

A new algorithm for reconstructing the three-dimensional flow field of the oceanic mesoscale eddy

Chao Yan(颜超), Jing Feng(冯径), Ping-Lv Yang(杨平吕), and Si-Xun Huang(黄思训)
Citation: Chin. Phys. B, 2021, 30 (**12**): 120204. DOI: 10.1088/1674-1056/ac29af
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Novel energy dissipative method on the adaptive spatial discretization for the Allen-Cahn equation

Jing-Wei Sun(孙竟巍), Xu Qian(钱旭), Hong Zhang(张弘), and Song-He Song(宋松和)
Chin. Phys. B, 2021, 30 (**7**): 070201. DOI: 10.1088/1674-1056/abe37b

Constructing reduced model for complex physical systems via interpolation and neural networks

Xuefang Lai(赖学方), Xiaolong Wang(王晓龙, and Yufeng Nie(聂玉峰)
Chin. Phys. B, 2021, 30 (**3**): 030204. DOI: 10.1088/1674-1056/abd92e

A meshless algorithm with the improved moving least square approximation for nonlinear improved Boussinesq equation

Yu Tan(谭渝) and Xiao-Lin Li(李小林)
Chin. Phys. B, 2021, 30 (**1**): 010201. DOI: 10.1088/1674-1056/abaed7

Atomic-scale simulations of material behaviors and tribology properties for BCC metal film

H D Aristizabal, P A Parra, P López, E Restrepo-Parra
Chin. Phys. B, 2016, 25 (**1**): 010204. DOI: 10.1088/1674-1056/25/1/010204

Designing shielded radio-frequency phased-array coils for magnetic resonance imaging

Xu Wen-Long, Zhang Ju-Cheng, Li Xia, Xu Bing-Qiao, Tao Gui-Sheng
Chin. Phys. B, 2013, 22 (**1**): 010203. DOI: 10.1088/1674-1056/22/1/010203

A new algorithm for reconstructing the three-dimensional flow field of the oceanic mesoscale eddy*

Chao Yan(颜超)[1,3], Jing Feng(冯径)[1], Ping-Lv Yang(杨平吕)[1], and Si-Xun Huang(黄思训)[1,2,†]

[1] Institute of Meteorology and Oceanography, National University of Defense Technology, Changsha 410073, China
[2] State Key Laboratory of Satellite Ocean Environment Dynamics, Second Institute of Oceanography, State Oceanic Administration, Hangzhou 310012, China
[3] Basic Department, Nanjing Tech University, Pujiang Institute, Nanjing 211112, China

(Received 20 July 2021; revised manuscript received 15 September 2021; accepted manuscript online 24 September 2021)

A new algorithm for reconstructing the three-dimensional flow field of the oceanic mesoscale eddies is proposed in this paper, based on variational method. Firstly, with the numerical differentiation Tikhonov regularizer, we reconstruct the continuous horizontal flow field on discrete grid points at each layer in the oceanic region, in terms of the horizontal flow field observations. Secondly, benefitting from the variational optimization analysis and its improvement, we reconstruct a three-dimensional flow field under the constraint of the horizontal flow and the vertical flow. The results of simulation experiments validate that the relative error of the new algorithm is lower than that of the finite difference method in the case of high grid resolution, which still holds in the case of unknown observational errors or in the absence of vertical velocity boundary conditions. Finally, using the reanalysis horizontal data sourcing from SODA and the proposed algorithm, we reconstruct three-dimensional flow field structure for the real oceanic mesoscale eddy.

Keywords: mesoscale eddy, numerical differentiation, Tikhonov regularization, variational optimization analysis

PACS: 02.60.Jh, 02.60.Cb, 02.30.Xx, 92.10.ak **DOI:** 10.1088/1674-1056/ac29af

1. Introduction

Mesoscale eddy refers to the widely distributed rotating water body in the ocean with diameters ranging from tens to hundreds of kilometers in spatial scale. It is easier to detect the horizontal velocity in the eddy motion and there has been a relatively large number of statistical laws.[1,2] The vertical velocity also plays an important role in the diagnosis and forecast of mesoscale eddy motion. As the vertical velocity in the oceanic large-scale motion is very small, which is only about one percent or one thousandth of the horizontal velocity[3,4] and cannot be easily measured directly by existing observation instruments, it can be diagnosed by other factors. For incompressible fluids, the horizontal velocity u, v and the vertical velocity w satisfy the following continuity equation:

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \tag{1}$$

When the horizontal velocity u, v and upper and lower boundary conditions of the vertical velocity are known, the method of variational optimization analysis can be used to calculate the vertical velocity.[5,6] However there exist two problems:

Problem 1 The method involves the calculation of the divergence of (u, v). If u, v adopt the observation data $\tilde{u}, \tilde{v}$, then on one hand $\tilde{u}, \tilde{v}$ are of low resolution because of sparse observation points, and on the other hand observational errors of $\tilde{u}, \tilde{v}$ will bring about large errors of $\partial\tilde{u}/\partial x, \partial\tilde{v}/\partial y$ after differentiating $\tilde{u}, \tilde{v}$ by the finite difference method, and finally lead to a large error of the vertical velocity via $\widetilde{D} = \partial\tilde{u}/\partial x + \partial\tilde{v}/\partial y$.

