented by $\rho'_{\rm A} = {\rm Tr_B}(\rho'_{\rm AB})$ and $\rho'_{\rm B} = {\rm Tr_A}(\rho'_{\rm AB})$, respectively. So the degree of first-order coherence of each subsystem can be written as

$$D_{\rm A}^2(\rho'_{\rm AB}) = d\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\chi_{j+(i-1)d}\right)^2 - 1, \tag{9a}$$

$$D_{\rm B}^2(\rho'_{\rm AB}) = d\sum_{j=1}^{d}\left(\sum_{i=1}^{d}\chi_{j+(i-1)d}\right)^2 - 1. \tag{9b}$$

Therefore, the mean coherence of the maximum first-order coherence is easily acquired with

$$\mathcal{D}_{\max} = \frac{D_{\rm A}^2(\rho'_{\rm AB}) + D_{\rm B}^2(\rho'_{\rm AB})}{d^2}. \tag{10}$$

This value is also known as the maximal accessible first-order coherence, which can be obtained by the global unitary transformation. The increased coherence comes from the decrease of mutual correlation between the two subsystems. Throughout the process, the purity of the bipartite composite system remains unchanged with the value of $\mathcal{P} = \sum_{k=1}^{d^2}\chi_i^2$. Simultaneously, the mutual correlation between subsystems reaches the minimum as

$$\mathcal{T}_{\min} = \mathcal{P} - \mathcal{D}_{\max}. \tag{11}$$

Next, we consider a typical mixed state, the isotropic state.[34] The isotropic state is of the form $\rho_{\rm AB} = p|\Psi\rangle\langle\Psi| + (1-p)\mathbb{I}_{d^2}/d^2$ in which $|\Psi\rangle = \sum_{j=1}^{d}|jj\rangle/\sqrt{d}$ is the $(d\otimes d)$-dimensional maximally entangled state, $\{|j\rangle\}_{j=1}^{d}$ denotes the computational basis on d-dimensional Hilbert space, and $p\in[0,1]$ is a parameter describing the mixing coefficient. The isotropic state here is also called Werner state for $d=2$. For the $(3\otimes3)$-dimensional case, the isotropic state is defined as

$$\rho_{\rm AB}^{\rm iso} = p|\Psi\rangle\langle\Psi| + (1-p)\mathbb{I}_9/9, \tag{12}$$

where the 3-dimensional maximally entangled state $|\Psi\rangle = (|11\rangle + |22\rangle + |33\rangle)/\sqrt{3}$. Obviously, the degree of first-order coherence of each subsystem $D_{\rm A}^2(\rho_{\rm AB}) = D_{\rm B}^2(\rho_{\rm AB}) = 0$, so the mean coherence of first-order coherence and the mutual correlation between the two subsystems can be rewritten as

$$\mathcal{D}(\rho_{\rm AB}^{\rm iso}) = 0, \quad \mathcal{T}(\rho_{\rm AB}^{\rm iso}) = \mathcal{P}(\rho_{\rm AB}^{\rm iso}) = \frac{1}{9}(1+8p^2). \tag{13}$$

The global unitary operator $U_1 = V^\dagger$ can be obtained by spectral decomposition of the isotropic state. According to Eqs. (10) and (11), we can calculate the maximum of mean coherence and the minimum of mutual correlation as

$$\mathcal{D}_{\max} = \frac{4}{9}p^2, \quad \mathcal{T}_{\min} = \frac{1}{9}\left(1+4p^2\right). \tag{14}$$

The mean coherence and mutual correlation as a function of parameter p of the isotropic state $\rho_{\rm AB}^{\rm iso}$ are shown in

Figs. 1(a) and 1(b). In these figures, the dashed lines and solid lines represent the value of the original state and evolutionary state by the global unitary operation, respectively. The gray areas represent allowable values achievable by arbitrary global unitary transformations. We prepare 80000 random unitary operators by numerical simulation, and each gray point in the figures represents the evolution state after a possible unitary transformation.

