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Wave function collapses and $1/n$ energy spectrum induced by a Coulomb potential in a one-dimensional flat band system

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We investigate the bound state problem in a one-dimensional flat band system with a Coulomb potential. It is found that, in the presence of a Coulomb potential of type I (with three equal diagonal elements), similarly to that in the two-dimensional case, the flat band could not survive. At the same time, the flat band states are transformed into localized states with a logarithmic singularity near the center position. In addition, the wave function near the origin would collapse for an arbitrarily weak Coulomb potential. Due to the wave function collapses, the eigen-energies for a shifted Coulomb potential depend sensitively on the cut-off parameter. For a Coulomb potential of type Ⅱ, there exist infinite bound states that are generated from the flat band. Furthermore, when the bound state energy is very near the flat band, the energy is inversely proportional to the natural number, e.g., $E_n \propto 1/n$, $n = 1, 2, 3, \ldots$ It is expected that the $1/n$ energy spectrum could be observed experimentally in the near future.

Keywords: wave function collapses, flat band, infinite bound states

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1. Introduction

A lot of interesting physics in flat band systems have attracted great interests.[1–16] For example, ferromagnetism transition,[17] localization,[18] super-Klein tunneling,[19–22] quantum Hall,[23] zitterbewegung,[24] preformed pairs,[25] strange metal,[26] and high-T_c superconductivity/superfluidity,[27,28] etc. have been investigated. A lot of the exotic physics in a flat band can be attributed to an infinitely large density of states of the flat band. For example, the fat band states have localized properties, which only occupy a finite number of unit cells in lattice model.[29] This is because although the eigen-states can be labeled by momentum in a translational invariant system, due to the flatness, an arbitrarily linear compositions of these states are also eigen-states, which can be localized. In addition, due to the zero dispersion of the flat band, the kinetic energy is extremely suppressed, an arbitrarily small interaction can dominate the physics. Therefore, the flat band can be very sensitive to a small perturbation, e.g., the magnetism transitions induced by the interaction,[30] a linear dependence of interaction strength of superconductor/superfluid order parameters.[31–36]

It is well known that behaviors of density of states near threshold of a continuous energy spectrum play crucial roles in formation of bound states.[37] Due to infinitely large density of density of states of a flat band, the existence conditions of bound states can be affected significantly. For example, in two-dimensional flat band system, Gorbar *et al.* found that an arbitrary weak Coulomb potential can destroy the flat band.[38] Furthermore, Pottelberge found that the flat band gradually becomes a continuous band (uncountable set) with the increasing Coulomb potential strength,[39] In a one-dimensional spin-1 flat band system, the bound state problem for short-ranged potential (delta potential and square well potential) has been solved by Zhang and Zhu.[40] It was found that for a square well potential of type Ⅱ, the infinitely large density of states and its ensuing $1/z$ singularity of Green function result in an infinite number of bound states and a hydrogen atom-like energy spectrum. Now a natural question arises: How does the long-ranged Coulomb potential of type Ⅱ change the energy spectrum in one dimension? For Coulomb potential of type I, what is the energy spectrum in a one-dimensional flat band system?

In this work, we investigate the bound state problem in a one-dimensional flat band system with a long-ranged Coulomb potential. It is found that for arbitrarily weak Coulomb potential of type I, similarly to that in the two-dimensional case,[38] the flat band is destroyed completely. Consequently, the flat band states evolve into the localized states which form a continuous band, which is similar to the finding by Pottelberge in two dimensions. However, we find that the localized properties of these new states in continuous band are very similar to the original flat band states. In addition, the eigen-energies of a shifted Coulomb potential depends sensitively on the cut-off parameters due to the singularity of Coulomb potential near the origin. For Coulomb potential of type Ⅱ, it is interesting that the long-ranged Coulomb potential also results in an infinite number of bound states (countable set), which is similar to that of shorted-range square well potential. However, it is found that the bound state energies are inversely proportional to the natural number, i.e.,

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$E_n \propto 1/n$, which is different from its shorted ranged counterpart, e.g., $E_n \propto 1/n^2$ in square well potential.

The article is organized as follows. In Section 2, the model Hamiltonian and the localized wave functions of the flat band are given. The localized properties are discussed with some modified Bessel functions. Next, we solve the bound state problems for Coulomb potentials of two types in Section 3. At the end, a summary is given in Section 4.

2. The model Hamiltonian with a flat band

In this work, we consider a three-component flat band Hamiltonian[40] in one dimension, i.e.,

$$\begin{aligned} H &= H_0 + V_{\rm p}(x), \\ H_0 &= -{\rm i}\hbar v_{\rm F} S_x \frac{\partial}{\partial x} + m S_z, \end{aligned} \tag{1}$$

where $V_{\rm p}(x)$ represents the potential energy, H_0 is the free-particle Dirac-type Hamiltonian, $v_{\rm F} > 0$ denotes the Fermi velocity, and $m > 0$ describes the particle mass. S_x and S_z are spin operators for spin-1 particles,[41] i.e.,

$$S_x = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{2}$$

in usual basis $|i\rangle$ with $i = 1, 2, 3$. In this article, we use the units of $v_{\rm F} = \hbar = 1$. When $V_{\rm p}(x) = 0$, the free particle Hamiltonian H_0 has three eigenstates and the eigenenergies (see Fig. 1), e.g.,

