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A nonlocal Boussinesq equation: Multiple-soliton solutions and symmetry analysis

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A nonlocal Boussinesq equation is deduced from the local one by using consistent correlated bang method. To study various exact solutions of the nonlocal Boussinesq equation, it is converted into two local equations which contain the local Boussinesq equation. From the N-soliton solutions of the local Boussinesq equation, the N-soliton solutions of the nonlocal Boussinesq equation are obtained, among which the $(N=2,3,4)$-soliton solutions are analyzed with graphs. Some periodic and traveling solutions of the nonlocal Boussinesq equation are derived directly from the known solutions of the local Boussinesq equation. Symmetry reduction solutions of the nonlocal Boussinesq equation are also obtained by using the classical Lie symmetry method.

Keywords: nonlocal Boussinesq equation, N-soliton solution, periodic waves, symmetry reduction solutions

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1. Introduction

Nonlocal equations have attracted much attention in nonlinear field since Ablowitz and Musslimani[1] introduced and investigated a PT symmetric nonlocal Schrödinger equation

$$\mathrm{i}q_t(x,t)=q_{xx}(x,t)\pm q(x,t)q^*(-x,t)q(x,t), \tag{1}$$

with $*$ denoting complex conjugate and q being a complex valued function of the real variables x and t. Equation (1) is integrable in the sense of having infinite number of conservation laws and some soliton solutions of it are obtained by using the inverse scattering transform method in Ref. [1]. Since then, many nonlocal equations with certain space time symmetries have been introduced and investigated, such as the nonlocal Korteweg–de-Vrise (KdV) systems,[2] the nonlocal Davey–Stewartson systems,[3] nonlocal Schrödinger equation,[4–7] and so on. Among the methods of solving nonlinear systems the Hirota's bilinear method[8,9] plays an important role, which is developed recently to get N-soliton solutions for both $(1+1)$-dimensional integrable equations[10] and $(2+1)$-dimensional integrable equations.[11–14] For nonlocal systems, the Hirota's bilinear method can also be used to get N-soliton solutions.[6,15,16] In the same time various methods for solving nonlocal nonlinear systems have also been developed, such as inverse scattering transformation,[1,17] Darboux transformations,[18,19] symmetry analysis,[20] *etc.*

In recent years, Lou proposed a concept of Alice–Bob (AB) system, in which two variables are linked by $B=\hat{f}A$ with $\hat{f}$ being a combination of parity ($\hat{P}$), time reversion ($\hat{T}$), and conjugation ($\hat{C}$)-related operator, to describe two events intertwined with each other and introduced $\hat{P}-\hat{T}-\hat{C}$ principle and $\hat{f}=\hat{f}^{-1}$ equivalence principle to construct and solve nonlocal AB systems from known local ones.[21] Many nonlocal AB systems, including AB–KdV system,[2,22] AB–mKdV[23,24] system, AB–KP system,[25] AB–Toda system,[2] AB–Schrödinger equation,[26] *etc.*, have been introduced and their integrable properties and exact solutions have also been studied a lot.

It is known that the Boussinesq equation plays an important role in describing long wave propagation in shallow water.[27–29] Here, we take the Boussinesq equation in the form

$$u_{tt}-u_{xx}-u_{xxxx}-3(u^2)_{xx}=0, \tag{2}$$

which is integrable and can be solved by using the inverse scattering method. In this paper, we study a nonlocal Boussinesq equation in the form of

$$\begin{aligned}&A_{tt}-\frac{9}{2}A_x^2-3A_xB_x+\frac{3}{2}B_x^2-\left(1+\frac{9}{2}A+\frac{3}{2}B\right)A_{xx}\\&+\frac{3}{2}(B-A)B_{xx}-A_{xxxx}=0,\end{aligned} \tag{3}$$

where

$$B=\hat{P}_{\mathrm{s}}\hat{T}_{\mathrm{d}}A=A(-x+x_0,-t+t_0). \tag{4}$$

The paper is organized as follows. In Section 2, we give multiple soliton solutions of the nonlocal Boussinesq equation (3) using known soliton solutions of the local Boussinesq equation. In Section 3, we obtain periodic solutions and traveling solutions of Eq. (3) directly from known solutions of the local Boussinesq equation. In Section 4, the symmetry reduction solutions of Eq. (3) are derived by using classical symmetry method. The last section devotes to a summary.

