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Pseudospin symmetric solutions of the Dirac equation with the modified Rosen–Morse potential using Nikiforov–Uvarov method and supersymmetric quantum mechanics approach

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Employing the Pekeris-type approximation to deal with the pseudo-centrifugal term, we analytically study the pseudospin symmetry of a Dirac nucleon subjected to equal scalar and vector modified Rosen–Morse potential including the spin–orbit coupling term by using the Nikiforov–Uvarov method and supersymmetric quantum mechanics approach. The complex eigenvalue equation and the total normalized wave functions expressed in terms of Jacobi polynomial with arbitrary spin–orbit coupling quantum number k are presented under the condition of pseudospin symmetry. The eigenvalue equations for both methods reproduce the same result to affirm the mathematical accuracy of analytical calculations. The numerical solutions obtained for different adjustable parameters produce degeneracies for some quantum number.

Keywords: Dirac equation, modified Rosen–Morse potential, Nikiforov–Uvarov method, supersymmetric quantum mechanics approach

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1. Introduction

About fifty years ago, the understanding of magic numbers was encapsulated within the nuclear shell model.[1,2] People found a quasi-degeneracy in single-nucleon states with quantum numbers $(n_r, l, j = l + 1/2)$ and $(n_r - 1, l + 2, j = l + 3/2)$, where n_r, l and j represent the radial, orbital and total angular momentum quantum numbers, respectively. This quasidegeneracy structure was defined as the so-called "pseudospin" symmetry basically expressed in terms of a "pseudo" orbital angular momentum $\tilde{l} = l + 1$ and a "pseudospin" $\tilde{s} = 1/2$. From then on, there has been a tremendous success in providing interpretation to a number of phenomena in nuclear physics, including the nuclear superdeformed configurations,[3,4] identical bands,[5–7] quantized alignment,[8] magnetic moments and transitions.[9–11] Researchers began to study this symmetry using some relativistic mean-field models as well as nonrelativistic phenomenological models.[12–23] Nevertheless, until about the late 1990s, this symmetry was found to emanate from Dirac Hamiltonian's relativistic symmetry where the scalar and the vector potentials are of equal magnitude and opposite sign.[24] More details and recent information about pseudospin can be seen in the following literatures.[25,26] Dirac equation is used for the description of spin-1/2 particles.[27] Spin symmetry is applicable in particle and high energy physics for the description of mesons. However, pseudo symmetry arises when the sum of the repulsive Lorentz vector potential $V(r)$ and the attractive Lorentz scalar potential $S(r)$ is constant, that is, $\Sigma(r) = V(r) + S(r)$.[28–30] On the other hand, the supersymmetric description can be regarded as an important supplement to the relativistic description for the pseudospin symmetry due to its ability to overcome the difficulties encountered in the conventional relativistic description of pseudospin symmetry, that is, the difference between two partner Hamiltonians involved in lower component of Dirac wave function cannot be considered small as compared to the potential difference.[31,32] Actually, during an attempt to give interpretation of magic numbers, many phenomenological potential models such as the square well, the harmonic oscillator and the Woods–Saxon potentials were widely used as scalar and vector potentials including the spin–orbit potentials in order to test this symmetry as aforementioned. Typical scientific research works can be attributed to Haxel, Jensen, Suess[33] and Mayer,[34] in which they independently introduced this type of spin–orbit potentials which largely splits the states with high orbital angular momentum. Stimulated by these studies, we attempt to examine the pseudospin symmetry using a modified Rosen–Morse potential within the framework of Nikiforov–Uvarov method and shape invariance formalism via supersymmetric quantum mechanics approach, which have not been considered yet to the best of our knowledge. It should be mentioned that the modified Rosen–Morse potential are proposed to fit the effect of inner-shell radii of two particles for quantum system in the