$$\begin{aligned} &\psi_{-,k}(x) = \frac{1}{2\sqrt{k^2+m^2}} \begin{pmatrix} \sqrt{k^2+m^2}-m \\ -\sqrt{2}k \\ \sqrt{k^2+m^2}+m \end{pmatrix} {\rm e}^{{\rm i}kx}, \\ &E_{-,k} = -\sqrt{k^2+m^2}; \\ &\psi_{0,k}(x) = \frac{1}{\sqrt{2(k^2+m^2)}} \begin{pmatrix} -k \\ \sqrt{2}m \\ k \end{pmatrix} {\rm e}^{{\rm i}kx}, \\ &E_{0,k} = 0; \\ &\psi_{+,k}(x) = \frac{1}{2\sqrt{k^2+m^2}} \begin{pmatrix} \sqrt{k^2+m^2}+m \\ \sqrt{2}k \\ \sqrt{k^2+m^2}-m \end{pmatrix} {\rm e}^{{\rm i}kx}, \\ &E_{+,k} = \sqrt{k^2+m^2}. \end{aligned} \tag{3}$$

It is found that a flat band with zero energy ($E_{0,k} = 0$) appears in between upper and lower bands. The possible bound states can exist in the gaps among the three bands, i.e., $0 < E < m$ and $-m < E < 0$ (the regions A and B in Fig. 1).

Due to the zero dispersion of flat band, an arbitrarily linear superpositions of the above wave function $\psi_{0,k}(x)$ are also eigen-states of H_0 (eigen-energy $E = 0$). Then some localized eigen-states can be constructed by a linear superposition, e.g.,

$$\begin{aligned} \psi(x-x_{\rm c}) &\equiv \begin{pmatrix} \psi_1(x-x_{\rm c}) \\ \psi_2(x-x_{\rm c}) \\ \psi_3(x-x_{\rm c}) \end{pmatrix} = \sum_k \psi_{0,k}(x)\, {\rm e}^{-{\rm i}kx_{\rm c}} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} {\rm d}k \frac{{\rm e}^{{\rm i}k(x-x_{\rm c})}}{\sqrt{2(k^2+m^2)}} \begin{pmatrix} -k \\ \sqrt{2}m \\ k \end{pmatrix} \\ &= \frac{m}{\pi} \begin{pmatrix} \frac{-{\rm isgn}(x-x_{\rm c})}{\sqrt{2}} K_1(m|x-x_{\rm c}|) \\ K_0(m|x-x_{\rm c}|) \\ \frac{{\rm isgn}(x-x_{\rm c})}{\sqrt{2}} K_1(m|x-x_{\rm c}|) \end{pmatrix}, \end{aligned} \tag{4}$$

where $K_0(x)$ and $K_1(x)$ are the zero- and first-order modified Bessel functions of the second kind,[42] respectively. The sign function ${\rm sgn}(x) = 1$ for $x > 0$; ${\rm sgn}(x) = -1$ for $x < 0$ and $x_{\rm c}$ describes the center position of the wave function $\psi(x-x_{\rm c})$ in flat band. Roughly speaking, the center position can be viewed as the particle position in the localized state (see Fig. 2).

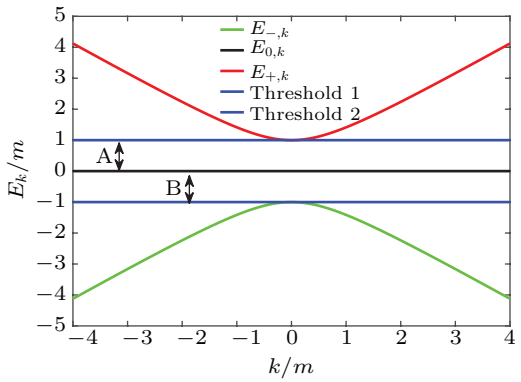


Fig. 1. The three energy bands. In the presence of Coulomb potential, some bound states may exist in regions A and B.

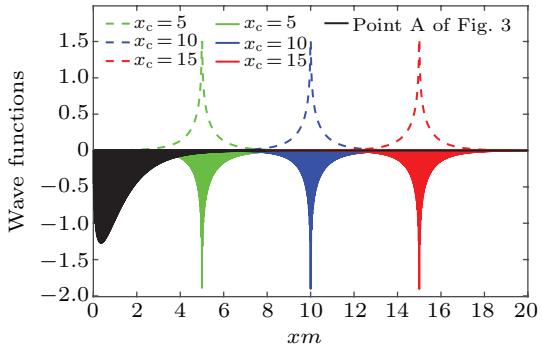


Fig. 2. Comparisons of wave functions $\phi_2(x-x_{\rm c})$ (un-normalized) between the trivial bound states in continuous band and the flat band states. The green, blue and red dashed lines are three flat band wave functions $\phi_2(x-x_{\rm c})$ for center position $x_{\rm c} = 5, 10$ and 15, respectively. Near the center position $x_{\rm c}$, the wave function has a logarithmic singularity, i.e., $\phi_2(x-x_{\rm c}) \sim -\log(|x-x_{\rm c}|) \to \infty$ as $x \to x_{\rm c}$. The green, blue and red solid lines are three corresponding wave functions for the trivial bound states in the continuous band with same center positions. The black solid line is the wave function of point indicated by black arrow of Fig. 3. Here the potential strength $\alpha = 0.25$.