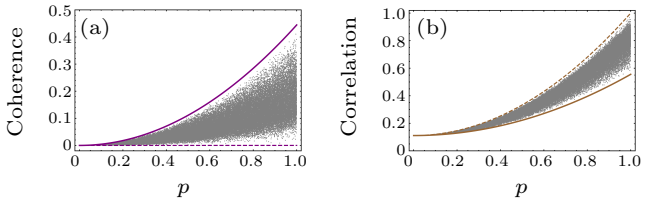


Fig. 1. The (a) mean coherence and (b) mutual correlation as a function of parameter p of the isotropic state. Purple dashed line and brown dashed line depict the values of the mean coherence and the mutual correlation of the original state. The maximum of mean coherence and the minimum of mutual correlation are marked by purple solid line and brown solid line, respectively. The gray areas represent allowable values achievable by arbitrary global unitary transformations.

3.2. Minimum first-order coherence

For the $(d\otimes d)$-dimensional composite Hilbert space, there are d^2 mutually orthogonal maximally entangled states denoted as $\{|\Psi_i\rangle\}_i^{d^2}$. The matrix $M=\{|\Psi_1\rangle,|\Psi_2\rangle,\ldots,|\Psi_{d^2}\rangle\}$ consists of the maximally entangled states. There exists a global unitary operator $U_2=MV^\dagger$ that acts on the initial $(d\otimes d)$-dimensional bipartite composite state $\rho_{\rm AB}$ and turns it into a new composite state $\rho''_{\rm AB}=U_2\rho_{\rm AB}U_2^\dagger=\sum_{i=1}^{d^2}\chi_i|\Psi_i\rangle\langle\Psi_i|$. $\rho''_{\rm AB}$ is called the Bell diagonal state, so that the first-order coherence of each subsystem and the mean coherence vanish and the mutual correlation can reaches the maximum

$$\mathcal{T}_{\max}=\mathcal{P}=\sum_{i=1}^{d^2}\chi_i^2. \tag{15}$$

In this process of the global unitary transformation, all the first-order coherence of the bipartite composite system is transformed into the mutual correlation and the purity of the system remains unchanged.

Now let us take the maximally coherent state as an example. A $(d\otimes d)$-dimensional maximally coherent state is given by $|\Phi\rangle=(1/d)\sum_{i,j=1}^{d}|ij\rangle$ and any $(d\otimes d)$-dimensional state may be prepared from $|\Phi\rangle$ with incoherent operations.[17,35] For the $(3\otimes 3)$-dimensional case, the density matrix of the maximally coherent state can be written as

$$\rho_{\rm AB}^{\rm mcs}=\frac{1}{9}\sum_{i,j=1}^{3}\sum_{m,n=1}^{3}|ij\rangle\langle mn|. \tag{16}$$

According to Eqs. (4a) and (4b), we can calculate the degree of first-order coherence of each subsystem as $D_{\rm A}^2(\rho_{\rm AB}^{\rm mcs})=D_{\rm B}^2(\rho_{\rm AB}^{\rm mcs})=2$. Thus the mean coherence and the mutual correlation can be calculated as

$$\mathcal{D}(\rho_{\rm AB}^{\rm mcs})=\frac{4}{9},\quad \mathcal{T}(\rho_{\rm AB}^{\rm mcs})=\frac{5}{9}. \tag{17}$$