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form of[35]

$$V(r) = D_{\rm e}\left(1 - \frac{{\rm e}^{\alpha r_{\rm e}} + \lambda}{{\rm e}^{\alpha r} + \lambda}\right)^2, \tag{1}$$

where the parameters $\lambda = {\rm e}^{\alpha r_{ij}}$, $D_{\rm e}$ is the dissociation energy and $r_{\rm e}$ is the equilibrium bond length. The symbol r_{ij} represents the nucleon distance depending on the definition of the dummy variable ij. If $i, j = {\rm e}$, then $r_{ij} = r_{\rm e}$, which is the equilibrium bond length and when $i, j = 1$, then $r_{ij} = r$, which is the internuclear distance. The potential of Eq. (1) is a long range potential used for the description of vibrational molecular energies for diatomic and polyatomic molecules as well as some physical systems. This potential approaches to zero as the internuclear distances approaches to infinity.[36] Many fantastic research has been reported for the relativistic and nonrelativistic quantum mechanics. Du *et al.*[37] studied the Schrödinger equation for a particle constrained to move on a rotating curved surface by using thin layer scheme with proper choice of gauge transformation for the wave function. Mahmoud and co-authors[38] recently examined the approximate solution to the time dependent Kratzer plus screened Coulomb potential in the Feinberg–Horodecki equation. Also, Ikot *et al.*[39] studied minimal length quantum mechanics of Dirac particles in non commutative space for spin 1/2 particles where they obtained energy spectra and energy eigen function as well as special cases. Gao and Zhang[40] obtained the analytical solutions to the Schrödinger equation with the Eckart potential using the Grenee–Aldrich approximation to centrifugal term within Nikiforov–Uvarov method. They obtained the discrete energy spectrum and the wave function expressed in terms of Jacobi polynomial. Their resulting energy equation agrees excellently with that obtained by other methods. This article addresses two important issues. Firstly, under the condition of pseudospin symmetry, the corresponding spinor wave functions of the Dirac equation with the modified Rosen–Morse potential is obtained by using the parametric Nikiforov–Uvarov method. Secondly, the existence of the pseudospin degeneracies of the Dirac equation is verified through numerical solutions obtained from the resulting energy eigen equation. This research paper is divided into six sections. Section 1 gives the brief introduction of the article. The solution to Dirac equation with pseudospin symmetry through the parametric Nikiforov–Uvarov method is presented in Section 2. The analytical determination of the normalization constant is presented in Section 3. The solution of the proposed potential through supersymmetric quantum mechanics approach (SUSY) is presented in Section 4. The numerical results and discussion are presented in Section 5. The article is finally concluded in Section 6.

2. Solutions to Dirac equation with pseudospin symmetry using parametric NU

The Nikiforov–Uvarov (NU) method is based on reducing the second order linear differential equation to a generalized equation of hyper-geometric type and provides exact solutions in terms of special orthogonal functions like Jacobi and Laguerre as well as corresponding energy eigenvalues.[41–43] The standard differential equation for parametric NU method according to Tezcan and Sever[44] is given as

$$\psi''(s) + \frac{(c_1 - c_2 s)}{s(1 - c_3 s)}\psi'(s) + \frac{1}{s^2(1 - c_3 s)^2}\left[-\Omega_1 s^2 + \Omega_2 s - \Omega_3\right]\psi(s) = 0. \tag{2}$$

The parametric constants are obtained as follows:

$$\begin{aligned} &c_4 = \frac{1}{2}(1 - c_1);\ \ c_5 = \frac{1}{2}(c_2 - c_3);\ \ c_6 = c_5^2 + \varepsilon_1;\\ &c_7 = 2c_4c_5 - \Omega_2;\ \ c_8 = c_4^2 + \Omega_3;\ \ c_9 = c_3c_7 + c_3^2c_8 + c_6;\\ &c_{10} = c_1 + 2c_4 + 2\sqrt{c_8};\ \ c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right);\\ &c_{12} = c_4 + \sqrt{c_8};\ \ c_{13} = c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right). \end{aligned} \tag{3}$$

The total eigenvalue equation is given as

$$\begin{aligned} &c_2 n - (2n+1)c_5(2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + n(n-1)c_3\\ &+ c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0. \end{aligned} \tag{4}$$

The corresponding total wave function is then given as

$$\begin{aligned} \Psi(s) \ =\ & N_{nl}s^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}}\\ & \times P_n^{\left(c_{10}-1,\ \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s). \end{aligned} \tag{5}$$