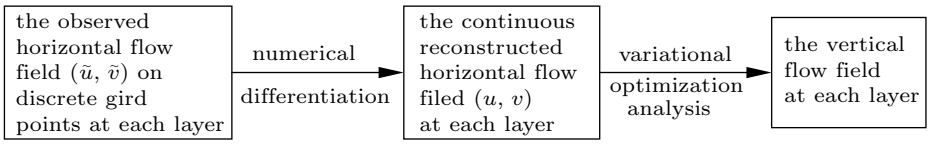
Problem 2 The method needs vertical velocity boundary conditions to calculate the vertical velocity. In the field of Oceanography, it is difficult to precisely measure the vertical flow field at sea level.

Based on the above considerations, a new algorithm for reconstructing the three-dimensional flow field of the oceanic mesoscale eddy based on variation is proposed in this paper. Firstly, in order to reduce the effect of noise and enhance the resolution of the horizontal flow field, we make use of numerical differentiation technique on the observation data $(\tilde{u}, \tilde{v})$. The numerical differentiation[7,8] proposed in recent decades for inverse problems in mathematical physics has been widely used in atmospheric science and various fields.[9–12] It is shown that the method of numerical differentiation, based on Tikhonov regularization, is feasible to analyze meteorological observation data and improve the recognition rate of small and medium scale weather systems. Secondly, we modify the method of variational optimization analysis by calculating the vertical velocity at sea level from the known vertical velocity at the seafloor to obtain the vertical velocity boundary conditions. Furthermore, we employ simulation experiments to test the sensitivity of the new algorithm to observational errors and vertical velocity boundary conditions, comparing with the result of finite difference method. Finally, we reconstruct the vertical flow field for the real oceanic two-dimensional horizontal flow field data.

2. New algorithm: mathematical theory and calculation scheme

The overall calculating procedure is depicted as follows:

*Project sported by the National Natural Science Foundation of China (Grant Nos. 41875045 and 61371119) and the Blue Project of Jiangsu Province, China.
†Corresponding author. E-mail: huangsxp@163.com
© 2021 Chinese Physical Society and IOP Publishing Ltd



2.1. One-dimensional numerical differentiation

It is a typical ill-posed problem to calculate partial derivatives of the observation horizontal flow field $(\tilde{u},\tilde{v})$ at each layer in mathematics, which can be overcome by making numerical differentiation on $(\tilde{u},\tilde{v})$. The so-called numerical differentiation is to calculate the approximate derivative of a function with function values at some discrete points, also known as the differentiation on observation data. The main problem about the numerical differentiation is that it is not suitable because a tiny observational error may lead to a large error of the numerical result. In the atmosphere and ocean research community, we often need to calculate the directional derivative in the horizontal direction for the further calculation of dispersion and vorticity. The observational error may lead to great differences between the results obtained by the difference method and the differential method respectively. In recent decades, the most important method for making the numerical differentiation is the Tikhonov regularization.

(I) Raise the question

Suppose $y = y(x)$, $x \in [0,1]$, $\Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ is an equidistant division of $[0,1]$ (non-equidistant division is also feasible), $\tilde{y}_i$ is the observation data at x_i. Denote $h_i = x_i - x_{i-1}$, and assume

1) $|\tilde{y}_i - y(x_i)| \leqslant \delta$, where δ is an observational error;

2) $\tilde{y}_0 = y(0)$, $\tilde{y}_n = y(1)$.

Construct $f_*(x)$ from the observation data $\tilde{y}_i$ $(i = 1,2,\ldots,n-1)$, maintaining a certain similarity between $f'_*(x)$ and $y'(x)$ satisfying the following two conditions:

$$\begin{aligned} J(f) &= \frac{1}{n-1}\sum_{j=1}^{n-1}(\tilde{y}_j - f(x_j))^2 + \alpha \left\|f''\right\|^2_{L^2(0,1)} = \min!, \\ &\frac{1}{n-1}\sum_{j=1}^{n-1}(\tilde{y}_j - f(x_j))^2 = \delta^2. \end{aligned} \tag{2}$$

Here, α represents the regularizing parameter, $f_*(0) = y_0$ and $f_*(1) = y_n$. It can be proved that if $\alpha = \delta^2$, then there exists a unique $f_*(x)$ of Eq. (2) with the following estimate:[13]

$$\left\|f'_* - y'\right\|^2_{L^2(0,1)} \leqslant \left(2h + 4\sqrt{\delta} + \frac{h}{\pi}\right)\|y''\|_{L^2(0,1)} + h + \sqrt{\delta}. \tag{3}$$

It is can be seen from Eq. (3) that when both n and δ are small enough, f'_* is sufficiently close to y', which means that the difference between derivatives of the reconstructed $f_*(x)$ and y are sufficiently small. Thus, $f_*(x)$ can be represented by the cubic natural spline function

$$\begin{aligned} f_*(x) &= a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \\ &j = 0,1,2,\ldots,n-1. \end{aligned} \tag{4}$$