Figure 2 gives the second component wave functions $\psi_2(x-x_{\rm c})$ in flat band for three different $x_{\rm c} = 5, 10, 15$ (three dashed lines). It is shown that these flat band states are localized eigen-states, in which localized length $1/m$ is determined by the particle mass m. Due to the logarithmic divergence of modified Bessel function $K_0(x)$ near the origin $x = 0$, the wave function shows a logarithmic singularity near the center position, i.e., $\psi_2(x-x_{\rm c}) \sim -\log(|x-x_{\rm c}|) \to \infty$ as $x \to x_{\rm c}$. In addition, in the following section, we will see that the singularity

of wave function is robust even in the presence of Coulomb potential (three solid lines in Fig. 2).

3. Bound states

In the following manuscript, we assume the potential energy $V_{\rm p}$ has the following diagonal form in usual basis $|i=1,2,3\rangle$, namely,

$$V_{\rm p}(x) = V_{11}(x)\otimes|1\rangle\langle 1| + V_{22}(x)\otimes|2\rangle\langle 2| + V_{33}(x)\otimes|3\rangle\langle 3| = \begin{bmatrix} V_{11}(x) & 0 & 0 \\ 0 & V_{22}(x) & 0 \\ 0 & 0 & V_{33}(x) \end{bmatrix}. \quad (5)$$

3.1. Coulomb potential of type I

In this subsection, we assume that the Coulomb potential satisfies $V_{11}(x)=V_{22}(x)=V_{33}(x)$ and

$$V_{11}(x) = \alpha/|x|, \quad (6)$$

where α describes the Coulomb potential strength. Without loss of generality, in most of this article (except for Fig. 4), we assume the potential strength to be positive, i.e., $\alpha>0$. Now the Schrödinger equation in the form of three component wave functions is

$$\begin{aligned} -{\rm i}\partial_x\psi_2(x)/\sqrt{2} &= [E-m-V_{11}]\psi_1(x), \\ -{\rm i}\partial_x[\psi_1(x)+\psi_3(x)]/\sqrt{2} &= [E-V_{11}]\psi_2(x), \\ -{\rm i}\partial_x\psi_2(x)/\sqrt{2} &= [E+m-V_{11}]]\psi_3(x). \end{aligned} \quad (7)$$

Similarly to that of short-ranged potentials,[40] we can reach an effective Schrödinger equation for the second component wave function $\psi_2(x)$, i.e,

$$\partial_x^2\psi_2(x) + \partial_x\left[\ln\left(\frac{E-V_{11}}{(E-V_{11})^2-m^2}\right)\right]\partial_x\psi_2(x) + [(E-V_{11})^2-m^2]\psi_2(x)=0. \quad (8)$$

Inserting the Coulomb potential in Eq. (8), for $x>0$, we obtain

$$\partial_x^2\psi_2(x) + \left[\frac{1}{x}+\frac{1}{x-\frac{\alpha}{E}}-\frac{1}{x-\frac{\alpha}{E+m}}-\frac{1}{x-\frac{\alpha}{E-m}}\right]\partial_x\psi_2(x) + \left[E^2-m^2-\frac{2E\alpha}{x}+\frac{\alpha^2}{x^2}\right]\psi_2(x)=0. \quad (9)$$

If the first order derivative term $\left[\frac{1}{x}+\frac{1}{x-\frac{\alpha}{E}}-\frac{1}{x-\frac{\alpha}{E+m}}-\frac{1}{x-\frac{\alpha}{E-m}}\right]\partial_x\psi_2(x)$ vanishes, the above equation would be the famous Klein–Gordon equation[43] of one dimension. As long as the Coulomb potential strength satisfies $\alpha^2\le 1/4$, the system has well-defined bound states. Furthermore, when $\alpha^2>1/4$, the resultant wave function would collapse and the system has no well-defined bound states for the Coulomb potential.

Now the non-vanishing first order derivative term would result in wave function collapses near the origin for an arbitrarily weak Coulomb potential. Consequently, the three-component flat band system has no well-defined bound states (near the origin $x=0$). This is because, near the origin $x=0$, the roots of indicial equation of the above second order differential equation [Eq. (9)] read (see Appendix A)

$$s = \pm{\rm i}\alpha. \quad (10)$$

Therefore the wave function near the origin $x=0$ has a form of

$$\psi_2(x)\propto x^{\pm{\rm i}\alpha} = \cos(\alpha\ln x)\pm{\rm i}\sin(\alpha\ln x), \quad (11)$$

which has infinite nodes near the origin $x=0$. It indicates that the wave function would collapse[44] for arbitrarily small Coulomb potential strength $\alpha\neq 0$. The Coulomb potential is too singular to have well-defined bound states in this three-component system. If one uses a shifted Coulomb potential, for example, $1/|x|\to 1/(|x|+x_0)$ with the cut-off parameter $0<x_0\ll 1$,[45] the bound states can exist. However, due to the wave function collapse for the true Coulomb potential, the eigen-energies would depend sensitively on the cut-off length x_0 (see Appendix A).

In order to further explore the properties of flat band states, we solve Eq. (7) numerically. The coordinate space is discretized with finite difference method. To be specific, we take the linear length of the system as $L=20$ and the number of spacial grids as $N=2000$. Therefore the spatial step $\Delta x = L/N = 0.01$, and a $3N\times 3N$ matrix can be established with zero boundary conditions at two ends. Then we diagonalize it to get the eigen-energies and eigen-states. The results are reported in Figs. 2 and 3.