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2. Multiple soliton solutions of the nonlocal Boussinesq equation (3)

According to the consistent correlated bang method in Ref. [30], the functions A and B in Eq. (3) can be written as

$$A = u + v, \quad B = u - v, \tag{5}$$

where the functions u and v satisfy the following parity condition:

$$\hat{P}_s\hat{T}_d u = u, \quad \hat{P}_s\hat{T}_d v = -v. \tag{6}$$

So the nonlocal Boussinesq equation (3) is changed to

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} = 0, \tag{7}$$

$$v_{tt} - 6vu_{xx} - 6uv_{xx} - 12u_x v_x - v_{xxxx} - v_{xx} = 0, \tag{8}$$

where $u = (A+B)/2$, $v = (A-B)/2$ and equation (7) is just the Boussinesq equation (2). To solve the nonlocal Boussinesq equation (3), we can use the solutions of the Boussinesq equation (2), which have N-soliton solution[27]

$$u = 2(\ln F)_{xx}, \quad F = \sum_{\nu} \exp\left(\sum_{j=1}^{N} \nu_j \xi_j + \sum_{1\le i<j}^{N} \nu_i \nu_j \theta_{ij}\right), \tag{9}$$

with $\nu_i = 0, 1$ $(i = 1, \ldots, N)$, the summations of it should be done for all permutations and

$$\xi_i = k_i x + l_i t + \xi_{i0},$$

$$\exp(\theta_{ij}) = \frac{\sqrt{1+k_i^2}\sqrt{1+k_j^2} - (2k_i^2 - 3k_ik_j + 2k_j^2 + 1)}{\sqrt{1+k_i^2}\sqrt{1+k_j^2} - (2k_i^2 + 3k_ik_j + 2k_j^2 + 1)}, \tag{10}$$

where $l_i = k_i\sqrt{1+k_i^2}$ and k_i, ξ_{i0}, $(i = 1, \ldots, N)$ are arbitrary constants.

In order to find $\hat{P}_s\hat{T}_d$ invariant solutions of u in Eq. (9), we first rewrite the variables as

$$\xi_j = \eta_j - \frac{1}{2}\sum_{i=1}^{j-1}\theta_{ij} - \frac{1}{2}\sum_{i=j+1}^{N}\theta_{ji}, \tag{11}$$

where

$$\eta_i = k_i\left(x - \frac{x_0}{2}\right) + l_i\left(t - \frac{t_0}{2}\right) + \eta_{0i} \tag{12}$$

with η_{0i} being arbitrary constants. According to the invariance of Eq. (9) under the transformation

$$F \to \beta \exp(Kx + \Omega t + X_0) F, \tag{13}$$

we finally get[23]

$$u = 2\left[\ln \sum_{\nu} K_\nu \cosh\left(\frac{1}{2}\sum_{j=1}^{N}\nu_j\eta_j\right)\right]_{xx} \equiv 2(\ln F_N)_{xx}, \tag{14}$$

where the summation of ν should be done for all non-dual permutations of $\nu_i = 1, -1$, $(i = 1, \ldots, N)$ and

$$K_\nu = \prod_{i>j}\sqrt{\sqrt{1+k_i^2}\sqrt{1+k_j^2} - (2k_i^2 - 3\nu_i\nu_j k_i k_j + 2k_j^2 + 1)}. \tag{15}$$

For $N = 2, 3$, we give their explicit expressions of F_N as

$$F_2 = \delta_{12,-}\cosh\left[\frac{1}{2}((k_1+k_2)x + (l_1+l_2)t)\right] + \delta_{12,+}\cosh\left[\frac{1}{2}((k_1-k_2)x + (l_1-l_2)t)\right], \tag{16}$$

$$\begin{aligned} F_3 = {} & \delta_{12,-}\delta_{13,-}\delta_{23,-}\cosh\left[\frac{1}{2}((k_1+k_2+k_3)x + (l_1+l_2+l_3)t)\right] + \delta_{12,+}\delta_{13,+}\delta_{23,-}\cosh\left[\frac{1}{2}((k_1-k_2-k_3)x + (l_1-l_2-l_3)t)\right] \\ & + \delta_{12,+}\delta_{13,-}\delta_{23,+}\cosh\left[\frac{1}{2}((k_1-k_2+k_3)x + (l_1-l_2+l_3)t)\right] + \delta_{12,-}\delta_{13,+}\delta_{23,+}\cosh\left[\frac{1}{2}((k_1+k_2-k_3)x + (l_1+l_2-l_3)t)\right], \end{aligned} \tag{17}$$

with

$$\delta_{ij,\pm} = \sqrt{\sqrt{1+k_i^2}\sqrt{1+k_j^2} - (2k_i^2 \pm 3k_ik_j + 2k_j^2 + 1)}. \tag{18}$$