The Dirac equation with an attractive scalar $S(r)$ and a repulsive vector potential $V(r)$ for a particle of mass M in units $\hbar = c = 1$ reads

$$\left[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta(M + S(r))\right]\Psi(r) = (E - V(r))\Psi(r), \tag{6}$$

where E is the relativistic energy of the system, $\boldsymbol{p}$ is the three dimensional momentum operator and $\boldsymbol{\alpha}$ and β are 4×4 Pauli matrices defined as $\alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Here, I is the 2×2 unitary matrix and σ_i are the three vector spin matrices given as $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -{\rm i} \\ {\rm i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. According to the principle of formulas (2)–(5) in literature,[19] a differential equation for lower component can be expressed as

$$\begin{aligned} &\left\{\frac{{\rm d}^2}{{\rm d}r^2} - \frac{k(k-1)}{r^2} - [M + E_{nk} - \Delta(r)][M - E_{nk} + \Sigma(r)]\right.\\ &\left. - \frac{\frac{{\rm d}\Sigma(r)}{{\rm d}r}\left(\frac{\rm d}{{\rm d}r} - \frac{k}{r}\right)}{M - E_{nk} + \Sigma(r)}\right\}G_{nk}(r) = 0, \end{aligned} \tag{7}$$

where $\Sigma(r) = V(r) + S(r)$ and $\Delta(r) = V(r) - S(r)$. To study the properties of the pseudospin symmetry, we take

$\mathrm{d}\Sigma(r)/\mathrm{d}r = 0$ or $\Sigma(r) = C =$ constant, and $\Delta(r)$ as the modified Rosen–Morse potential and inserting it into Eq. (7) leads to a Schrödinger like equation

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{k(k-1)}{r^2} - E_{nk}^2 - M^2 - C(M+E_{nk}) + C(M-E_{nk}) + (M-E_{nk}+C)V(r)\right\}G_{nk}(r) = 0. \tag{8}$$

Let $\beta = (M - E_{nk} + C)$, $P = M + E_{nk}$, $L = k(k-1)$ and $\tilde{E}_{nk} = E_{nk}^2 + M^2$, Eq. (8) now reduces to

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{L}{r^2} + \tilde{E}_{nk} - PC + \beta V(r)\right\}G_{nk}(r) = 0. \tag{9}$$

Substituting Eq. (1) into Eq. (9) gives

$$\frac{\mathrm{d}^2 G_{nk}(r)}{\mathrm{d}r^2} + \left[-\frac{L}{r^2} + \tilde{E}_{nk} - PC + \beta D_\mathrm{e}\left(1 - \frac{\mathrm{e}^{\alpha r_\mathrm{e}} + \lambda}{\mathrm{e}^{\alpha r} + \lambda}\right)^2\right]G_{nk}(r) = 0. \tag{10}$$

This equation can only be solved analytically when $k = 1$. Therefore, applying Pekeris-type approximation[45] to the pseudo-centrifugal term $k(k-1)/r^2$, which has been proven effective in solving the scattering state problem.[46] The approximate formula can be expressed as

$$\frac{1}{r^2} \approx d_0 + \frac{d_1}{\mathrm{e}^{\alpha r} + \lambda} + \frac{d_2}{(\mathrm{e}^{\alpha r} + \lambda)^2}, \tag{11}$$

where the parameters

$$d_0 = \frac{\mathrm{e}^{-2\alpha r_\mathrm{e}}\left[3(\mathrm{e}^{\alpha r_\mathrm{e}}+\lambda)^2 + \alpha r_\mathrm{e}(-2\mathrm{e}^{\alpha r_\mathrm{e}}\lambda + \lambda^2 + \mathrm{e}^{2\alpha r_\mathrm{e}}(-3+\alpha r_\mathrm{e}))\right]}{\alpha^2 r_\mathrm{e}^4},$$
$$d_1 = \frac{2\mathrm{e}^{-2\alpha r_\mathrm{e}}(\mathrm{e}^{\alpha r_\mathrm{e}}+\lambda)^2[-\lambda(3+\alpha r_\mathrm{e}) + \mathrm{e}^{\alpha r_\mathrm{e}}(-3+2\alpha r_\mathrm{e})]}{\alpha^2 r_\mathrm{e}^4},$$
$$d_2 = -\frac{\mathrm{e}^{-2\alpha r_\mathrm{e}}(\mathrm{e}^{\alpha r_\mathrm{e}}+\lambda)^3[\mathrm{e}^{\alpha r_\mathrm{e}}(\alpha r_\mathrm{e}-3) - \lambda(3+\alpha r_\mathrm{e})]}{\alpha^2 r_\mathrm{e}^4}. \tag{12}$$