It is found that once the Coulomb potential is turned on, the flat band gradually evolves into a continuous band (see the blue lines in Fig. 3), and then the flat band is destroyed completely by an arbitrarily weak Coulomb potential, which is similar to two-dimensional result.[38,39] At the same time, the original flat band states have been transformed into the states of the continuous band.

We plot the wave functions for three typical states in the continuous band (solid lines in Fig. 2). It is found that the wave functions of these states are also localized with three center positions $x_{\rm c}=5$, 10, 15, respectively. These localized states basically inherit the localized properties of original flat band states (see the dashed lines of Fig. 2), for example, the logarithmic singularity near the $x_{\rm c}$. This is because for the Coulomb potential of type I, the three diagonal matrix elements of potential are equal (in the usual basis) [see Eq. (6)]. One can also view the potential energy in the energy band basis $(|+,k\rangle, |0,k\rangle$ and

$|-,k\rangle$), consequently the diagonal form with three equal matrix elements is also unchanged approximately. Therefore for the three energy bands (flat band, upper and lower bands), they experience three same Coulomb potentials.

Due to the spatial localization of the flat band states, once the potential is turned on, the energies of these localized states at x only shift a value of $V(x)$, i.e., $E=0\to E=0+V(x)=\alpha/x$. Consequently, the localized properties are basically same to the original flat band states. Therefore, in this sense, we call them the "trivial bound states". It should be remarked that these bound sates (localized at $x_c=\alpha/E$) are far away from the origin $x=0$, then the wave functions are not affected by the wave function collapses near the origin.

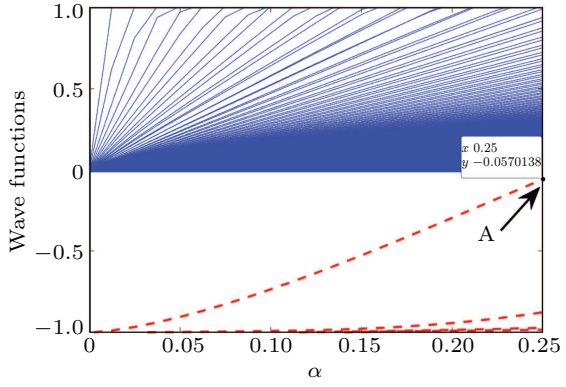


Fig. 3. The bound state energy of Coulomb potential of type I. The bound state energies are obtained with finite difference method. The flat band gradually becomes a continuous band (the blue solid lines). The red dashed lines are the bound states near the origin for the shifted Coulomb potential $1/(|x|+x_0)$.

In addition, for positive (negative) potential strength α, some states in lower (upper) band can be pushed (pulled) into the gaps, and then they form bound states (see the red dashed lines in Fig. 3). We should point out that these bound states are not true bound states for a true Coulomb potential with $1/x$ singularity. These bound states are due to our finite difference method. In our numerical method, the spatial step Δx is always finite, so we could not faithfully simulate the true $1/|x|$ singularity of Coulomb potential near the origin $x=0$. In fact, the true Coulomb potential would result in the wave function collapses. Consequently, any well-defined bound states near the origin $x=0$ could not exist. Due to the finiteness of spatial step Δx, the $1/|x|$ singularity near the origin has been shifted, e.g., $1/|x|\to 1/(|x|+x_0)$ with the cut-off parameter $x_0\ll 1$. Then, the bound states can exist in the shifted Coulomb potential. Due to collapse nature of wave function near origin, the eigen-energy depends sensitively on the cut-off length x_0 (see Appendix). The wave function $[\psi_2(x)]$ for a typical bound state (corresponding to the point in Fig. 3 indicated by black arrow) in the shifted potential is reported in Fig. 2 (see the black solid line). In comparison with the trivial bound states, the wave function of the bound state near the origin has no logarithmic singularity.

3.2. Coulomb potential of type II

In this subsection, we assume the potential energy satisfies $V_{11}(x)=V_{33}(x)\equiv 0$ and

$$V_{22}(x)=\alpha/|x|, \tag{12}$$

where α describes the potential strength. Now the Schrödinger equation can be rewritten as

$$\begin{aligned}
&-\mathrm{i}\partial_x\psi_2(x)/\sqrt{2}=[E-m]\psi_1(x),\\
&-\mathrm{i}\partial_x[\psi_1(x)+\psi_3(x)]/\sqrt{2}=[E-V_{22}(x)]\psi_2(x),\\
&-\mathrm{i}\partial_x\psi_2(x)/\sqrt{2}=[E+m]\psi_3(x).
\end{aligned} \tag{13}$$

Similarly to the above subsections, we obtain an effective Schrödinger equation (a second order differential equation)

$$\partial_x^2\psi_2(x)+\frac{[E-V_{22}(x)](E^2-m^2)}{E}\psi_2(x)=0. \tag{14}$$

Next, let us write it into a form of an ordinary second order Schrödinger equation, e.g.,

$$-\partial_x^2\psi_2(x)+\tilde{V}\psi_2(x)=\tilde{E}\psi_2(x), \tag{15}$$

where effective total energy $\tilde{E}$ and effective potential energy $\tilde{V}$ are given by

$$\begin{aligned}
&\tilde{E}=E^2-m^2,\\
&\tilde{V}(x)=\tilde{E}-\frac{[E-V_{22}(x)](E^2-m^2)}{E}=\frac{V_{22}(x)(E^2-m^2)}{E},
\end{aligned} \tag{16}$$