The solutions of Eq. (8) with Eq. (6) can be obtained from the symmetry of the Boussinesq equation (7), here we give a simple example

$$v = -c_1 u_t - c_2 u_x \tag{19}$$

with arbitrary constants c_1 and c_2. It can be seen from the expression of u in Eq. (14) that it is an even function under the operator of $\hat{P}_s\hat{T}_d$, which means v in Eq. (19) being an odd function, so the condition of Eq. (6) is satisfied. Now we have a special form of solution of Eq. (3)

$$A = u - c_1 u_t - c_2 u_x, \tag{20}$$

with Eq. (14).

Substituting Eqs. (16) and (17) into Eq. (20) with Eq. (14) we get 2-soliton and 3-soliton solutions of the nonlocal Boussinesq equation (3), which are shown in Figs. 1(a)–1(c). Similarly, we can get any arbitrary N soliton solutions of

Eq. (3), Fig. 1(d) is a plot of 4-soliton solution. In Fig. 1, the parameters are fixed as follows: $k_1=1$, $k_2=4$, $x_0=t_0=\eta_{10}=\eta_{20}=0$, $c_1=1, c_2=1$ for Fig. 1(a); $k_1=1$, $k_2=3$, $k_3=4$, $x_0=t_0=\eta_{10}=\eta_{20}=\eta_{30}=0$, $c_1=1, c_2=2$ for Figs. 1(b) and 1(c); and $k_1=-1$, $k_2=-2$, $k_3=4$, $k_4=6$, $x_0=t_0=\eta_{10}=\eta_{20}=\eta_{30}=\eta_{40}=0$, $c_1=4$, $c_2=-2$ for Fig. 1(d). From Fig. 1, we see that for any 2-soliton, 3-soliton, and 4-soliton solutions, the interaction between solitons are elastic with phases changed.

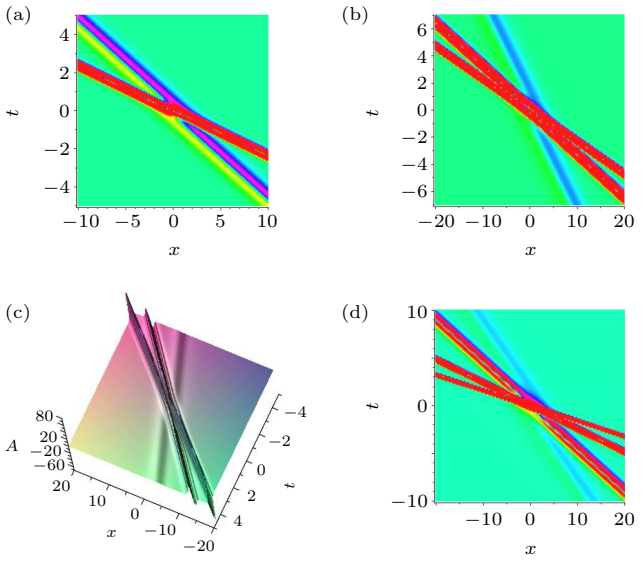


Fig. 1. The density plots of N soliton solutions of Eq. (3) with (a) $N=2$, (b) $N=3$, (d) $N=4$, panel (c) is a three-dimensional (3D) plot of $N=3$ case.

3. Periodic wave solutions and traveling wave solutions of the nonlocal Boussinesq equation (3)

To solve the Boussinesq equation (7), the author of Ref. [27] obtained some kinds of exact solutions, including soliton solutions, traveling wave solutions, plane periodic solutions, *etc.* Here, we use these known solutions, which all satisfy the parity condition (6), to obtain corresponding solutions of the nonlocal Boussinesq equation (3).