(II) Calculate a_j, b_j, c_j and d_j

Denote

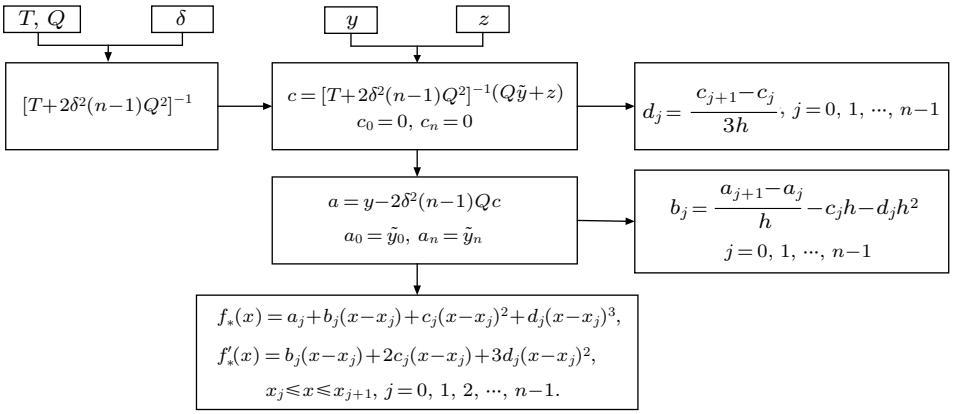
$$T = h \cdot \begin{pmatrix} 4/3 & 1/3 & 0 & \cdots & 0 & 0 & 0 \\ 1/3 & 4/3 & 1/3 & \cdots & 0 & 0 & 0 \\ 0 & 1/3 & 4/3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 4/3 & 1/3 & 0 \\ 0 & 0 & 0 & \cdots & 1/3 & 4/3 & 1/3 \\ 0 & 0 & 0 & \cdots & 0 & 1/3 & 4/3 \end{pmatrix}_{(n-1)\times(n-1)},$$

$$Q = \frac{1}{h}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}_{(n-1)\times(n-1)},$$

$$z = (\tilde{y}_0, 0, 0, \ldots, 0, \tilde{y}_n)^{\mathrm{T}}/h,\ \ c = (c_1, c_2, \ldots, c_{n-1})^{\mathrm{T}},$$

$$a = (a_1, a_2, \ldots, a_{n-1})^{\mathrm{T}},\ \ y = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{n-1})^{\mathrm{T}}.$$

$f_*(x)$ can be reconstructed by calculating a_j, b_j, c_j and d_j as follows:



Remark 1 Since we obtain the cubic spline expression (4) of the function $f_*(x)$ on each sub-interval, the observation field can be reconstructed and refined to any resolution.

It is required to find exact values of $y(x)$ at $x=0$ and $x=1$ for the calculation of $f_*(x)$, which is impossible. Therefore, it is feasible to take $\tilde{y}_0$, $\tilde{y}_n$ as y_0, y_n, with errors only at $x=0$, $x=1$, and take $f_*(x)$ on $[x_1, x_{n-1}]$, in order to guarantee sufficient accuracy.

2.2. Reconstruct the vertical field from the horizontal flow field

Problem Assume that the observation flow field are $\tilde{u}(x,y,z)$ and $\tilde{v}(x,y,z)$, while the vertical observation data is unknown. The flow field area is set to be $\Omega \times [0,H]$ and the analyzed flow field (u,v,w) is required to satisfy

$$J=\int_0^H \int_\Omega \left[(u-\tilde{u})^2+(v-\tilde{v})^2\right] \mathrm{d}x\mathrm{d}y\mathrm{d}z = \min!$$

and meet the following constraint:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Suppose it is known that $w|_{z=0}=w_0$ and $w|_{z=H}=w_H$, inspired by Sasaki's work,[5] the following equalities hold by the method of variation:

$$u=\tilde{u}-\frac{\partial \lambda}{\partial x},\quad v=\tilde{v}-\frac{\partial \lambda}{\partial y}. \tag{5}$$

The λ meets

$$\Delta\lambda=\frac{1}{H}\int_0^H \tilde{D}\mathrm{d}z+\frac{1}{H}(w_H-w_0),\quad \lambda|_{\partial\Omega}=0, \tag{6}$$

where $\tilde{D}=\partial\tilde{u}/\partial x+\partial\tilde{v}/\partial y$ can be calculated by the numerical differentiation.

The relaxation method can be used to find out $\lambda(x,y)$. Then u,v can be obtained by Eq. (5) and the vertical flow field w can be diagnosed as

$$w(x,y,z)=w_0-\int_0^z \tilde{D}\mathrm{d}z+z\Delta\lambda. \tag{7}$$

Remark 2 For the ocean floor, it can be set as $w_0=0$. But w_H cannot be observed generally, which is always handled as $w_H=-\int_0^H \tilde{D}\mathrm{d}z$ approximately.

3. Simulation experiment

In order to validate the effectiveness of the proposed algorithm, we first employ simulation experiments. Let $\Omega=[0,2\pi]\times[0,2\pi]\times[0,0.2\pi]$, (u,v,w) gives the real flow field

$$\begin{cases} u^{\mathrm{T}}=5\sin x\cos y\cos 10z, \\ v^{\mathrm{T}}=5\cos x\sin y\cos 10z, \\ w^{\mathrm{T}}=-\cos x\cos y\sin 10z. \end{cases} \tag{8}$$

Obviously $\partial u^{\mathrm{T}}/\partial x+\partial v^{\mathrm{T}}/\partial y+\partial w^{\mathrm{T}}/\partial z=0$.