and $\tilde{E}<0$ for bound states. In the determining effective total energy, we use the fact that potential energy $V_{22}(x)$ goes to zero as $x\to\pm\infty$. It shows that when $\alpha/E>0$, the effective potential $\tilde{V}(x)=V_{22}(x)(E^2-m^2)/E<0$ is attractive, then there would exist bound states. The above effective potential can be written as an effective Coulomb potential, e.g.,

$$\tilde{V}(x)=\frac{V_{22}(x)(E^2-m^2)}{E}=\frac{\tilde{\alpha}}{|x|}, \tag{17}$$

where effective potential strength

$$\tilde{\alpha}=\frac{(E^2-m^2)\alpha}{E}. \tag{18}$$

According to the one-dimensional hydrogen-atomic energy formula, we get the quantized total energy, i.e.,

$$\tilde{E}_n\equiv\tilde{E}=-\frac{\tilde{\alpha}^2}{4n^2}, \tag{19}$$

where quantum number $n=1,2,3,\ldots$ Further using Eqs. (18) and (19), we obtain the bound state energy (see Fig. 4)

$$E_{\mathrm{B}}=E_n\equiv E=\frac{m\alpha}{\sqrt{4n^2+\alpha^2}}. \tag{20}$$

It is found that there exist infinite bound states generated from the flat band. For a given n, when $\alpha\to\infty$, the energy approaches the thresholds of continuous spectrum, i.e., $E_n\to$

$\pm m$, which is similar to that of short-ranged potential.[40] The wave function (see Fig. 5)

$$\psi_{2,n}(x) = x\mathrm{e}^{-\sqrt{m^2-E_n^2}x}L^1_{n-1}(2\sqrt{m^2-E_n^2}x), \tag{21}$$

where

$$L_n^{n'}(z) = \sum_{k=0}^{n} C_{n+n'}^{n-k}(-z)^k/k! = \frac{z^{-n'}\,\mathrm{e}^z\,\mathrm{d}^n[z^{n'+n}\,\mathrm{e}^{-z}]}{n!\,\mathrm{d}z^n}$$

is the generalized Laguerre polynomial,[42] $C_n^k = \frac{n!}{(n-k)!k!}$ is the binomial coefficient, and factorial $n! = 1\times 2\times 3\times\cdots\times n$.

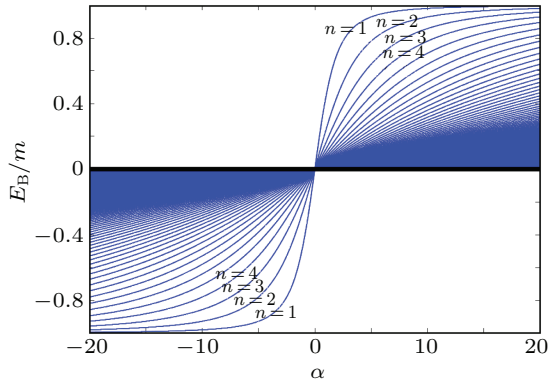


Fig. 4. The bound state energy of Coulomb potential of type Ⅱ [Eq. (21)].

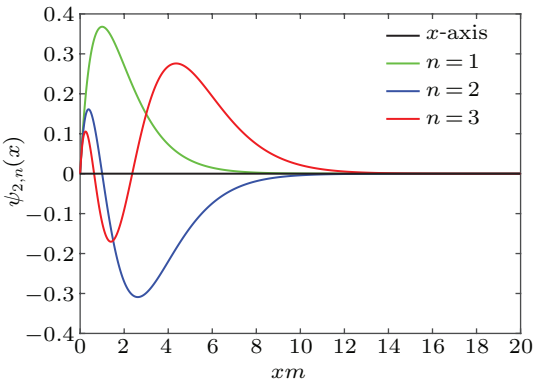


Fig. 5. The three (un-normalized) wave functions of Coulomb potential potential of type Ⅱ [see Eq. (21)], with quantum number $n = 1,2,3$. Here the potential strength $\alpha = 0.25$.

When $\alpha \ll n$, the energy can be approximated with

$$E_{\mathrm{B}} = E_n = E \simeq \frac{m\alpha}{2n}, \tag{22}$$

where quantum number $n = 1,2,3,\ldots$. For every n, due to the presence of parity symmetry ($x\to -x$), the eigen-energy is doubly degenerate, which corresponds two eigen-states with an odd parity and and an even parity. When the potential strength α is small, the eigen-energy is proportional to α, which is similar to the short-ranged potential case.[40] In addition, it shows that, in the presence of long-ranged Coulomb potential, the bound state energy is inversely proportional to the natural number, i.e., $E_n \propto 1/n$, which is different from that ($E_n \propto 1/n^2$) in the case of short-ranged square well potential. $E_n \propto 1/n$ of the Coulomb potential makes level distances between the adjacent states apart farther than that $1/n^2$ of short-ranged square well. It indicates that the long-ranged potential has more abilities to pull some states out the flat band than its short-ranged counterpart.

4. Conclusion

In summary, we have investigated the bound states in a spin-1 Dirac model with a long-ranged Coulomb potential. It is found in the presence of Coulomb potential of type I, the flat band is destroyed completely. Furthermore, the flat band evolves into a continuous band with the increasing of potential strength. The states in the continuous band have a similar logarithmic singularity as that of the flat band states. In addition, the Coulomb potential causes the wave function collapses near the origin. Due to collapses near the origin, the eigen-energies of shifted Coulomb potential depend sensitively on the cut-off length.