Substituting Eq. (11) into Eq. (10) gives

$$\frac{\mathrm{d}^2 G_{nk}(r)}{\mathrm{d}r^2} + \left[\tilde{E}_{nk} - \left(Ld_0 + \frac{Ld_1}{\mathrm{e}^{\alpha r} + \lambda} + \frac{Ld_2}{(\mathrm{e}^{\alpha r} + \lambda)^2}\right) - PC + \beta D_\mathrm{e}\left(1 - \frac{\mathrm{e}^{\alpha r_\mathrm{e}} + \lambda}{\mathrm{e}^{\alpha r} + \lambda}\right)^2\right]G_{nk}(r) = 0. \tag{13}$$

Let $(\mathrm{e}^{\alpha r} + \lambda)$ be expressed in the form of $\mathrm{e}^{\alpha r}(1 + \lambda\,\mathrm{e}^{-\alpha r})$ and $s = -\lambda\,\mathrm{e}^{-\alpha r}$, Eq. (13) can be transformed from r-dimension to S-dimension as follows:

$$\frac{\mathrm{d}^2 G_{nk}(r)}{\mathrm{d}s^2} + \frac{(1-s)}{s(1-s)}\frac{\mathrm{d}G_{nk}(r)}{\mathrm{d}s} + \frac{1}{s^2(1-s)^2}\left[-(\Lambda_1 + \Lambda_2 + \Lambda_3 - \Lambda_4 + \Lambda_5 - \Lambda_6)s^2 + (2\Lambda_1 + \Lambda_2 - 2\Lambda_4 + \Lambda_5)s - (\Lambda_1 - \Lambda_4)\right]G_{nk}(s) = 0, \tag{14}$$

where

$$\Lambda_1 = \frac{Ld_0}{\alpha^2},\ \Lambda_2 = \frac{Ld_1}{\lambda\alpha^2},\ \Lambda_3 = \frac{Ld_2}{\lambda^2\alpha^2},$$
$$\Lambda_4 = \frac{(\tilde{E}_{nk} - PC + \beta D_\mathrm{e})}{\alpha^2},\ \Lambda_5 = \frac{2\beta D_\mathrm{e}}{\alpha^2\lambda}(\mathrm{e}^{\alpha r_\mathrm{e}} + \lambda),$$
$$\Lambda_6 = \frac{\beta D_\mathrm{e}}{\alpha^2\lambda^2}(\mathrm{e}^{\alpha r_\mathrm{e}} + \lambda)^2. \tag{15}$$

Comparing Eq. (14) to Eq. (2), we can obtain the following polynomials:

$$\Omega_1 = \Lambda_1 + \Lambda_2 + \Lambda_3 - \Lambda_4 + \Lambda_5 - \Lambda_6,$$
$$\Omega_2 = 2\Lambda_1 + \Lambda_2 - 2\Lambda_4 + \Lambda_5,$$
$$\Omega_3 = \Lambda_1 - \Lambda_4. \tag{16}$$

By making use of Eq. (3), the following parameter constants are obtained:

$$\left.\begin{aligned}
&c_1 = 1,\ \ c_2 = c_3 = 1,\ \ c_4 = 0,\ \ c_5 = -\tfrac{1}{2},\\
&c_6 = \tfrac{1}{4} + (\Lambda_1 + \Lambda_2 + \Lambda_3 - \Lambda_4 + \Lambda_5 - \Lambda_6),\\
&c_7 = (-2\Lambda_1 - \Lambda_2 + 2\Lambda_4 - \Lambda_5),\\
&c_8 = (\Lambda_1 - \Lambda_4),\\
&c_9 = \left(\tfrac{1}{4} + \Lambda_3 - \Lambda_6\right);\ \ c_{10} = 1 + 2\sqrt{\Lambda_1 - \Lambda_4},\\
&c_{11} = 2 + 2\left(\sqrt{\tfrac{1}{4} + \Lambda_3 - \Lambda_6} + \sqrt{\Lambda_1 - \Lambda_4}\right),\\
&c_{12} = \left(\sqrt{\Lambda_1 - \Lambda_4}\right),\\
&c_{13} = -\tfrac{1}{2} - \left(\sqrt{\tfrac{1}{4} + \Lambda_3 - \Lambda_6} + \sqrt{\Lambda_1 - \Lambda_4}\right).
\end{aligned}\right\} \tag{17}$$