Suppose that the observation horizontal flow field $(\tilde{u},\tilde{v})$ is generated by adding a random error to the real flow field $(u^{\mathrm{T}},v^{\mathrm{T}})$, which subjects to the uniform distribution on the interval $[-\delta,\delta]$. We employ the following four simulation experiments:

Scheme A_1: Calculate the horizontal dispersion $\tilde{D}$ of the observation flow field by the finite difference method, and then obtain the vertical velocity by solving the equation $\partial w/\partial z=-\tilde{D}$ by the finite difference method.

Scheme A_2: Calculate the horizontal dispersion $\tilde{D}$ of the observation flow field by the finite difference method, obtain λ by solving Eq. (6), substitute it into Eq. (5) to produce (u,v), and finally calculate the vertical velocity by Eq. (7).

Scheme A_3: Substitute the observation flow field with the reconstructed flow field, calculate the horizontal dispersion $\tilde{D}$ of the reconstructed flow field by the method of numerical differentiation, and then obtain the vertical velocity by solving the equation $\partial w/\partial z=-\tilde{D}$ by the finite difference method.

Scheme B: The implementation of the new algorithm. Substitute the observation flow field with the reconstructed flow field, calculate the horizontal dispersion $\tilde{D}$ of the reconstructed flow field by the method of numerical differentiation, obtain λ by solving Eq. (6), substitute it into Eq. (5) to produce (u,v), and finally calculate the vertical velocity by Eq. (7).

Definition 1 The relative error between the vertical velocity and the real vertical velocity is defined as

$$\rho_w=\left(\sum_{i,j,k}(w-w_e)^2\right)^{1/2}\Big/\left(\sum_{i,j,k}(w_e^2)\right)^{1/2}.$$

3.1. Scheme comparison with definite δ

Conclusion 1 In the case of calculating derivatives with then same method, the method based on variation is superior to the traditional finite difference method for the vertical velocity calculation. Given $\delta=0.05$, Table 1 shows relative errors of calculation with different grid resolutions. The relative error of the scheme A_2 is much smaller than that of the scheme A_1, and the relative error of the scheme B is much smaller than that of schemes A_1, A_2, A_3.

Conclusion 2 The relative error of the scheme A_1 and scheme A_2 increases with the increment of resolution, while the relative error of the scheme B and scheme A_3 is almost unchanged, which further verified that the numerical differentiation is not suitable, and a small disturbance value of the observation flow field may lead to a large calculation error.

As seen from Table 1, the relative error of the scheme B is smaller than that of the scheme A_2. When the grid resolution is $160\times160\times40$, the relative error of the scheme B is 84% smaller than that of the scheme A_2. The results show that for the vertical velocity calculation, the regularization method can effectively reduce the relative error compared with the method based on variation.

Table 1. Relative error of vertical velocity at different grid resolutions with $\delta = 0.05$.

	$40\times40\times40$	$80\times80\times40$	$120\times120\times40$	$160\times160\times40$
Scheme-A_1	14.1%	15.6%	17.7%	20.2%
Scheme-A_2	3.2%	6.0%	8.8%	11.6%
Scheme-A_3	13.5%	13.5%	13.5%	13.5%
Scheme-B	2.1%	1.9%	1.8%	1.8%

In summary, the scheme B is the optimal scheme for calculating the vertical velocity. When the grid resolution is $160\times160\times40$, the relative error of the scheme B is reduced by 91% compared with the traditional finite difference scheme A_1, which means that the new algorithm is very effective.

3.2. Scheme comparison with unknown δ

Although the regularizing parameter is taken as $\alpha = \delta^2$, the observational error δ is unknown in many practical situations. Therefore, it is necessary to test the calculation results of the scheme B according to the change of the presumed observational error δ'. Suppose the true observational error is $\delta = 0.05$, and the regularizing parameter is $\alpha = \delta'^2$.

Table 2 shows the relative error of vertical velocity obtained by the scheme B with different δ'. As can be seen from the table, even if the exact value of δ is unknown, the vertical velocity obtained by the new algorithm is still more accurate than that obtained by the finite difference method, as long as δ' is within a certain range. In some cases, such as $\delta' = 0.010$ and $\delta' = 0.005$, the relative error is smaller than that in the case of $\delta' = \delta = 0.05$. Therefore, according to the results shown in Table 2, as long as δ' is within the range of $[0.01\delta, 2\delta]$, the relative error of the scheme B is smaller than that of the scheme A_1.

Table 2. Relative error of the scheme B with different δ'.

δ'	$40\times40\times40$	$80\times80\times40$	$120\times120\times40$	$160\times160\times40$
0.0005	7.04%	7.85%	5.81%	5.01%
0.001	6.01%	4.31%	3.44%	3.19%
0.005	2.03%	1.55%	1.38%	1.27%
0.010	1.45%	1.10%	1.00%	0.93%
0.05	2.10%	1.90%	1.80%	1.80%
0.100	6.32%	6.14%	6.11%	6.11%
0.250	28.09%	28.18%	28.21%	28.24%
0.500	60.72%	60.92%	61.00%	61.04%

3.3. Scheme comparison without boundary conditions

The vertical velocity can be obtained by neither of the four methods when boundary conditions are unknown at $z = 0$ and $z = H$. In many practical problems, it is difficult to obtain vertical velocity boundary conditions at both the top and the bottom. Therefore, it is necessary to solve the problem of missing boundary conditions when calculating the vertical velocity by the scheme A_2 or the scheme B. Assume that the vertical velocity at $z = 0$ is known and the vertical velocity at $z = H$ is unknown. Then the following scheme can be taken to solve the problem.