In the presence of Coulomb potential of type Ⅱ, there exists an infinite number of bound states, which is generated from the flat band. Furthermore, when the bound state energies are near the zero energy of the flat band, the energies are inversely proportional to the natural number, e.g., $E_n \propto 1/n$, which is different from that of its shorted-ranged counterpart, i.e., $E_n \propto 1/n^2$ in square well potential. It is expected that the wave function collapses and $1/n$ energy spectrum could be verified experimentally[46–49] in future. Finally, the Coulomb potential of type Ⅲ would also result in the $1/n$ energy spectrum,[50] and bound state in the continuous spectrum (BIC).[51]

Appendix A: The wave function collapses near the origin in the presence of Coulomb potential of type I

For $x > 0$, the effective Schrödinger equation (9) is

$$\partial_x^2\psi_2(x) + \left[\frac{1}{x} + \frac{1}{x-\frac{\alpha}{E}} - \frac{1}{x-\frac{\alpha}{E+m}} - \frac{1}{x-\frac{\alpha}{E-m}}\right]\partial_x\psi_2(x)$$
$$+ \left[E^2 - m^2 - \frac{2E\alpha}{x} + \frac{\alpha^2}{x^2}\right]\psi_2(x) = 0. \tag{A1}$$

Near the origin $x = 0$, the above equation is reduced to

$$\partial_x^2\psi_2(x) + \frac{1}{x}\partial_x\psi_2(x) + \frac{\alpha^2}{x^2}\psi_2(x) \simeq 0. \tag{A2}$$

Furthermore, one can assume that the wave function has the following form:

$$\psi_2(x) = x^s, \tag{A3}$$

where s is the root of the indicial equation of the above differential equation near the origin $x = 0$. Substituting it in Eq. (A2), we obtain an indicial equation, i.e.,

$$s(s-1) + s + \alpha^2 = 0. \tag{A4}$$

Thus, we get two distinct roots

$$s = \pm \mathrm{i}\alpha. \tag{A5}$$

Consequently the wave function near the origin is

$$\psi_2(x) \simeq x^s = x^{\pm \mathrm{i}\alpha} = \cos[\alpha \ln(x)] \pm \mathrm{i} \sin[\alpha \ln(x)], \tag{A6}$$

which has infinite nodes near the origin $x = 0$. It indicates that the wave function would collapse near the origin $x = 0$[49] for arbitrarily weak Coulomb potential $\alpha \neq 0$. A similar collapse induced by strong Coulomb potential is also found in the two-dimensional gapped[38] and gapless[50] spin-1 system.

Near the origin $x = 0$ (a regular singularity), in principle, the above equation Eq. (A1) can be solved with series expansion (Frobenius method).[42] However, the resultant recurrence relation between the expansion coefficients is very complex. In the following, in order to see the processes of collapse, we would solve a simplified equation to simulate the collapse physics qualitatively.

In the following, we assume that the potential strength is positive, i.e., $\alpha > 0$ (there is a similar discussion for $\alpha < 0$). First of all, we see that the effective Schrödinger equation (A1) has four regular singularities, i.e., $x = 0$, $\frac{\alpha}{E}$, $\frac{\alpha}{E+m}$, $\frac{\alpha}{E-m}$. For $x > 0$ and $|E|/m < 1$, the most nearest singularity relative to the origin $x = 0$ is $x = x_1 \equiv \frac{\alpha}{E+m} > 0$. At this singularity, the coefficient of Eq. (A1) becomes infinity, the wave function should vanish. If we only consider the solution between interval $[0, x_1]$, we may neglect other singularity terms, then the equation becomes

$$\partial_x^2 \psi_2(x) + \left[\frac{1}{x} - \frac{1}{x - \frac{\alpha}{E+m}}\right] \partial_x \psi_2(x) + \left[E^2 - m^2 - \frac{2E\alpha}{x} + \frac{\alpha^2}{x^2}\right] \psi_2(x) = 0. \tag{A7}$$

There are also some difficulties to solve Eq. (A7), then we further drop the singularity term $\frac{1}{x - \frac{\alpha}{E+m}}$. Then the equation becomes

$$\partial_x^2 \psi_2(x) + \frac{1}{x} \partial_x \psi_2(x) + \left[E^2 - m^2 - \frac{2E\alpha}{x} + \frac{\alpha^2}{x^2}\right] \psi_2(x) = 0. \tag{A8}$$

At the same time, we need a zero boundary condition at $x = x_1 = \frac{\alpha}{E+m}$ to solve the above eigen-value problem.

Equation (A8) has a similar collapse near the origin $x = 0$ as Eq. (A1). The merit of it lies on that it can be solved with some special functions.[53] Its general solution is

$$\psi_2(x) = c_1 x^{\mathrm{i}\alpha} \mathrm{e}^{-\sqrt{m^2 - E^2} x} {}_1F_1(A, B, 2\sqrt{m^2 - E^2} x) + c_2 x^{-\mathrm{i}\alpha} \mathrm{e}^{-\sqrt{m^2 - E^2} x} {}_1F_1(A^*, B^*, 2\sqrt{m^2 - E^2} x), \tag{A9}$$

where ${}_1F_1(A, B, z) = \sum_{k=0}^{\infty} \frac{(A)_k z^k}{k!(B)_k}$ is confluent hypergeometric function,[42] $(A)_k = A \times (A+1) \times (A+2) \times \cdots \times (A+k-1)$, and c_1 (c_2) are two arbitrary constants. $A = \frac{1}{2} + \mathrm{i}\alpha + \frac{\alpha E}{\sqrt{m^2 - E^2}}$, $B = 1 + 2\mathrm{i}\alpha$, and A^* is the complex conjugate of A.