By making use of Eq. (4) with much algebraic simplification, the pseudospin eigenvalue equation can be represented in the form

$$(E_{nk}^2 - M^2) - [k(k-1)d_0 + (M + E_{nk})C - \beta D_\mathrm{e}] + \frac{\alpha^2}{4}\left(n + \delta + \frac{\frac{Q_1}{\alpha^2}}{n+\delta}\right)^2 = 0, \tag{18}$$

where $Q_1 = \frac{1}{\lambda^2}[\beta D_\mathrm{e}(\mathrm{e}^{\alpha r_\mathrm{e}} + \lambda)(\mathrm{e}^{\alpha r_\mathrm{e}} - 2\lambda) - k(k-1)\lambda d_1 - k(k-1)d_2]$, and $\delta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{k(k-1)d_2 - \beta D_\mathrm{e}(\mathrm{e}^{\alpha r_\mathrm{e}} + \lambda)}{\alpha^2\lambda^2}}$. The corresponding total pseudospin wave function expressed in terms of Jacobi polynomial can be calculated by using Eq. (5) as follows:

$$G_{nk}(s) = N_{nl}\, s^{\chi_1}(1-s)^{\chi_2} P_n^{2\chi_1, 2\chi_2 - 1}(1-2s), \tag{19}$$

where $\chi_1 = \left(\sqrt{\Lambda_1 - \Lambda_4}\right)$, and $\chi_2 = \frac{1}{2} + \sqrt{\frac{1}{4} + \Lambda_3 - \Lambda_6}$.

3. Analytical calculation of the normalization constant

The pseudospin wave function as given in Eq. (19) can be normalized by using the condition

$$\int_0^\infty |\, G_{nk}(r)\,|^2\, \mathrm{d}r = 1. \tag{20}$$

Substituting Eq. (19) into Eq. (20) gives

$$N_{nl}^2 \int_0^\infty |\, s^{\chi_1}(1-s)^{\chi_2} P_n^{[2\chi_1, 2\chi_2 - 1]}(1-2s)\,|^2\, \mathrm{d}r = 1. \tag{21}$$

Recall that $s = -\lambda\,\mathrm{e}^{-\alpha r}$, then the wave function is assumed to be bound at $r \in (0, \infty)$ and $s \in (-\lambda, 0)$. If $\lambda = -1$ the

wave function is bounded in $s \in (1,0)$ such that the integral of Eq. (21) takes the form

$$-\frac{N_{nl}^2}{\alpha}\int_1^0 s^{2\chi_1-1}(1-s)^{2\chi_2}\,|\,P_n^{[2\chi_1,2\chi_2-1]}(1-2s)\,|^2\,\mathrm{d}s=1. \quad (22)$$

Let $z=1-2s$ such that the boundaries of integration change from $s \in (1,0)$ to $z \in (-1,1)$, then Eq. (22) reduces to

$$\frac{N_{nl}^2}{2\alpha}\int_{-1}^1\left[\frac{1-z}{2}\right]^{2\chi_1-1}\left[\frac{1+z}{2}\right]^{2\chi_2}\,|\,P_n^{[2\chi_1,2\chi_2-1]}(z)\,|^2\,\mathrm{d}z=1. \quad (23)$$

Equation (23) can be evaluated by using the standard integral[47,48]

$$\int_{-1}^1\left[\frac{1-w}{2}\right]^x\left[\frac{1+w}{2}\right]^y\,|\,P_n^{x,y-1}(w)\,|^2\,\mathrm{d}w = \left[\frac{2^{x+y+1}\Gamma(x+n+1)\Gamma(y+n+1)}{n!\Gamma(x+y+n+1)\Gamma(x+y+2n+1)}\right]. \quad (24)$$

Substituting Eq. (24) into Eq. (23), the normalized constant is obtained as

$$N_{nl}=\sqrt{\frac{2\alpha n!\Gamma(2\chi_1+2\chi_2+n)\Gamma(2\chi_1+2\chi_2+2n)}{2^{2\chi_1+2\chi_2}\Gamma(2\chi_1+n)\Gamma(2\chi_2+n+1)}}. \quad (25)$$

Hence, the total normalized pseudospin wave function is given as

$$G_{nk}(s) = \sqrt{\frac{2\alpha n!\Gamma(2\chi_1+2\chi_2+n)\Gamma(2\chi_1+2\chi_2+2n)}{2^{2\chi_1+2\chi_2}\Gamma(2\chi_1+n)\Gamma(2\chi_2+n+1)}} \times s^{\chi_1}(1-s)^{\chi_2}P_n^{[2\chi_1,2\chi_2-1]}(1-2s). \quad (26)$$