Scheme B_1 Calculate the vertical velocity at $z = H$ by the scheme A_3 and take it as the boundary condition. Then recalculate the vertical velocity by the scheme B.

When $\delta = 0.05$, the vertical velocity at $z = 0.25\pi$ is not zero by Eq. (7).

As can be seen from Table 3, even the boundary condition is predicted by the finite difference method of the scheme B_1, the variation-based method is still superior to the pure finite difference method of the scheme A_3.

Table 3. Relative error without boundary condition with $\delta = 0.05$.

	$40\times40\times50$	$80\times80\times50$	$120\times120\times50$	$160\times160\times50$
Scheme-A_3	12.3%	12.3%	12.3%	12.3%
Scheme-B_1	6.2%	6.4%	6.5%	6.5%

4. Reconstruction experiment on the vertical velocity of the oceanic mesoscale eddy

The experimental area is selected in 16.8750°N–21.8750°N, 290.1250°–295.6250°W, depth of 5 m to 299.93 m below sea level (20 layers in total), as shown in Fig. 1(a). The horizontal oceanic data is sourcing from SODA (https://www2.atmos.umd.edu/ocean/) for January in 2001, SODA is the reanalysis data jointly released by the Department of Atmospheric and Oceanographic Sciences at the University of Maryland and the Department of Oceanography at TAMU, has a horizontal resolution of $0.25^\circ \times 0.25^\circ$.

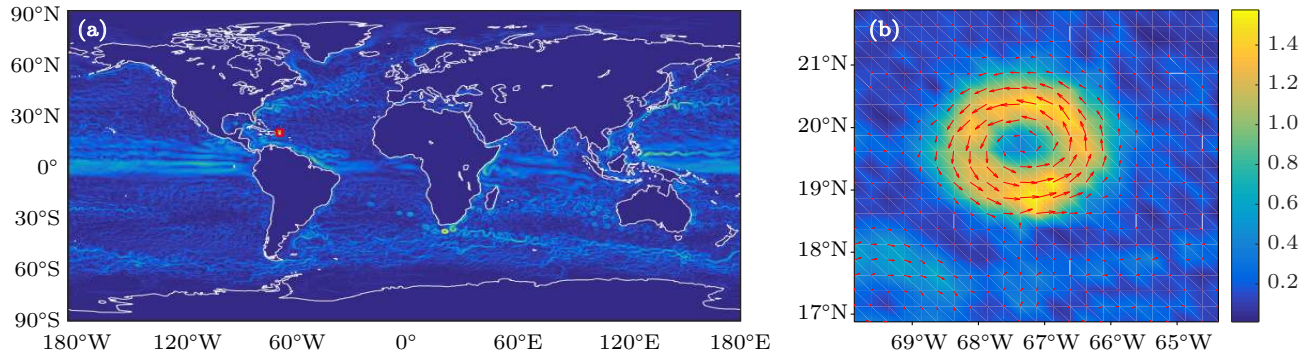


Fig. 1. (a) The selected flow field and (b) the vector graph in this paper.

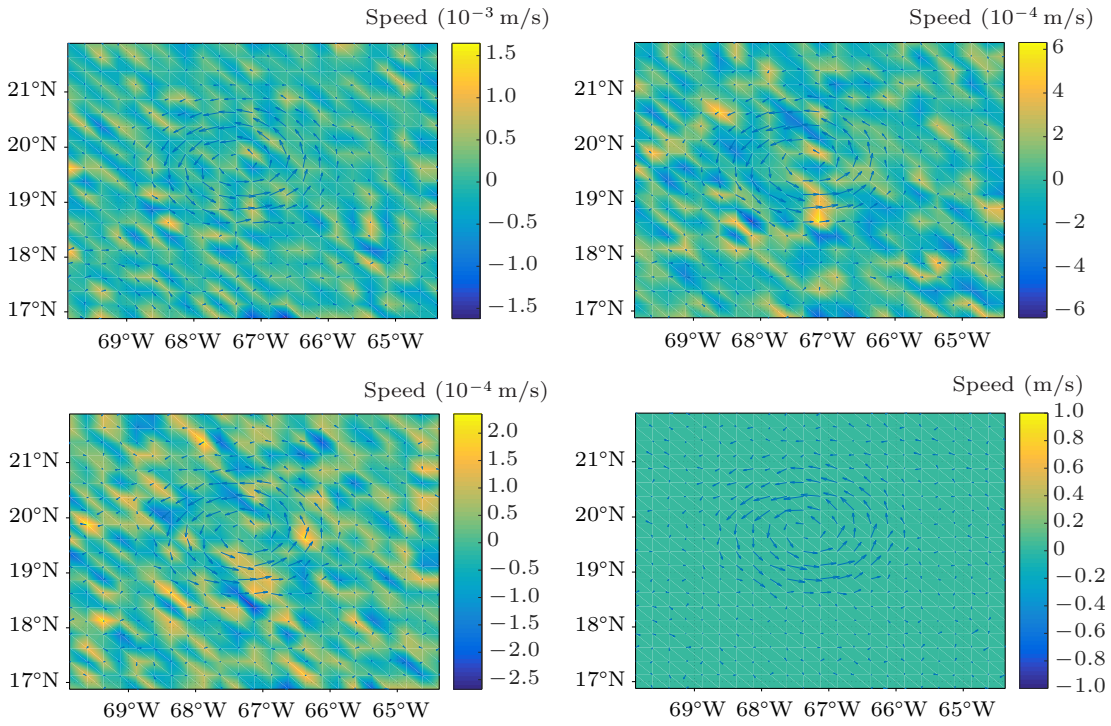


Fig. 2. The distribution of vertical velocity at depth of 5 m, 115.87 m, 222.71 m and 299.93 m by the finite difference method with $\delta = 0.15$.