In $(3\otimes 3)$-dimensional composite Hilbert space, the 9 mutually orthogonal maximally entangled states can be given as[36]

$$\begin{aligned}
|\Psi_1\rangle&=(|11\rangle+|22\rangle+|33\rangle)/\sqrt{3},\\
|\Psi_2\rangle&=(|11\rangle+{\rm e}^{2\pi{\rm i}/3}|22\rangle+{\rm e}^{4\pi{\rm i}/3}|33\rangle)/\sqrt{3},\\
|\Psi_3\rangle&=(|11\rangle+{\rm e}^{4\pi{\rm i}/3}|22\rangle+{\rm e}^{2\pi{\rm i}/3}|33\rangle)/\sqrt{3},\\
|\Psi_4\rangle&=(|12\rangle+|23\rangle+|31\rangle)/\sqrt{3},\\
|\Psi_5\rangle&=(|12\rangle+{\rm e}^{2\pi{\rm i}/3}|23\rangle+{\rm e}^{4\pi{\rm i}/3}|31\rangle)/\sqrt{3},\\
|\Psi_6\rangle&=(|12\rangle+{\rm e}^{4\pi{\rm i}/3}|23\rangle+{\rm e}^{2\pi{\rm i}/3}|31\rangle)/\sqrt{3},\\
|\Psi_7\rangle&=(|13\rangle+|21\rangle+|32\rangle)/\sqrt{3},\\
|\Psi_8\rangle&=(|13\rangle+{\rm e}^{2\pi{\rm i}/3}|21\rangle+{\rm e}^{4\pi{\rm i}/3}|32\rangle)/\sqrt{3},\\
|\Psi_9\rangle&=(|13\rangle+{\rm e}^{4\pi{\rm i}/3}|21\rangle+{\rm e}^{2\pi{\rm i}/3}|32\rangle)/\sqrt{3}.
\end{aligned} \tag{18}$$

These states form the matrix $M=\{|\Psi_1\rangle,|\Psi_2\rangle,\ldots,|\Psi_9\rangle\}$, thus we can get the expression of the global unitary operator $U_2=MV^\dagger$, which can help us obtain the minimum of mean coherence and the maximum of mutual correlation. After simple calculation, the effect of U_2 is to transform the maximally coherent state $\rho_{\rm AB}^{\rm mcs}$ into the maximally entangled state $|\Psi_1\rangle\langle\Psi_1|$. Therefore, the mean coherence disappears and the mutual correlation reaches the maximum $\mathcal{T}_{\max}=1$. We say that all the lost coherence migrate into correlation.

4. Coherence migration in open quantum systems

The dynamics of open quantum systems studies the interaction between the quantum system and its surrounding environment. A common method to investigate the dynamics of an open quantum system is to regard the principal system and the interacting environment as a larger composite closed quantum state, whose evolution can be described by a unitary transform.[25] Suppose we have a principal quantum system $\rho_{\rm S}$ in d-dimensional Hilbert space, which is subjected to a dynamical evolution. Meanwhile, we assume that $\{|i_{\rm E}\rangle\}_{i=1}^{n}$ is an orthonormal basis for the state space of environment, which can be modeled in a Hilbert space of no more than n $(n\le d^2)$ dimensions. There is no loss of generality in assuming that the environment starts in a pure state. So that $\rho_{\rm E}=|1_{\rm E}\rangle\langle 1_{\rm E}|$ represents the initial state of interacting environment. The system-environment bipartite composite state is a typical $(d\otimes n)$-dimensional product state $\rho_{\rm SE}=\rho_{\rm S}\otimes|1_{\rm E}\rangle\langle 1_{\rm E}|$. Assume that the global unitary operator acting on the composite state is denoted by U, whose specific form depends on the actual physical situation and system-environment interaction. Then the evolution state of the principal system can be obtained by performing a partial trace over the environment,

$$\rho_{\rm S}(t)={\rm Tr}_{\rm E}[U(\rho_{\rm S}\otimes|1_{\rm E}\rangle\langle 1_{\rm E}|)U^\dagger]=\sum_i K_i\rho_{\rm S}K_i^\dagger, \tag{19}$$

where $K_i \equiv \langle i_{\rm E}|U|i_{\rm E}\rangle$ is customarily known as the Kraus operator on the state space of the principal system, which satisfies a normalization condition $\sum_i K_i^\dagger K_i = \mathbb{I}$. Further, we can easily obtain the evolution state of the environment and the composite state,

$$\rho_{\rm E}(t) = \sum_{i,j} {\rm Tr}(K_i \rho_{\rm S} K_j^\dagger)|i_{\rm E}\rangle\langle j_{\rm E}|, \tag{20}$$

$$\rho_{\rm SE}(t) = \sum_{i,j} K_i \rho_{\rm S} K_j^\dagger \otimes |i_{\rm E}\rangle\langle j_{\rm E}|. \tag{21}$$