(i) Soliton solution 1

When we take

$$u=\frac{1}{2}(c^2-1)\,\mathrm{sech}^2(X_1),\quad (c^2>1),$$
$$X_1=\frac{1}{2}\sqrt{c^2-1}\left[x-\frac{1}{2}x_0-c(t-\frac{1}{2}t_0)\right], \tag{21}$$

by substituting it into Eq. (20) we get a soliton solution of Eq. (3)

$$A=\frac{1}{2}\,\mathrm{sech}^2(X_1)[\tanh(-X_1)(c^2-1)^{\frac{3}{2}}(cc_1-c_2)+c^2-1]. \tag{22}$$

(ii) Soliton solution 2

When we take

$$u=-\frac{c^2-1}{6}(1-3\tanh^2(X_2))\quad (c^2<1),$$
$$X_2=\frac{1}{2}\sqrt{1-c^2}\left(x-\frac{1}{2}x_0-c(t-\frac{1}{2}t_0)\right), \tag{23}$$

by substituting it into Eq. (20) we get a soliton solution of Eq. (3)

$$A=-\frac{1}{6}(c^2-1)\Big[\sqrt{1-c^2}(c_1c-c_2)(\tanh^3(X_2)-3\tanh(X_2))-3\tanh^2(X_2)+1\Big]. \tag{24}$$

(iii) Travelling wave solution 1

We take

$$u=-\frac{1}{2}(c^2-1)\,\mathrm{csch}^2(X_1),\quad (c^2>1), \tag{25}$$

which leads to

$$A=-\frac{1}{2}\mathrm{csch}^2(X_1)\Big[\coth(X_1)(c^2-1)^{3/2}(c_1c-c_2)+c^2-1\Big]. \tag{26}$$

(iv) Travelling wave solution 2

We take

$$u=-\frac{c^2-1}{6}(1-3\coth^2(X_2)),\quad (c^2<1), \tag{27}$$

which leads to

$$A=-\frac{c^2-1}{6}\Big[3\sqrt{1-c^2}(c_1c-c_2)\coth(X_2)(\coth^2(X_2)-1)-3\coth^2(X_2)+1\Big]. \tag{28}$$

(v) Plane periodic wave solution 1

When taking

$$u=\frac{c^2-1}{2}\sec^2(X_2),\quad (c^2<1), \tag{29}$$

we get

$$A=\frac{c^2-1}{2}\sec^2(X_2)[(c_1c-c_2)\sqrt{1-c^2}\tan(X_2)+1]. \tag{30}$$

(vi) Plane periodic wave solution 2

When taking

$$u=-\frac{c^2-1}{6}(1+3\tan^2(X_1)),\quad (c^2>1), \tag{31}$$

we get

$$A=\frac{1-c^2}{2}\Big[\sqrt{c^2-1}(c_1c-c_2)\tan(X_1)(\tan^2(X_1)+1)+\tan^2(X_1)+\frac{1}{3}\Big]. \tag{32}$$

(vii) Plane periodic wave solution 3

When taking

$$u=\frac{c^2-1}{2}\csc^2(X_2),\quad (c^2<1), \tag{33}$$

we get

$$A=\frac{c^2-1}{2}\Big[\cot(X_2)\sqrt{1-c^2}(c_1c-c_2)+1\Big]\csc^2(X_2). \tag{34}$$

(viii) Plane periodic wave solution 4

When taking

$$u=-\frac{c^2-1}{6}(1+3\cot^2(X_1)),\quad (c^2>1), \tag{35}$$

we get

$$\begin{aligned}A\ =\ &-\frac{1}{2}(c^2-1)\Big[\sqrt{c^2-1}(c_1c-c_2)\cot^3(X_1)+\cot^2(X_1)\\&+\sqrt{c^2-1}(c_1c-c_2)\cot(X_1)+\frac{1}{3}\Big].\end{aligned} \tag{36}$$

Figure 2 is a 3D plot of two single soliton solutions of the nonlocal Boussinesq equation (3): Fig. 2(a) for Eq. (22) with parameters being fixed as

$$c_1=1,\ \ c_2=2,\ \ c=2,\ \ x_0=0,\ \ t_0=0; \tag{37}$$

Fig 3 for Eq. (24) with parameters being fixed as

$$c_1=1,\ \ c_2=2,\ \ c=0.5,\ \ x_0=0,\ \ t_0=0. \tag{38}$$

Figure 3 is density plots of two plane periodic solutions of the nonlocal Boussinesq equation (3): Fig. 3(a) for Eq. (30) with parameters being fixed as Eq. (38); Fig. 3(b) for Eq. (32) with parameters being fixed as Eq. (37).