In order to get the eigen-energy of Eq. (A8), we assume near the origin $x = 0$, the wave function at $x = x_0 \ll 1$ is also zero. It is equivalent that the Coulomb potential is shifted, i.e., $1/|x| \to 1/(|x| + x_0)$. Using the zero boundary conditions at $x_1 = \alpha/(E+m)$ and x_0, the eigen-energy equation can be obtained as follows:

$$\begin{aligned}
\psi_2(x_0) &= c_1 x_0^{\mathrm{i}\alpha} \mathrm{e}^{-\sqrt{m^2 - E^2} x_0} {}_1F_1(A, B, 2\sqrt{m^2 - E^2} x_0) \\
&\quad + c_2 x_0^{-\mathrm{i}\alpha} \mathrm{e}^{-\sqrt{m^2 - E^2} x_0} {}_1F_1(A^*, B^*, 2\sqrt{m^2 - E^2} x_0) \\
&= 0, \\
\psi_2(x_1) &= c_1 x_1^{\mathrm{i}\alpha} \mathrm{e}^{-\sqrt{m^2 - E^2} x_1} {}_1F_1(A, B, 2\sqrt{m^2 - E^2} x_1) \\
&\quad + c_2 x_1^{-\mathrm{i}\alpha} \mathrm{e}^{-\sqrt{m^2 - E^2} x_1} {}_1F_1(A^*, B^*, 2\sqrt{m^2 - E^2} x_1) \\
&= 0.
\end{aligned} \tag{A10}$$

In order to have non-trivial solutions of c_1 and c_2, the corresponding determinant should vanish, i.e.,

$$\mathrm{Det} \begin{vmatrix} x_0^{\mathrm{i}\alpha} {}_1F_1(A, B, 2\sqrt{-\tilde{E}} x_0) & x_0^{-\mathrm{i}\alpha} {}_1F_1(A^*, B^*, 2\sqrt{-\tilde{E}} x_0) \\ x_1^{\mathrm{i}\alpha} {}_1F_1(A, B, 2\sqrt{-\tilde{E}} x_1) & x_1^{-\mathrm{i}\alpha} {}_1F_1(A^*, B^*, 2\sqrt{-\tilde{E}} x_1) \end{vmatrix} = 0, \tag{A11}$$

where $\tilde{E} = E^2 - m^2$.

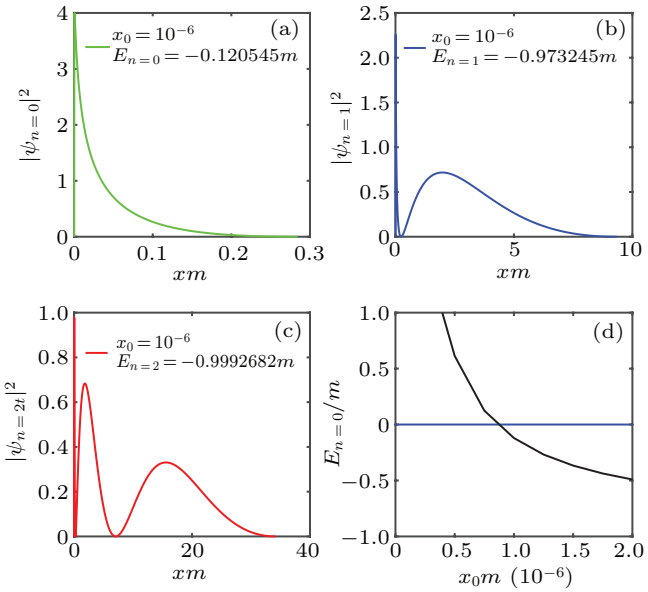


Fig. A1. The squares of (un-normalized) wave functions and eigen-energy for a shifted Coulomb potential $\alpha/(|x| + x_0)$. The panel (a), (b) and (c) are the squares of three wave functions with $x_0 = 10^{-6}$, and the Coulomb potential strength $\alpha = 0.25$. The panel (d) shows the increasing eigen-energy E_0 with the decreasing x_0. Here the quantum number n is the number of nodes of wave function (except for two ends of interval, x_0 and x_1). The quantum number for the "ground state" is $n = 0$. Due to the potential strength $\alpha > 0$ (repulsive potential), the ground state ψ_0 has highest energy among these bound states.

The results are reported in Fig. A1. The panel (d) shows that the ground state energy E_0 grows up with the decreasing cut-off parameter x_0. Due to the wave function collapses near

the origin $x = 0$, the eigen-energies depend sensitively on the cut-off x_0. In addition, the weights of wave functions accumulate gradually near the origin. This reflect that the wave function collapses, and the system has no ground state for true Coulomb potential.