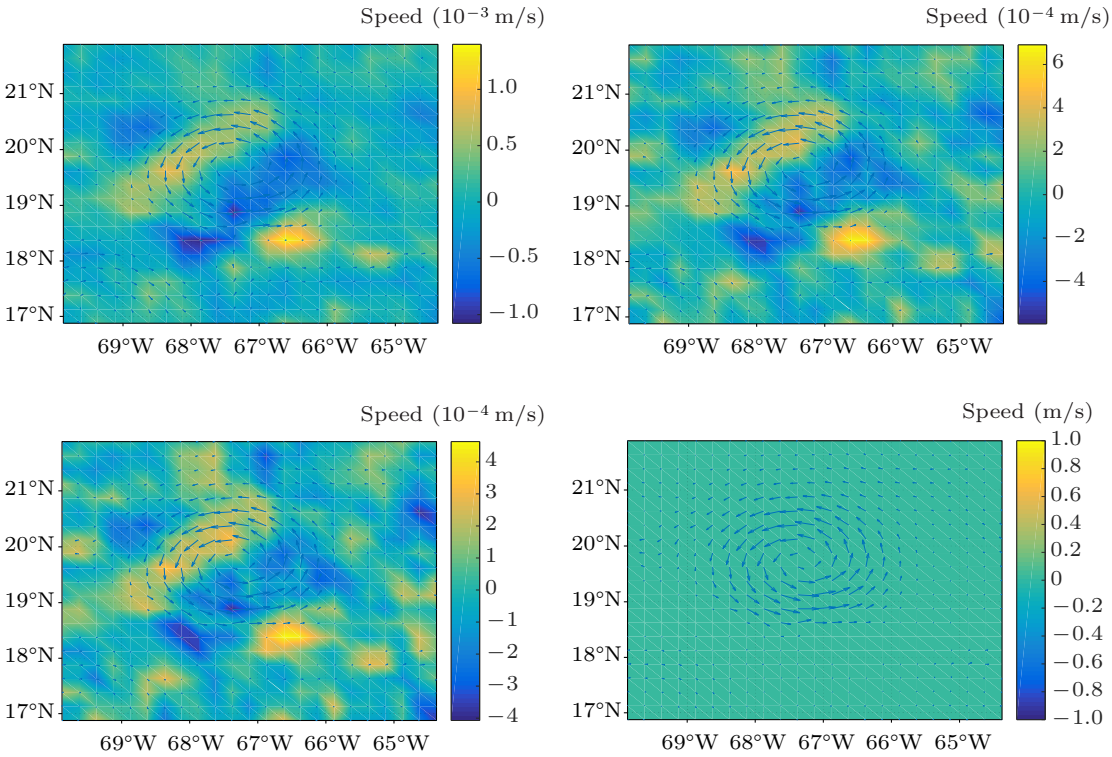


Fig. 3. The distribution of vertical velocity at depth of 5 m, 115.87 m, 222.71 m and 299.93 m by the new algorithm with $\delta = 0.15$.

As seen from Fig. 1(a), there exists a mesoscale eddy in the region with a diameter of 70–100 km, the vector graph of the eddy is shown in Fig. 1(b), it is a cyclonic eddy, and there is a strong south-to-north ocean current flowing into this eddy in the south, while the horizontal velocity is relatively slow in other' region except for a small eddy below left.

We use the new algorithm and the finite difference method respectively to calculate the vertical velocity w. Firstly, we take 299.93 m below sea level as the bottom layer. The vertical velocity of this layer is approximately 0 by analyzing the horizontal flow field of the layer and we can get the vertical velocity of the first layer, considered as the boundary condition,