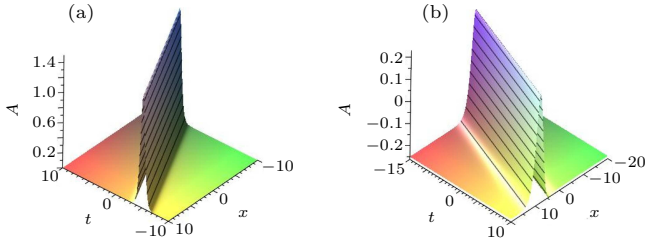


Fig. 2. The 3D plot of two single soliton solutions of Eq. (3): panel (a) is for Eq. (22) with parameters being fixed as Eq. (38); panel (b) is for Eq. (24) with parameters being fixed as Eq. (37).

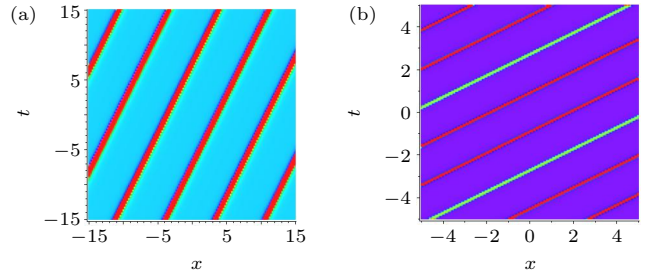


Fig. 3. The density plot of two plane periodic solutions of Eq. (3): panel (a) is for Eq. (30) with parameters being fixed as Eq. (37); panel (b) is for Eq. (32) with parameters being fixed as Eq. (38).

4. Symmetry reduction solutions of the nonlocal Boussinesq equation (3)

Symmetry analysis plays an important role in solving nonlinear equations, in this section we seek Lie point symmetry of the nonlocal Boussinesq equation (3) by using the classic Lie group method.[31,32] To this end, we assume that the Lie point symmetry of Eq. (3) is in the form

$$V=X\frac{\partial}{\partial x}+T\frac{\partial}{\partial t}+\Gamma\frac{\partial}{\partial A}+\Lambda\frac{\partial}{\partial B}, \tag{39}$$

where X, T, Γ, Λ are functions of x, t, A, B, which means that equation (3) is invariant under the transformation

$$\{x,t,A,B\}\to\{x+\varepsilon X, t+\varepsilon T, A+\varepsilon\Gamma, B+\varepsilon\Lambda\}, \tag{40}$$

with the infinitesimal parameter ε. Equivalently, the symmetry of Eq. (39) can be written in a function form as

$$\sigma_A=XA_x+TA_t-\Gamma, \tag{41a}$$

$$\sigma_B=XB_x+TB_t-\Lambda. \tag{41b}$$

From definition of symmetry, σ_A, σ_B in Eqs. (41a) and (41b) satisfy linearized equations of Eq. (3) and its $\hat{P}_s\hat{T}_d$ counterpart by exchanging A and B in Eq. (3)

$$\begin{aligned}&\sigma_{A,tt}-3(3\sigma_{A,x}+\sigma_{B,x})A_x-3(\sigma_{A,x}-\sigma_{B,x})B_x\\&-\frac{3}{2}(3\sigma_A+\sigma_B)A_{xx}-\frac{3}{2}(\sigma_A-\sigma_B)B_{xx}-\sigma_{A,xxxx}\\&-\left(1+\frac{9}{2}A+\frac{3}{2}B\right)\sigma_{A,xx}-\frac{3}{2}(A-B)\sigma_{B,xx}=0,\end{aligned} \tag{42a}$$

$$\begin{aligned}&\sigma_{B,tt}-3(3\sigma_{B,x}+\sigma_{A,x})B_x-3(\sigma_{B,x}-\sigma_{A,x})A_x\\&-\frac{3}{2}(3\sigma_B+\sigma_A)B_{xx}-\frac{3}{2}(\sigma_B-\sigma_A)A_{xx}-\sigma_{B,xxxx}\\&-\left(1+\frac{9}{2}B+\frac{3}{2}A\right)\sigma_{B,xx}-\frac{3}{2}(B-A)\sigma_{A,xx}=0,\end{aligned} \tag{42b}$$

$$\sigma_A=\sigma_B(-x+x_0,-t+t_0). \tag{42c}$$

To solve for X, T, A, B, in Eq. (39), by substituting Eqs. (41a) and (41b) into Eqs. (42a) and (42b) and eliminating A_{tt}, B_{tt} by Eqs. (3) and its $\hat{P}_s\hat{T}_d$ counterpart, respectively, after vanishing all the coefficients of the independent partial derivatives of variables A, B, a system of over determined linear equations are obtained. After solving these equations using software like Maple and considering the relation (42c), we obtain