4. Pseudospin solutions within supersymmetric quantum mechanics approach

Within the concept of SUSY, Eq. (13) should be arranged in the form

$$\frac{\mathrm{d}^2G_{nk}(r)}{\mathrm{d}r^2}+\left[\tilde{E}_{nk}-V_{\mathrm{eff.}}(r)\right]G_{nk}(r)=0, \quad (27)$$

where the effective potential $V_{\mathrm{eff.}}(r)$ is given as

$$V_{\mathrm{eff.}}(r) = \left(Ld_0+\frac{Ld_1}{\mathrm{e}^{\alpha r}+\lambda}+\frac{Ld_2}{(\mathrm{e}^{\alpha r}+\lambda)^2}\right)+PC -\beta D_\mathrm{e}\left(1-2\left(\frac{\mathrm{e}^{\alpha r_\mathrm{e}}+\lambda}{\mathrm{e}^{\alpha r}+\lambda}\right)+\frac{(\mathrm{e}^{\alpha r_\mathrm{e}}+\lambda)^2}{(\mathrm{e}^{\alpha r}+\lambda)^2}\right). \quad (28)$$

The superpotential for the proposed potential is

$$\omega(r)=B-\frac{A\,\mathrm{e}^{\alpha r}}{\mathrm{e}^{\alpha r}+\lambda}. \quad (29)$$

The square of the superpotential is given as

$$\omega^2(r)=B^2-\frac{2AB\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)}+\frac{A^2\,\mathrm{e}^{2\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)^2}. \quad (30)$$

The two supersymmetric partner potentials can be obtained by using the following equations:

$$V_+(r) = \omega^2(r)+\omega'(r) = B^2-\frac{2AB\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)}+\frac{A^2\,\mathrm{e}^{2\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)^2}-\frac{A\alpha\lambda\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)^2}, \quad (31)$$

$$V_-(r) = \omega^2(r)-\omega'(r) = B^2-\frac{2AB\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)}+\frac{A^2\,\mathrm{e}^{2\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)^2}+\frac{A\alpha\lambda\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)^2}. \quad (32)$$

Equations (31) and (32) obey the shape invariance condition given as

$$V_+(r,a_j)=V_-(r,a_{j+1})+R_{a_j}. \quad (33)$$

The ground state energy can be calculated by solving the associated Riccati equation

$$V_-(r)=V_{\mathrm{eff.}}(r)-\tilde{E}_{0k}. \quad (34)$$

Substituting Eqs. (28) and (32) into Eq. (34) gives

$$B^2-\frac{2AB\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)}+\frac{A^2\,\mathrm{e}^{2\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)^2}+\frac{A\alpha\lambda\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}+\lambda)^2} = \left(Ld_0+\frac{Ld_1}{\mathrm{e}^{\alpha r}+\lambda}+\frac{Ld_2}{(\mathrm{e}^{\alpha r}+\lambda)^2}\right)+PC -\beta D_\mathrm{e}\left(1-2\left(\frac{\mathrm{e}^{\alpha r_\mathrm{e}}+\lambda}{\mathrm{e}^{\alpha r}+\lambda}\right)+\frac{(\mathrm{e}^{\alpha r_\mathrm{e}}+\lambda)^2}{(\mathrm{e}^{\alpha r}+\lambda)^2}\right)-\tilde{E}_{0k}. \quad (35)$$

Solving Eq. (35) gives rise to three simultaneous equations

$$\begin{aligned}
&\tilde{E}_{0k}=Ld_0+PC-\beta D_\mathrm{e}-(A-B)^2,\\
&2PC\lambda+2\beta D_\mathrm{e}(b-\lambda)+Ld_1+2\lambda Ld_0-2\tilde{E}_{0k}\lambda\\
&\quad=2\lambda B^2-2AB\lambda+A\alpha\lambda,\\
&PC\lambda^2+Ld_0\lambda^2+Ld_1\lambda+Ld_2-\beta D_\mathrm{e}\left(\lambda^2-2b\lambda+b^2\right)-\tilde{E}_{0k}\lambda^2\\
&\quad=B^2\lambda^2,
\end{aligned} \quad (36)$$

where $b=(\mathrm{e}^{\alpha r_\mathrm{e}}+\lambda)$. The ground state energy E_{0k} will be calculated by using Eq. (36). The solution of Eq. (36) satisfies the following conditions:

$$A=\alpha\left(\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{Ld_2-\beta D_\mathrm{e}b^2}{\alpha^2\lambda^2}}\right), \quad (37)$$

$$B=\frac{1}{2}\left[A-\frac{\frac{1}{\lambda^2}\left(\beta D_\mathrm{e}b(b-2\lambda)-Ld_1\lambda-Ld_2\right)}{A}\right]. \quad (38)$$