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References

[1] Sutherland B 1986 *Phys. Rev. B* **34** 5208
[2] Vidal J, Mosseri R and Doucot B 1998 *Phys. Rev. Lett.* **81** 5888
[3] Bergman D L, Wu C and Balents L 2008 *Phys. Rev. B* **78** 125104
[4] Bercioux D, Urban D F, Grabert H and Häusler W 2009 *Phys. Rev. A* **80** 063603
[5] Leykam D, Andreanov A and Flach S 2018 *Adv. Phys. X* **3:1** 1473052
[6] Dóra B, Kailasvuori J and Moessner R 2011 *Phys. Rev. B* **84** 195422
[7] Raoux A, Morigi M, Fuchs J N, Piéchon F and Montambaux G 2014 *Phys. Rev. Lett.* **112** 026402
[8] Liu Z, Liu F and Wu Y S 2014 *Chin. Phys. B* **23** 077308
[9] Peotta S and Törmä P 2015*Nat. Commun.* **6** 8944
[10] Cao Y *et al.* 2018 *Nature* **556** 43
[11] Hazra T, Verma N,and Randeria M 2019 *Phys. Rev. X* **9** 031049
[12] Xu H Y and Lai Y C 2020 *Phys. Rev. Res.* **2** 013062
[13] Wang F and Ran Y 2011 *Phys. Rev. B* **84** 241103
[14] Chen Y R, Xu Y, Wang J, Liu J F and Ma Z 2019 *Phys. Rev. B* **99** 045420
[15] Wu Y R and Zhang Y C 2021 *Chin. Phys. B* **30** 060306
[16] Wu J *et al* 2021 *Chin. Phys. B* **30** 077104
[17] Mielke A 1999 *Phys. Rev. Lett.* **82** 4312
[18] Leykam D, Bodyfelt J D, Desyatnikov A S and Flach S 2017 *Eur. Phys. J. B* **90** 1
[19] Shen R, Shao L B, Wang B and Xing D Y 2010 *Phys. Rev. B* **81** 041410
[20] Urban D F, Bercioux D, Wimmer M, and Häusler W 2011 *Phys. Rev. B* **84** 115136
[21] Fang A, Zhang Z Q, Louie S G and Chan C T 2016 *Phys. Rev. B* **93** 035422
[22] Betancur-Ocampo Y, Cordourier-Maruri G, Gupta V and de Coss R 2017 *Phys. Rev. B* **96** 024304
[23] Yang S, Gu Z C, Sun K and Das Sarma S 2012 *Phys. Rev. B* **86** 241112
[24] Biswas T and Ghosh T K 2018 *J. Phys.: Condens. Matter* **30** 075301
[25] Tovmasyan M, Peotta S, Liang L, Törmä P and Huber S D 2018 *Phys. Rev. B* **98** 134513
[26] Volovik G E 2019 *JETP Lett.* **110** 352
[27] Julku A, Peltonen T J, Liang L, Heikkilä T T and Törmä P 2020 *Phys. Rev. B* **101** 060505
[28] Kopnin N B, Heikkila T T and Volovik G E 2011 *Phys. Rev. B* **83** 220503
[29] Mukherjee S *et al.* 2015 *Phys. Rev. Lett.* **114** 245504
[30] Zhang S, Hung H H and Wu C 2010 *Phys. Rev. A* **82** 053618
[31] Iglovikov V I *et al.* 2014 *Phys. Rev. B* **90** 094506
[32] Julku A *et al.* 2016 *Phys. Rev. Lett.* **117** 045303
[33] Liang L *et al.* 2017 *Phys. Rev. B* **95** 024515
[34] Iskin M 2019 *Phys. Rev. A* **99** 053608
[35] Wu Y R, Zhang X F, Liu C F, Liu W M and Zhang Y C 2021 *Sci. Rep.* **11** 13572
[36] Xu F, Zhang L and Jiang L Y 2021 *Chin. Phys. B* **30** 067401
[37] Economou E N 2006 *Green's Functions in Quantum Physics* 3rd edn (Berlin: Springer-Verlag)
[38] Gorbar E V, Gusynin V P and Oriekhov D O 2019 *Phys. Rev. B* **99** 155124
[39] Pottelberge R V 2020 *Phys. Rev. B* **101** 197102
[40] Zhang Y C and Zhu G B 2022 *J. Phys. B: At. Mol. Opt. Phys.* (accepted)
[41] Zhang Y C, Song S W, Liu C F and Liu W M 2013 *Phys. Rev. A* **87** 023612
[42] Wang Z X and Guo D R 1989 *Special Functions* (Singapore: World Scientific)
[43] Strange P 1998 *Relativistic Quantum Mechanics* 1st edn (Cambridge: Cambridge University Press)
[44] Landau L D and Lifshitz E M 1977 *Quantum mechanics: Non-relativistic theory* 3rd edn (Orxford: Pergamon Press)
[45] Downing C A and Portnoi M E 2014 *Phys. Rev. A* **90** 052116
[46] Huang X, Lai Y, Hang Z H, Zheng H and Chan C T 2011 *Nat. Mater.* **10** 582
[47] Chan C T, Hang Z H and Huang X 2012 *Adv. Optoelectron.* **2012** 313984
[48] Wang Y *et al.* 2013 *Science* **340** 734
[49] Mao J *et al.* 2016 *Nat. Phys.* **12** 545
[50] Zhang Y C 2022 *Phys. Scr.* **97** 015401
[51] Zhang Y C 2021 *Bound States in the Continuum (BIC) Protected By Self-Sustained Potential Barriers in a Flat Band System* (Preprint at Works Across Apps and Sites)
[52] Han C D, Xu H Y, Huang D and Lai Y C 2019 *Phys. Rev. B* **99** 245413
[53] Dong S H 2003 *Phys. Scr.* **67** 89