There is no loss of generality in assuming that the principal system is a general initial single qubit state $\rho_{\rm S} = (\mathbb{I}_2 + \boldsymbol{r}\cdot\boldsymbol{\sigma})/2$, where $\boldsymbol{\sigma}\{\sigma_1,\sigma_2,\sigma_3\}$ denotes the vector of Pauli matrices, and $\boldsymbol{r}\{r_1,r_2,r_3\}$ is the Bloch vector of the state with $r_i = {\rm Tr}(\rho_{\rm S}\sigma_i)$. Meanwhile, the depolarizing channel is an important type of quantum noise, which causes the qubit depolarizing with probability p. That is, it is replaced by the completely mixed state $\mathbb{I}/2$, while the qubit is left untouched with probability $1-p$. The Kraus operators of the depolarizing channel are given by

$$K_1 = \sqrt{1-\frac{3p}{4}}\mathbb{I}_2, \quad K_2 = \sqrt{\frac{p}{4}}\sigma_1,$$
$$K_3 = \sqrt{\frac{p}{4}}\sigma_2, \quad K_4 = \sqrt{\frac{p}{4}}\sigma_3, \tag{22}$$

where the probability $p = 1 - {\rm e}^{-\gamma_0 t}$ can be considered as a time-dependent parameter to describe the strength of channel noise, and γ_0 is the corresponding decay factor. Therefore, we should assume that the environment is 4-dimensional, and $\{|1_{\rm E}\rangle, |2_{\rm E}\rangle, |3_{\rm E}\rangle, |4_{\rm E}\rangle\}$ is an orthonormal basis for the state space of the environment. The system-environment state $\rho_{\rm SE} = \rho_{\rm S}\otimes|1_{\rm E}\rangle\langle 1_{\rm E}|$ is a $(2\otimes 4)$-dimensional bipartite composite state.

By use of Eqs. (19) and (20), we can calculate the evolution states of principal system and environment as

$$\rho_{\rm S}(t) = \{(1-p)r_1, (1-p)r_2, (1-p)r_3\}, \tag{23}$$

$$\begin{aligned}\rho_{\rm E}(t) = \frac{1}{4}\big[&(4-3p)|1\rangle\langle 1| + p(|2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|)\\ &+\sqrt{p(4-3p)}(r_1|1\rangle\langle 2| + r_2|1\rangle\langle 3| + r_3|1\rangle\langle 4|)\\ &-{\rm i}pr_3|2\rangle\langle 3| + {\rm i}pr_2|2\rangle\langle 4| - {\rm i}pr_1|3\rangle\langle 4|\\ &+({\rm H.c.})_{\rm nondiag}\big],\end{aligned} \tag{24}$$

where $({\rm H.c.})_{\rm nondiag}$ means the Hermitian conjugate of only the nondiagonal terms. The analytical form of system-environment evolution state $\rho_{\rm SE}(t)$ is too complex to express conveniently. By Eqs. (4a), (4b), (5), and (6), we can calculate the first-order coherence for the evolution states of principal system and environment, the mean coherence and the mutual correlation as

$$D_{\rm S}^2(t) = (1-p)^2|\boldsymbol{r}|^2, \tag{25a}$$
$$D_{\rm E}^2(t) = 3 - p(2-p)\left(3-|\boldsymbol{r}|^2\right), \tag{25b}$$
$$\mathcal{D}(t) = \left[3 - 3p(2-p) + |\boldsymbol{r}|^2\right]/8, \tag{25c}$$
$$\mathcal{T}(t) = \left[1 + 3p(2-p) + 3|\boldsymbol{r}|^2\right]/8. \tag{25d}$$