The excited supersymmetric state energy is obtained by using the shape invariant condition. Hence, Eq. (34) can be written in the form

$$V_-(r,a)=V_{\mathrm{eff.}}(r)-\tilde{E}_{0k}. \quad (39)$$

Substituting Eqs. (34) and (36) into Eq. (39) with simplification gives

$$V_-(r,a)=\frac{2\beta D_\mathrm{e}b+Ld_1}{(\mathrm{e}^{\alpha r}+\lambda)}+\frac{Ld_2-\beta D_\mathrm{e}b^2}{(\mathrm{e}^{\alpha r}+\lambda)^2}+(A-B)^2. \quad (40)$$

Substituting the parameter of Eq. (37) into Eq. (40) with simplification gives

$$V_{-}(r,a)=\frac{2\beta D_{\mathrm{e}}b+Ld_1}{(\mathrm{e}^{\alpha r}+\lambda)}+\frac{Ld_2-\beta D_{\mathrm{e}}b^2}{(\mathrm{e}^{\alpha r}+\lambda)^2}+\frac{1}{4}\left(A+\frac{Q_2}{A}\right)^2. \tag{41}$$

Let $A=a_0$, $a_1=a_0+1$, $a_2=a_0+2$, and $a_n=a_0+\alpha n$, then the following equation can be obtained:

$$V_{+}(r,a_0)=\frac{2\beta D_{\mathrm{e}}b+Ld_1}{(\mathrm{e}^{\alpha r}+\lambda)}+\frac{Ld_2-\beta D_{\mathrm{e}}b^2}{(\mathrm{e}^{\alpha r}+\lambda)^2}+\frac{1}{4}\left(a_0+\frac{Q_2}{a_0}\right)^2, \tag{42}$$

$$V_{-}(r,a_1)=\frac{2\beta D_{\mathrm{e}}b+Ld_1}{(\mathrm{e}^{\alpha r}+\lambda)}+\frac{Ld_2-\beta D_{\mathrm{e}}b^2}{(\mathrm{e}^{\alpha r}+\lambda)^2}+\frac{1}{4}\left(a_1+\frac{Q_2}{a_1}\right)^2. \tag{43}$$

Using Eqs. (42) and (43), one can write the following recurrence relation:

$$R(a_1)=V_{+}(r,a_0)-V_{-}(r,a_1)=\frac{1}{4}\left(a_0+\frac{Q_2}{a_0}\right)^2-\frac{1}{4}\left(a_1+\frac{Q_2}{a_1}\right)^2; \tag{44}$$

$$\vdots$$

$$R(a_n)=V_{+}(r,a_{n-1})-V_{-}(r,a_n)=\frac{1}{4}\left(a_{n-1}+\frac{Q_2}{a_{n-1}}\right)^2-\frac{1}{4}\left(a_n+\frac{Q_2}{a_n}\right)^2. \tag{45}$$

The higher order supersymmetric energy can be obtained as

$$\overline{E}_{nk}=\frac{1}{4}\left(a_0+\frac{Q_2}{a_0}\right)^2-\frac{1}{4}\left(a_n+\frac{Q_2}{a_n}\right)^2. \tag{46}$$

The total energy which is the sum of higher order energy and ground state energy can be calculated as

$$\sum_{k=1}^{n}R(a_k)=\overline{E}_{nk}+\tilde{E}_{0k}=\tilde{E}_{nk}. \tag{47}$$

Substituting Eqs. (36) and (46) into Eq. (47) gives the total supersymmetric energy as

$$\tilde{E}_{nk}=\frac{1}{4}\left(a_0+\frac{Q_2}{a_0}\right)^2-\frac{1}{4}\left(a_n+\frac{Q_2}{a_n}\right)^2+Ld_0+PC-\beta D_{\mathrm{e}}-(A-B)^2. \tag{48}$$

Recall that $(A-B)^2=\frac{1}{4}\left(A+\frac{Q_2}{A}\right)^2=\frac{1}{4}\left(a_0+\frac{Q_2}{a_0}\right)^2$. Equation (46) finally reduces to

$$\tilde{E}_{nk}=-\frac{1}{4}\left(a_n+\frac{Q_2}{a_n}\right)^2+Ld_0+PC-\beta D_{\mathrm{e}}. \tag{49}$$

The eigenvalue Eq. (49) obtained by using the supersymmetric invariance method is the same as that obtained by using the Nikiforov–Uvarov method. This affirms the high mathematical accuracy of our analytical calculations.