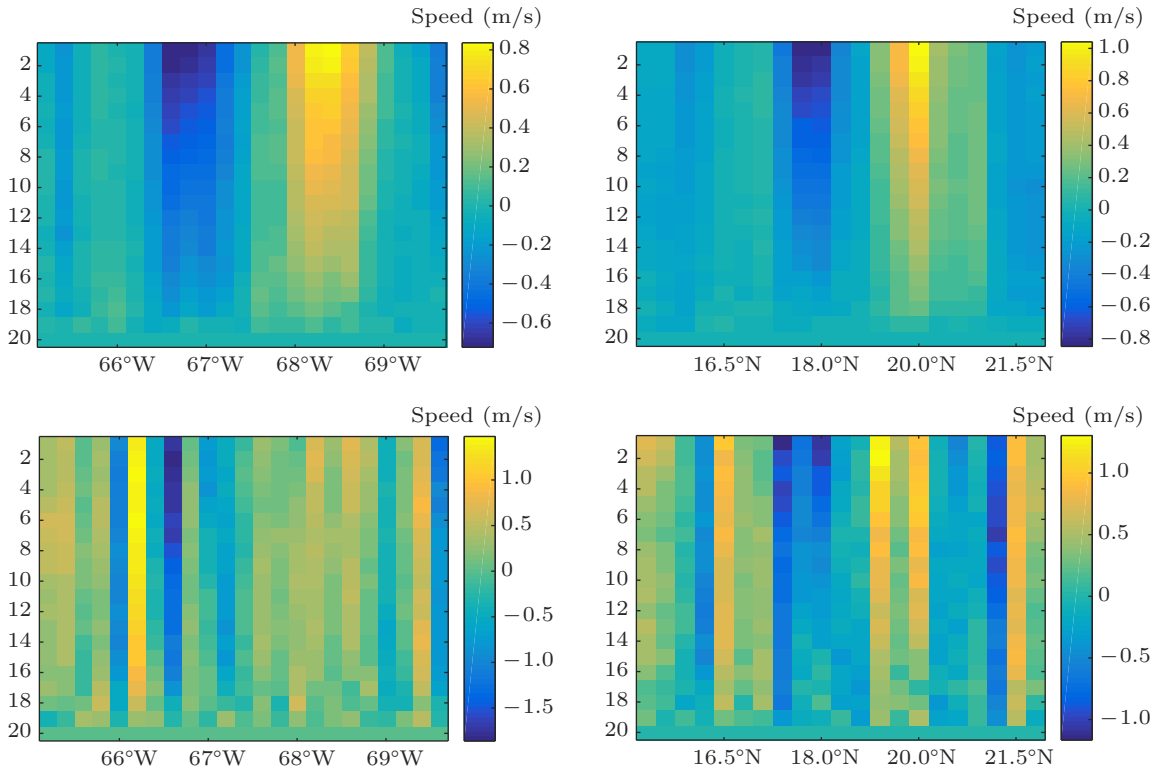


Fig. 4. The vertical velocity profiles.

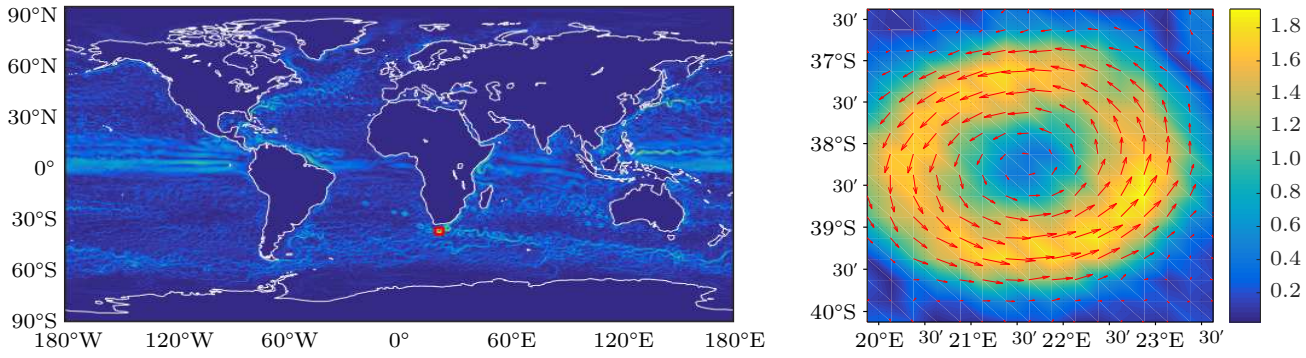


Fig. 5. The position and the vector graph of an anticyclonic eddy.

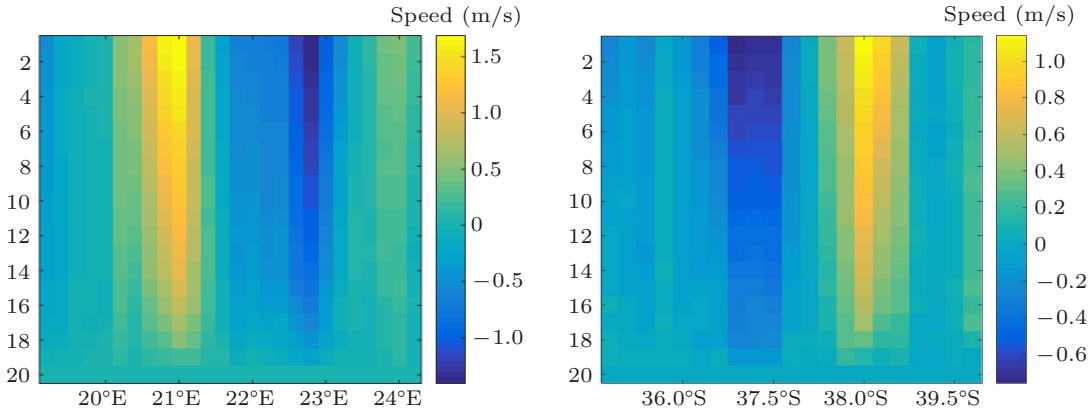


Fig. 6. The vertical velocity profiles of the anticyclonic eddy.

by further use of finite difference method. Based on the above boundary conditions, we take the observed data shown in Fig. 1(b) as "real data" and discuss the new algorithm and the finite difference method respectively by introducing a disturbance $\delta = 0.15$ as follows:

1) Calculate the vertical velocity at each layer by using

the finite difference method with a small disturbance $\delta = 0.15$. Figure 2 shows the distribution of vertical velocity at depth of 5 m, 115.87 m, 222.71 m and 299.93 m.