From the the above equations, we can find that the first-order coherence of the principal system $D_{\rm S}^2(t)$ decreases from the initial $|\boldsymbol{r}|^2$ to 0, while the first-order coherence of the environment $D_{\rm E}^2(t)$ decreases from 3 to $|\boldsymbol{r}|^2$ with the increase of time. Therefore, the mean coherence $\mathcal{D}(t)$ also monotonically decreases from the initial $(3+|\boldsymbol{r}|^2)/8$ to $|\boldsymbol{r}|^2/8$. Meanwhile, the mutual correlation $\mathcal{T}(t)$ increases monotonously from the initial $(1+3|\boldsymbol{r}|^2)/8$ to $(4+3|\boldsymbol{r}|^2)/8$. Note that we regard the principal system and the interacting environment as a larger composite closed quantum state, hence, the evolution of the open quantum system can be described by a unitary transform. Therefore, the conservation law for mean coherence and mutual correlation is still satisfied, i.e., $\mathcal{D}+\mathcal{T} = \mathcal{P} = (1+|\boldsymbol{r}|^2)/2$. It means that with the evolution of time, the reduced mean coherence is all transferred to the mutual correlation of the composite state.

Assume that the initial principal system is a general single qubit state with $\rho_{\rm S} = \boldsymbol{r}\{0.7, 0.5, 0.4\}$. The coherence as a function of time-dependent parameter $\gamma_0 t$ under the depolarizing channel is shown in Figs. 2(a) and 2(b). As can be seen from Fig. 2(a), with the passage of time, the first-order coherence of the principal system gradually decreases from the initial 0.9 to 0, while the first-order coherence of the environment gradually decreases from the initial 3 to 0.9. We notice from Figs. 2(a) and 2(b), although the first-order coherence of the environment may exceed 1, the mean coherence and mutual correlation of the open quantum systems are always within the range of $[0,1]$. The changes of coherence and correlation are completely symmetrical, and the center of symmetry is $\mathcal{P}/2 = 0.475$.

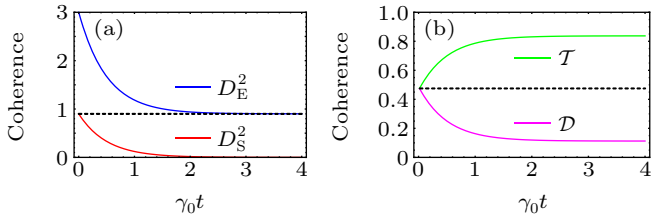


Fig. 2. (a) The first order coherence of the principal system (red line) and environment (blue line), (b) the mean coherence (purple line), and the mutual correlation (green line) as a function of time-dependent parameter $\gamma_0 t$ under the depolarizing channel (assume the initial state is $\rho_{\rm S} = \boldsymbol{r}\{0.7, 0.5, 0.4\}$).

5. Summary and discussion

In summary, we have generalized the framework of the conservation law for first-order coherence and mutual correlation to an arbitrary $(m\otimes n)$-dimensional bipartite composite state by introducing an extended Bloch decomposition form of the state. Then, we investigate coherence migration under global unitary transformations in high-dimensional systems, and generalize two kinds of unitary operators, which can help us obtain the maximum or minimum first-order coherence when they act on the quantum states. Moreover, we take depolarizing channels as a typical example to discuss coherent migration in open systems.