5. Numerical results and discussion

The eigenvalue Eq. (18) can not directly explain the pseudospin symmetry of the single nucleon spectrum. By solving the energy level equation numerically, we can find the energy level characteristics of the single nucleon spectrum under pseudospin symmetry. The numerical computation is carried out with the following set of parameters $D_{\mathrm{e}}=2000, \alpha=0.2, M=1, C=-2, r_{\mathrm{e}}=1.6, \lambda=2$, as shown in Table 1. It is worth mentioning that each quantum state has both positive and negative numerical values though predominantly negative to ascertain bound state condition and this categorizes Eq. (18) to be a transcendental equation. In this work we only report one set of numerical values for all quantum state. The quantum state of degeneracies as shown in Table 1 are in excellent agreement to the work of Okon *et al.* as reported in Ref. [30]. A lot of quantum degeneracies were observed in the numerical computation. Few among them were $(1s_{1/2}=0d_{3/2})$, $(1p_{3/2}=0f_{5/2})$, $(1d_{5/2}=0g_{7/2})$, $(2s_{1/2}=1d_{3/2})$, $(2p_{3/2}=1f_{5/2})$. Degeneracies are expected to occur in approximate bound state solutions of relativistic equations especially the Dirac and Klein–Gordon equations. The numerical data obtained from Table 1 produces a fascinating result for a Dirac nucleon. Firstly, the shell model describes how much energy is required to move nucleon from one orbit to another. The orbital angular quantum number l determines the shape of an orbital of a Dirac nucleon and the angular distribution with respect to specific sub-shell which are the s, p, d and f sub shell. The energy at s-sub shell for a Dirac nucleon is expected to be higher than p, d and f sub-orbital shell because of its closeness to the nucleus of an atom. Our results from Table 1 authenticates the existing theory of sub-orbital energy. $(1s_{1/2})$ has the pseudospin energy of -1.060406226 which is greater than the energy of $(1p_{3/2})$ with pseudospin value of -1.174078710 as expected in general perspective. Therefore, the trend of table confirms the accuracy of both numerical and analytical results.

Table 1. The bound state energy eigenvalues $E_{n_r,k}$ of pseudospin symmetry for several values n_r, k. The parameters $D_{\mathrm{e}}=2000, \alpha=0.2, M=1, C=-2, \lambda=2, r_{\mathrm{e}}=1.6$.

$\tilde{l}$	$n_r,k<0$	l,j	$E_{n_r,k<0}$	$n_r-1,k>0$	$l+2,j+1$	$E_{n_r-1,k>0}$
1	$1,-1$	$1s_{1/2}$	-1.060406226	0,2	$0d_{3/2}$	-1.060406226
2	$1,-2$	$1p_{3/2}$	-1.174078710	0,3	$0f_{5/2}$	-1.174078710
3	$1,-3$	$1d_{5/2}$	-1.338236425	0,4	$0g_{7/2}$	-1.338236425
4	$1,-4$	$1s_{7/2}$	-1.550816544	0,5	$0s_{9/2}$	-1.550816544
1	$2,-1$	$2s_{1/2}$	-1.062254026	1,2	$1d_{3/2}$	-1.062254026
2	$2,-2$	$2p_{3/2}$	-1.180686269	1,3	$1f_{5/2}$	-1.180686269
3	$2,-3$	$2d_{5/2}$	-1.352212746	1,4	$1g_{7/2}$	-1.352212746
4	$2,-4$	$2s_{7/2}$	-1.574553602	1,5	$1s_{9/2}$	-1.574553602
1	$3,-1$	$3s_{1/2}$	-1.063223028	2,2	$2d_{3/2}$	-1.063223028
2	$3,-2$	$3p_{3/2}$	-1.184549912	2,3	$2f_{5/2}$	-1.184549912
3	$3,-3$	$3d_{5/2}$	-1.360870303	2,4	$2g_{7/2}$	-1.360870303
4	$3,-4$	$3s_{7/2}$	-1.589875212	2,5	$2s_{9/2}$	-1.589875212

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