2) Calculate the vertical velocity at each layer by using the new algorithm proposed in this paper with a small disturbance $\delta = 0.15$. Figure 3 shows the distribution of vertical velocity at depth of 5 m, 115.87 m, 222.71 m and 299.93 m.

Comparing Fig. 3 with Fig. 2, when the disturbance exists, the vertical velocity calculated by the new algorithm is more reasonable and feasible than that of the difference method. For the vertical velocity of mesoscale eddies, some parts rise, while some parts decline. The cause is related to the eddies surrounding environment, the phenomenon explained in the literature[14] is similar with our results.

The vertical velocity profiles of the flow field obtained by the two methods are shown in Fig. 4 (for display purposes, we multiply the value of the vertical velocity by 1000). The two figures on the top are profiles of vertical velocity obtained by the new algorithm, from longitude and latitude direction respectively. The two profiles at the bottom of Fig. 4 are obtained at the same longitude and latitude by the finite difference method. It can be seen from the profiles at the same position that the new algorithm is obviously better than the finite difference method.

Since the mesoscale eddy include cyclone and anticyclonic eddy, we have used the new algorithm to construct the three-dimensional flow field of cyclone eddy, similarly, we select an anticyclonic eddy and use the new algorithm to construct its three-dimensional flow field. As shown in Fig. 5, the eddy located in the Cape of good hope, is an anticyclonic eddy. The horizontal oceanic data is also sourcing from SODA.

Based on the new algorithm, we calculate the vertical velocity of this anticyclonic eddy. The profiles of the vertical velocity are shown in Fig. 6. It can be seen from the two profiles that there is a strong upwelling and a downwelling in this anticyclonic eddy, which is in line with the motion law of the fluid in the eddy.

5. Conclusion and perspectives

In this paper, a new algorithm for calculating the vertical velocity of the incompressible flow field based on variation is proposed. The partial derivative is calculated by using the one-dimensional numerical differentiation method based on Tikhonov regularization. Comparing with the traditional finite difference method, we can draw the following conclusions:

1) In the case of calculating derivatives with the same method, the method based on variation is superior to the traditional finite difference method for the vertical velocity calculation.

2) When the observational error exists in the horizontal flow field, the one-dimensional numerical differentiation method can effectively reduce the relative error of the vertical velocity.

3) In the case of an unknown observational error, the vertical velocity calculated by the new algorithm is still more accurate than that of the finite difference method if the ratio of the guess value to exact value is between 0.1 and 2.

4) The missing upper and lower boundary conditions of vertical velocity can be obtained approximately by the one-dimensional numerical differentiation method and finite difference method. The relative error of vertical velocity recalculated by variation is also well controlled.

In summary, the vertical velocity calculation algorithm proposed in this paper is obviously better than the traditional finite difference method. This method can be widely used in the calculation of the vertical velocity in the ocean, meteorology and fluid experiment observations. Furthermore, by combining this method with the study of ocean dynamics and using the long-term data from SODA, the evolution of the three-dimensional structure of mesoscale eddies can be deeply studied.

Acknowledgments

The horizontal oceanic data used in this study are released by the SODA, which can be accessed in https://www2.atmos.umd.edu/ocean/. Inspired by the discussion with Wang Liang, the code of the proposed algorithm has been better optimized.

References

[1] Huang R Q, Xie L L, Zheng Q N, *et al.* 2020 *Acta Oceanologica Sinica* **39** 91
[2] Cheng Z, Zhou M, Zhong Y, *et al.* 2020 *Acta Oceanologica Sinica* **39** 36
[3] Lu M Z, Hou Z M and Zhou Y 2004 *Dynamic Meteorology* (Beijing: China Meteorological Press) p. 48
[4] Zeng Z Y 1997 *Meteorological Data. Assimilation* (Taipei: Bo Hai Hall Press) p. 68
[5] Sasaki Y K 1969 *Meteor. Soc Jpn.* **47** 115
[6] Huang S X, Teng J J, Lan W R and Xiang J 2005 *Chinese Journal of Theoretical and Applied Mechanics* **37** 399 (in Chinese)
[7] Wang Y B 2005 *Numerical Differentiation and Its Application* (Ph. D. Dissertation) (Shanghai: Fudan University) (in Chinese)
[8] Wang, Y B, Jia X Z and Cheng J 2002 *Inverse Problems* **18** 1461
[9] Cai Q F, Huang S X, Gao S T, Zhong K and Li Z Q 2008 *Acta Phys. Sin.* **57** 3912 (in Chinese)
[10] Wang, Y G, Cai Q F and Huang S X 2010 *Acta Phys. Sin.* **59** 4359 (in Chinese)
[11] Wang Y B and Ting W 2005 *J. Math. Anal. Appl.* **312** 121
[12] Wang Y B, Hon Y C and Cheng J 2006 *J. Inv. Ill-Posed Prob.* **14** 205
[13] Hanke M and Scherzer O 2001 *Amer Math Monthly* **108** 512
[14] Yang S, Xing J, Sheng J, *et al.* 2020 *Ocean Dynamics* **70** 879