It is worth discussing that in order to describe the nonclassical correlation of bipartite composite systems, we generalize the definition of the mutual correlation between the two subsystems in Ref. [24] as Eq. (6). The research on the nonclassical correlation of bipartite composite systems is not only that, but also the violation of the CHSH inequality,[21] the discord,[37] the intrinsic concurrence,[39] the nonlocal advantage of quantum coherence,[39–41] and so on. These nonclassical correlations can be regarded as a useful quantum resource.[42] It is still an open problem to study the transformation of coherent and other quantum resources in high-dimensional systems. We expect that our paper would present a useful idea to regulate the quantum resource for quantum information processing.

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Appendix A

To prove that the global unitary operator $V^\dagger$ can make the high-dimensional bipartite composite state $\rho_{\rm AB}$ obtain the maximum accessible first-order coherence, we first consider the pure state. For an arbitrary $(d\otimes d)$-dimensional pure state $\rho_{\rm AB}=|\psi_{\rm AB}\rangle\langle\psi_{\rm AB}|$, its subsystems can be expressed as $\rho_{\rm A}={\rm Tr_B}(|\psi_{\rm AB}\rangle\langle\psi_{\rm AB}|)$ and $\rho_{\rm B}={\rm Tr_A}(|\psi_{\rm AB}\rangle\langle\psi_{\rm AB}|)$. The purity of the subsystem A is the same as the subsystem B,[25] and we assume that ${\rm Tr}(\rho_{\rm A}^2)={\rm Tr}(\rho_{\rm B}^2)=p$. Therefore, the mean coherence can be calculated by

$$\mathcal{D}(\rho_{\rm AB})=\frac{D_{\rm A}^2+D_{\rm B}^2}{d^2}=\frac{2dp-2}{d^2}. \tag{A1}$$

Since the purity p satisfies the relation $1/d\le p\le 1$, the maximal accessible first-order coherence occurs $p=1$, i.e.,

$$\mathcal{D}_{\max}=\frac{2d-2}{d^2}. \tag{A2}$$

There is a unitary operator U which acts on the pure state $\rho_{\rm AB}=|\psi_{\rm AB}\rangle\langle\psi_{\rm AB}|$ and transforms it into a separable state $\rho'_{\rm AB}=U\rho_{\rm AB}U^\dagger=|\psi'_{\rm A}\rangle\langle\psi'_{\rm A}|\otimes|\psi'_{\rm B}\rangle\langle\psi'_{\rm B}|$. The subsystems of the separable state $\rho'_{\rm AB}$ are also pure states $\rho'_{\rm A}=|\psi'_{\rm A}\rangle\langle\psi'_{\rm A}|$ and $\rho'_{\rm B}=|\psi'_{\rm B}\rangle\langle\psi'_{\rm B}|$. Obviously, ${\rm Tr}(\rho'^2_{\rm A})={\rm Tr}(\rho'^2_{\rm B})=1$, the mean coherence of $\rho'_{\rm AB}$ reaches the maximum $\mathcal{D}(\rho'_{\rm AB})=\mathcal{D}_{\max}$. In other words, the separable pure states reach the maximum coherence $\mathcal{D}_{\max}$. In particular, considering such unitary operator $U=V^\dagger$, the final state can be given as $\rho'_{\rm AB}=V^\dagger\rho_{\rm AB}V=|1_{\rm A}\rangle\langle 1_{\rm A}|\otimes|1_{\rm B}\rangle\langle 1_{\rm B}|$.