$$\begin{aligned}&X=\frac{1}{2}e_1x,\ \ T=e_1t,\ \ \Lambda=-(e_1+e_2)A+e_2B-\frac{1}{6}e_1,\\&\Gamma=-(e_1+e_2)B+e_2A-\frac{1}{6}e_1,\end{aligned}$$

where e_1 and e_2 are arbitrary constants. So the explicit expressions of Eqs. (41a) and (41b) are

$$\sigma_A\ =\ \frac{e_1}{2}xA_x+e_1tA_t+(e_1+e_2)B-e_2A+\frac{1}{6}e_1, \tag{43a}$$

$$\sigma_B\ =\ \frac{e_1}{2}xB_x+e_1tB_t+(e_1+e_2)A-e_2B+\frac{1}{6}e_1. \tag{43b}$$

The invariant solutions of the nonlocal Boussinesq equation (3) can be obtained by assuming $\sigma_A=\sigma_B=0$ in Eqs. (43a)

and (43b), which is equivalent to solving the characteristic equation

$$\frac{\mathrm{d}x}{\frac{e_1}{2}x}=\frac{\mathrm{d}t}{e_1 t}=\frac{\mathrm{d}A}{-(e_1+e_2)B+e_2A-\frac{1}{6}e_1}=\frac{\mathrm{d}B}{-(e_1+e_2)A+e_2B-\frac{1}{6}e_1}. \tag{44}$$

After some calculations, we get the symmetry reduction solutions of the nonlocal Boussinesq equation (3)

$$A=\frac{G}{t}+t^{(e_1+2e_2)/e_1}H-\frac{1}{6}, \tag{45}$$

$$B=\frac{G}{t}-t^{(e_1+2e_2)/e_1}H-\frac{1}{6}, \tag{46}$$

where G, H are invariant functions of $\xi=x/\sqrt{t}$.

Substituting Eqs. (45) and (46) into the nonlocal Boussinesq equation (3) and its $\hat{P}_{\mathrm{s}}\hat{T}_{\mathrm{d}}$ counterpart, we get the symmetry reduction equations for G and H

$$24G_\xi^2-7\xi G_\xi-(\xi^2-24G)G_{\xi\xi}-8G+4G_{\xi\xi\xi\xi}=0, \tag{47}$$

$$\begin{aligned}&48e_1^2H_\xi G_\xi+e_1\xi(e_1+8e_2)H_\xi+24e_1^2G_{\xi\xi}H\\&+e_1^2(24G-\xi^2)H_{\xi\xi}+4e_1^2H_{\xi\xi\xi\xi}\\&-8e_2(e_1+2e_2)H=0,\end{aligned} \tag{48}$$

$$H(\xi)=(-1)^{(2e_2)/e_1}H(\mathrm{i}\xi),\ \ G(\xi)=-G(\mathrm{i}\xi), \tag{49}$$

with $G(-\xi)=G(\xi)$, $H(\xi)=(-1)^{4e_2/e_1}H(-\xi)$ under the condition $e_1\neq -e_2$.

As a concrete example, under the condition $e_1=-e_2/2$, we see that equations (47)–(49) admit the following solution:

$$G=\frac{1}{6}\xi^2+d_1,\ \ H=d_2\xi^4+\frac{12}{5}d_2, \tag{50}$$

with arbitrary constants d_1 and d_2. Substituting Eq. (50) into Eqs. (45) and (46), we have a special solution of Eq. (3)

$$\begin{aligned}A&=\frac{30d_2x^4-5t^5+5t^3x^2+72d_2t^2}{30t^5},\\B&=-\frac{30d_2x^4+5t^5-5t^3x^2+72d_2t^2}{30t^5},\end{aligned} \tag{51}$$

which can be verified easily.

5. Conclusion

In summary, a nonlocal Boussinesq equation is investigated by converting it into two local equations. Using a known N-soliton solutions of the local Boussinesq equation, a general form of multiple soliton solutions is obtained, among which the $N=2$, 3, 4 soliton solutions are plotted and analyzed. Some kinds of single soliton solutions, travelling wave solutions, and plane periodic solutions of the nonlocal Boussinesq equation are derived directly from known solutions of the local Boussinesq equation. The symmetry reduction solutions of the nonlocal Boussinesq equation are also obtained by applying the classic Lie symmetry method on it.

Acknowledgments

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