For an arbitrary $(d\otimes d)$-dimensional mixed state $\rho_{\rm AB}$, it can be expressed as a probabilistic mixed form of its eigenstates, i.e., $\rho_{\rm AB}=\sum_{k=1}^{d^2}\chi_k|\psi_k\rangle\langle\psi_k|$, where $\langle\psi_k|\psi'_{k'}\rangle=\delta_{kk'}$. Here, we use $\{\boldsymbol{x},\boldsymbol{y}\}$ and $\{\boldsymbol{x}_k,\boldsymbol{y}_k\}$ to represent the extended Bloch vectors of the states $\rho_{\rm AB}$ and its eigenstates $|\psi_k\rangle\langle\psi_k|$, respectively. It is obvious that these Bloch vectors satisify the relations $\boldsymbol{x}=\sum_{k=1}^{d^2}\chi_k\boldsymbol{x}_k$ and $\boldsymbol{y}=\sum_{k=1}^{d^2}\chi_k\boldsymbol{y}_k$. Equations (4a) and (4b) shows that the first-order coherence is only related to the Bloch vectors, i.e., $D(\rho_{\rm AB})=(1/2d)(|\boldsymbol{x}|^2+|\boldsymbol{y}|^2)=(1/2d)\sum_{k,k'=1}^{d^2}\chi_k\chi'_k(\boldsymbol{x}_k\cdot\boldsymbol{x}'_k+\boldsymbol{y}_k\cdot\boldsymbol{y}'_k)$. Next, we consider the final state $\rho'_{\rm AB}$ which is formed by performing a global unitary operator U,

$$\rho'_{\rm AB}=U\rho_{\rm AB}U^\dagger=\sum_{k=1}^{d^2}\chi_k|\psi'_k\rangle\langle\psi'_k|, \tag{A3}$$

where $|\psi'_k\rangle=U|\psi_k\rangle$. Obviously the prue state $|\psi'_k\rangle$ is the eigenvector of the final state $\rho'_{\rm AB}$, since $\langle\psi'_k|\psi''_k\rangle=\langle\psi_k|U^\dagger U|\psi'_k\rangle=\langle\psi_k|\psi'_k\rangle=\delta_{kk'}$ for all k and k'. Thus, the coherence of the final state $\rho'_{\rm AB}$ can be given by $D(\rho'_{\rm AB})=(1/2d)(|\boldsymbol{x}'|^2+|\boldsymbol{y}'|^2)=(1/2d)\sum_{k,k'=1}^{d^2}\chi_k\chi_{k'}(\boldsymbol{x}'_k\cdot\boldsymbol{x}'_{k'}+\boldsymbol{y}'_k\cdot\boldsymbol{y}'_{k'})$. This shows that the coherence of the state $\rho'_{\rm AB}$ is positively correlated with the coherence of these pure states $\{|\psi'_k\rangle\}_{k=1}^{d^2}$. According to the condition of the pure state $|\psi_{\rm AB}\rangle\langle\psi_{\rm AB}|$ achieving the maximum coherence, one obtain that the state $\rho'_{\rm AB}$ reaches the maximum coherence if and only if these pure states $\{|\psi'_k\rangle\}_{k=1}^{d^2}$ are separable (the condition can also be written as $|\boldsymbol{x}'_k|^2=|\boldsymbol{y}'_k|^2=(2d-2)/d$ for all k). In particular, considering such unitary operator $U=V^\dagger$, these pure states $\{|\psi'_k\rangle\}_{k=1}^{d^2}$ are exactly right separable, i.e., $|\psi'_k\rangle=|i_{\rm A}\rangle\otimes|j_{\rm B}\rangle$ $(i,j=1,2,\ldots,d)$. Therefore, the subsystems of the state $\rho'_{\rm AB}$ can be written as

$$\rho'_{\rm A}=\sum_{i=1}^{d}\sum_{j=1}^{d}\chi_{j+(i-1)d}|i_{\rm A}\rangle\langle i_{\rm B}|, \tag{A4a}$$

$$\rho'_{\rm B}=\sum_{j=1}^{d}\sum_{i=1}^{d}\chi_{j+(i-1)d}|j_{\rm A}\rangle\langle j_{\rm B}|. \tag{A4b}$$

So the maximal accessible first-order coherence can be obtained by

$$\mathcal{D}_{\max}=\frac{d\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\chi_{j+(i-1)d}\right)^2+d\sum_{j=1}^{d}\left(\sum_{i=1}^{d}\chi_{j+(i-1)d}\right)^2-2}{d^2}. \tag{A5}$$

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